

# SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

## KALYANAPURAM RANGACHARI PARTHASARATHY **An additional remark on unitary evolutions in Fock space**

*Séminaire de probabilités (Strasbourg)*, tome 25 (1991), p. 37-38

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# AN ADDITIONAL REMARK ON UNITARY EVOLUTIONS IN FOCK SPACE

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This is a continuation of our discussions in [1]. Adopting the notations of Section 2 in [1] express the Hilbert space  $h = h_0 \otimes (\mathcal{O}e_{-\infty} \oplus k \oplus \mathcal{O}e_{\infty})$  as a vector space of elements of the form

$$f = \begin{pmatrix} f_- \\ f_0 \\ f_+ \end{pmatrix}, f_{\pm} \in h_0, f_0 \in h_0 \otimes k.$$

Any bounded operator  $L$  in  $h$  can now be expressed as a  $3 \times 3$  matrix of appropriate operators. Let  $U$  be a unitary operator in  $h_0 \otimes k$ ,  $\ell$  a bounded operator from  $h_0$  into  $h_0 \otimes k$  and let  $H$  be a bounded selfadjoint operator in  $h_0$ . Define the operator  $L = L(U, \ell, H)$  in  $h$  by

$$L = \begin{pmatrix} 0 & -\ell^* & -iH - \frac{1}{2}\ell^*\ell \\ 0 & U - 1 & U\ell \\ 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

Then  $L \in \mathcal{I}(h)$ , i.e.,  $Lf \otimes e_{-\infty} = L^*f \otimes e_{\infty} \equiv 0$  and furthermore

$$L^b L + L^b + L = LL^b + L^b + L = 0,$$

the superscript  $b$  indicating the involution described in [1]. Thus there exists a unitary operator valued adapted process  $U_L$  satisfying

$$U_L(0) = 1, dU_L = (d\Lambda_L)U_L \quad (2)$$

in the Hilbert space  $h_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes k)$ ,  $\Gamma$  indicating the boson Fock second quantization. Then for any  $X \in \mathcal{B}(h_0)$ , putting  $j_t(X) = U_L^* X U_L$  we get

$$dj_t(X) = U_L^*(t) d\Lambda_{\theta(X)} U_L(t) \quad (3)$$

where

$$\theta(X) = L^b X + XL + L^b XL = \begin{pmatrix} 0 & \ell^* U^* XU - X \ell^* & \mathcal{L}(X) \\ 0 & U^* XU - X & U^* XU \ell - \ell X \\ 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}(\ell^* \ell X + X \ell^* \ell - 2\ell^* X \ell). \quad (5)$$

$\{j_t, t \geq 0\}$  defined by (3)-(5) is an Evans-Hudson flow whose vacuum expectation  $\mathbb{E}_0$  is given by

$$\mathbb{E}_0 j_t(X) = e^{t\mathcal{L}}(X), X \in \mathcal{B}(h_0).$$

Now consider the special case when  $k = L^2(\Omega, \mathcal{F}, P)$  is a separable probability space and  $h_1 = h_0 \otimes k = L^2(P, h_0)$ , the Hilbert space of norm square integrable  $h_0$ -valued maps on  $(\Omega, \mathcal{F}, P)$ . Suppose the operators  $U, \ell$  in (1) are of the form

$$(Uf)(\omega) = U(\omega)f(\omega), (\ell u)(\omega) = \ell(\omega)u$$

where  $U(\cdot)$  is a  $h_0$ -unitary operator valued map and  $\ell(\cdot)$  is a  $\mathcal{B}(h_0)$ -valued map on  $(\Omega, \mathcal{F}, P)$ . If  $U(\omega)\ell(\omega) = M(\omega)$  then (5) assumes the form

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2} \int_{\Omega} [M(\omega)^* M(\omega) X + X M(\omega)^* M(\omega) - 2M(\omega)^* X M(\omega)] dP(\omega) \quad (6)$$

Suppose  $h_0 = L^2(\mathcal{X}, \mathcal{S}, \mu)$  where  $(\mathcal{X}, \mathcal{S}, \mu)$  is a  $\sigma$ -finite separable measurable space and for any  $\phi \in L^\infty(\mu)$

$$U(\omega)^* \phi U(\omega) = \phi \circ T(\omega)$$

where  $T(\omega)$  is a  $\mu$ -measure class preserving transformation on  $\mathcal{X}$  for each  $\omega \in \Omega$  and  $L^\infty(\mu)$  is viewed as the abelian  $*$  subalgebra of  $\mathcal{B}(h_0)$ . Furthermore let  $\ell(\omega)$  be multiplication by  $\ell(x, \omega)$  in  $L^2(\mu)$  and  $H = 0$ . Then (6) becomes

$$\ell(\phi)(x) = \int_{\Omega} |\ell(x, \omega)|^2 \{\phi(T(\omega)x) - \phi(x)\} dP(\omega) \quad (7)$$

and  $\{j_t|_{L^\infty(\mu)}, t \geq 0\}$  is an Evans-Hudson flow describing a classical Markov flow with generator given by (7).

### References

- [1]. K.R. Parthasarathy : Realisation of a class of Markov processes through unitary evolutions in Fock space, Preprint, Indian Statistical Institute, Delhi (1990).