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An extension of Krein's inverse spectral theorem  
to strings with nonreflecting left boundaries

by

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**Abstract:** Krein's inverse spectral theorem describes the spectral measures  $\tau$  of the differential operators  $D_m D_x$  with boundary condition  $f'_-(0) = 0$ , if  $m$  runs through all nondecreasing functions on  $[0, \infty)$ . This result will be extended to boundary conditions of the type  $af'_-(0) - f(0) = 0$  ( $a \in [0, \infty)$ ).

Other conditions as in Krein's theorem appear.

**Key words:** gap-diffusions, quasidiffusions, generalized second order differential operator, spectral measures, local times, Krein's inverse spectral theorem, Krein's correspondence

60J35, 60J60, 34B20

## 1. Introduction

It is well-known that every nondecreasing function  $m$  on  $[0, \infty)$  performed with appropriate boundary conditions at zero and at  $1 := \sup \text{supp } m$  (a so-called string) generates a strong Markov process  $(X_t)$  on  $\text{supp } m$ , where  $\text{supp } m$  denotes the set of points where  $m$  increases. This process has as its (selfadjoint) infinitesimal generator in  $L_2(m)$  the generalized second order differential operator  $D_m D_x$  together with the mentioned boundary conditions.  $(X_t)$  is called a quasi- (or gap-) diffusion with speed measure  $m$ . Examples are diffusions and birth- and death-processes. Several probabilistic quantities of  $(X_t)$  as e.g. transition densities, first hitting time densities, Lévy-measures of the inverse local time at zero, can be expressed in terms of spectral measures  $\tau^{(m)}$  of  $D_m D_x$  under different boundary conditions, see e.g. Ito, McKean [2], Küchler [7], [8], Küchler, Salminen [9].

An essential result concerning these spectral measures is M.G. Krein's inverse spectral theorem, in a more extended form known as Krein's correspondence theorem, see Kac, Krein [3], Kotani, Watanabe [6]. Roughly speaking it states that the mapping  $m \rightarrow \tau^{(m)}$  is a one-to-one and onto correspondence between the strings  $m$  with the "reflecting" boundary condition  $f^-(0) := f^+(0-) = 0$  and the set of all measures  $\tau$  on  $[0, \infty)$  that integrate  $(1 + \mu)^{-1}$  thereon, see Theorem 2.2 below. What we are going to do is to study the situation for the boundary conditions

$$af^-(0) - f(0) = 0,$$

where  $a \in [0, \infty)$  is fixed. (The case above corresponds to  $a = \infty$ .)

If  $a \in (0, \infty)$  ("elastic killing boundary"), then there is still a one-to-one and into correspondence (Theorem 2.4). If  $a = 0$ , then  $m \rightarrow \tau^{(m)}$  maps the strings  $m$  with the "killing" boundary condition  $f(0) = 0$  onto the set of measures on  $(0, \infty)$  that integrate  $[\mu(1+\mu)]^{-1}$ , but not one-to-one. In Theorem 3.2 we shall describe the preimages for every  $\tau$  which form one-parametric families.

As an application we get the description of all measures  $\nu$  that can appear as the Lévy-measure of the inverse local times at zero for quasidiffusions (see Remark 3.6). This result was proved by other (probabilistic) means in Knight [5]. Here we shall present an analytical approach.

Moreover, a generalization of Lemma 1 of Karlin, McGregors paper [4] concerning birth- and death-processes to strings is given (see Corollary 3.7).

## 2. Strings, spectral measures and Krein's theorem

Here we shall summarize some facts from the theory of generalized second order differential operators  $D_m D_x$ . For details the reader is referred to Kac, Krein [3] or Dym, McKean [1], the latter uses another terminology.

By  $R$  and  $K$  we denote the real axis and the complex plane, respectively.  $R_+$  stands for  $[0, \infty)$ ,  $K_-$  for  $K \setminus R_+$ . Put  $\bar{R}_+ := [0, \infty]$  and  $\frac{1}{0} := \infty$ ,  $\frac{1}{\infty} := 0$ . Let  $m$  be a nondecreasing right-continuous extended real-valued function on  $R$  with  $m(x) \equiv 0$ ,  $x < 0$ . Define  $E_m$  to be the set of points where  $m$  increases and is finite:

$$E_m := \{x \in R_+ \mid \exists \varepsilon_0 > 0: m(x-\varepsilon) < m(x+\varepsilon) < \infty \quad \forall \varepsilon \in (0, \varepsilon_0)\}.$$

We shall assume  $E_m \neq \emptyset$  and denote by the same letter  $m$  the measure generated by the function  $m$ . Such a measure  $m$  is called a speed measure.

Introduce  $c$ ,  $l$  and  $r$  by

$$c := \inf E_m = \inf \{x \geq 0 \mid m(x) > 0\},$$

$$l := \sup E_m \leq \infty,$$

$$r := \sup \{x \geq 0 \mid m(x) < \infty\} \leq \infty.$$

We have  $0 \leq c \leq l \leq r$  and put  $h := r - l$  if  $l < \infty$ . Otherwise  $h$  is irrelevant and for convenience we put  $h = 0$  in this case. Note that  $h = 0$  if  $l < \infty$  and  $m(l) = \infty$ . If  $l + m(l) < \infty$  and  $m(\{l\}) > 0$ , then  $h$  must be greater than zero. The number  $r = l + h$  is called the length of the string.

Sometimes we shall write  $c_m, l_m, \dots$  to express that these numbers come from  $m$ .

By  $\mathcal{V}$  we denote the set of all real functions  $f$  on  $R$  having a representation

$$f(x) = \bar{a} + \bar{b} \cdot x + \int_0^x (x-s)g(s)m(ds) \quad (2.1)$$

for some measurable  $g$  on  $R$  and some reals  $\bar{a}, \bar{b}$ .

Note that every  $f \in \mathcal{V}$  is continuous and linear on the open intervals of  $R \setminus E_m$ .

On  $\mathcal{V}$  we define a generalized second order differential operator  $D_m D_x$  by  $D_m D_x f = g$ , details can also be found in K  chler [7], [8].

For every fixed  $a \in [0, \infty]$  the restriction  $A_a$  of  $D_m D_x$  to

$$\Delta_a := \{f \in \mathcal{V} \cap L_2(m) \mid D_m D_x f \in L_2(m), af^-(0) - f(0) = 0\} \quad (2.2)$$

(for  $a = \infty$  we mean  $f^-(0) = 0$ ) is a nonnegative selfadjoint operator in  $L_2(m)$ .

(By  $f^+$  and  $f^-$  we denote the right- and left-hand-side derivative of  $f$ , respectively.)

Note that  $f \in \mathcal{V} \cap L_2(m)$  implies  $f(r) = 0$  if  $r = l + h < \infty$ .

Because of the linearity of  $f$  on the intervals of  $R \setminus E_m$  this can be written as a boundary condition  $hf^+(1) + f(1) = 0$  with  $f^+(1) = 0$  if  $h = \infty$ . Otherwise, the boundary condition appearing in (2.2) can also be included in  $f \in \mathcal{V} \cap L_2(m)$  if we change  $m$  to the left of  $-a$  into  $m(x) = -\infty, x < -a$ .

In the following,  $m$  will be understood in this way.

This change of  $m$  charges  $-a$  with infinite mass. The original

measure  $m$  on  $R_+$  remains unchanged by this procedure if  $a > 0$ . In case  $a = 0$ , the value of  $m(\{0\})$  is not reconstructable. But this does not disturb the corresponding spectral theory, as we will see below. Thus we suppose  $m(0) = 0$  if we consider  $a = 0$ . Now  $m$  has infinite mass at  $-a$  (and  $r$  if  $r < \infty$ ) and thus  $f \in \mathcal{V} \cap L_2(m)$  implies also  $f(-a) = 0$ , i.e.  $af^-(0) - f(0) = 0$ .

Therefore, the selfadjoint operators  $A_a$  are characterized by the (changed) function  $m$ , or by  $(m, a)$ . We call the pair  $(m, a)$  a string and denote it by  $S_a(m)$ . If the length  $r = l + h$  is infinite, then we say that the string  $S_a(m)$  is infinite. Depending on  $l + m(l-) < \infty$  or  $= \infty$  the string  $S_a(m)$  is called regular or singular. The resolvent operator  $R_{\lambda, a} := (A_a - \lambda I)^{-1}$  exists for  $\lambda \in (-\infty, 0)$ , and it can be shown analogously to Dym, McKean [1] that  $R_{\lambda, a}$  is given by

$$(R_{\lambda, a} f)(x) = \int_0^1 r_{\lambda, a}(x, y) f(y) m(dy), \quad f \in L_2(m),$$

where

$$r_{\lambda, a}(x, y) := \frac{\Phi_a^\uparrow(x \wedge y, \lambda) \Phi_a^\downarrow(x \vee y, \lambda)}{W}.$$

Here  $\Phi_a^\uparrow$  and  $\Phi_a^\downarrow$  denote the solutions  $f \in \mathcal{V}$  of

$$D_m D_x f + \lambda f = 0$$

satisfying the boundary conditions

$$\Phi_a^\uparrow(0, \lambda) = 1, \quad a \in (0, \infty]; \quad \Phi_0^{\uparrow-}(0, \lambda) = 1, \quad (2.3)$$

$$a \Phi_a^{\uparrow-}(0, \lambda) - \Phi_a^\uparrow(0, \lambda) = 0, \quad a \in [0, \infty); \quad \Phi_\infty^{\uparrow-}(0, \lambda) = 0, \quad (2.4)$$

and

$$\Phi_a^{\downarrow-}(0, \lambda) = -1, \quad \text{and} \quad (2.5)$$

$$h \Phi_a^{\downarrow+}(1, \lambda) + \Phi_a^\downarrow(1, \lambda) = 0. \quad (2.6)$$

Note that  $\Phi_a^\uparrow(\cdot, \lambda)$  is increasing and  $\Phi_a^\downarrow(\cdot, \lambda)$  is decreasing for fixed  $\lambda < 0$ .

$W$  denotes the Wronskian:

$$W = W(\lambda) := \Phi_a^{\uparrow-} \Phi_a^\downarrow - \Phi_a^\uparrow \Phi_a^{\downarrow-}$$

Several times we will use that  $\Phi_a^\uparrow(\cdot, \lambda)$  is the uniquely determined solution of

$$\Phi(x, \lambda) = 1 + \frac{x}{a} - \lambda \int_0^x (x-s) \Phi(s, \lambda) m(ds), \quad x \in [0, r) \quad (2.7)$$

for  $a \in (0, \infty]$ , and of

$$\Phi(x, \lambda) = x - \lambda \int_0^x (x-s) \Phi(s, \lambda) m(ds), \quad x \in [0, r) \quad (2.8)$$

for  $a = 0$ .

Similarly,  $\Phi^\downarrow(\cdot, \lambda)$  is the unique solution of

$$\Phi(x, \lambda) = \Phi(0, \lambda) - x - \lambda \int_0^x (x-s) \Phi(s, \lambda) m(ds), \quad x \in [0, r). \quad (2.9)$$

**DEFINITION 2.1:** Assume  $S_a(m)$  is a string with  $a \in [0, \infty]$ . Then a measure  $\tau$  on  $[0, \infty)$  is called a spectral measure of  $S_a(m)$ , if

$$r_{\lambda, a}(x, y) = \int_0^\infty \frac{\Phi_a^\uparrow(x, \mu) \Phi_a^\uparrow(y, \mu)}{\mu - \lambda} d\tau(\mu), \quad \lambda < 0; x, y \in E_m.$$

The set  $\text{supp } \tau$  is called the spectrum of  $S_a(m)$ .

As for the case of  $a = \infty$ , treated in Kac, Krein [3] and Dym, McKean [1], one can show that for every string  $S_a(m)$  a unique spectral measure  $\tau$  exists (on  $(0, \infty)$  if  $a \neq 0$ ). It will often be denoted by  $\tau_a^{(m)}$ . (We shall identify measures  $\tau$  on  $R_+$  and their generating function  $\mu \rightarrow \tau([0, \mu])$ .)

Note that  $\Phi_0^\uparrow(\cdot, \lambda)$ , and therefore  $\tau_0^{(m)}$  does not depend on the mass of  $m$  at zero. Thus, considering a string  $S_0(m)$  we shall always suppose that  $m(0) = 0$ .

If the string  $S_a(m)$  is regular, then  $\tau_a^{(m)}$  is given by

$$\tau_a^{(m)}(\mu) = \sum_{k=0}^{\infty} \tau_a^{(m)}(\{\mu_k\}) \cdot \mathbb{1}_{[0, \mu_k]}(\mu) \quad (2.10)$$

where  $(\mu_k)_{k \geq 0}$  denotes the sequence of solutions of

$$h \Phi_a^{\uparrow, +}(1, \mu) + \Phi_a^\uparrow(1, \mu) = 0$$

and

$$\tau_a^{(m)}(\{\mu_k\}) = \left[ \int_0^1 [\Phi_a^\uparrow(x, \mu_k)]^2 dm \right]^{-1}.$$

We have

$$0 \leq \mu_0 < \mu_1 < \dots < \mu_n < \dots \quad \text{and} \quad \sum_{n \geq 1} \mu_n^{-1} < \infty.$$

The following theorem answers the question which measures may appear as spectral measures for strings  $S_\infty(m)$ . Its second part is M.G. Krein's inverse spectral theorem, (i) and (ii) together are known as Krein's correspondence (Kotani, Watanabe [6]).

**THEOREM 2.2:**

(i) For every string  $S_\infty(m)$  its spectral measure  $\tau = \tau_\infty^{(m)}$  satisfies

$$\int_{0-}^{\infty} \frac{d\tau(\mu)}{1+\mu} < \infty. \quad (2.11)$$

(ii) For every measure  $\tau$  on  $R_+$  with  $\tau(R_+) > 0$  and (2.11) there exists one and only one string  $S_\infty(m)$  with  $c_m = 0$  having  $\tau$  as its spectral measure.

Note that the condition  $c_m = 0$  in (ii) ensures the unicity of  $m$ . Indeed, all "shifted" strings  $S_\infty(m(\cdot - c))$  ( $c > 0$ ) have the same spectral measure, compare Proposition 2.3 below.

For every string  $S_\infty(m)$  its characteristic function  $\Gamma_m(\cdot)$  is given by (see Chapter 4 below)

$$\Gamma_m(\lambda) := c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu - \lambda} = \lim_{\lambda \uparrow r} \frac{\Phi_0^\uparrow(x, \lambda)}{\Phi_\infty^\uparrow(x, \lambda)}, \quad \lambda \in K_-. \quad (2.12)$$

Because of the definition of the spectral measure we obtain

$$\Phi^\downarrow(0, \lambda) = r_{\lambda, \infty}(0, 0) = \Gamma_m(\lambda), \quad \lambda < 0. \quad (2.13)$$

Letting  $\lambda \uparrow 0$  in (2.12) we get the formula

$$c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu} = r = 1 + h \quad (2.14)$$

with the understanding that  $h = 0$  if  $1 + m(1-) = \infty$ .

Krein's theorem says that  $\Gamma_m(\cdot)$  determines  $S_\infty(m)$  uniquely.

In Chapter 4 below we shall see that it holds

$$-\frac{1}{\Gamma_m(\lambda)} = \lambda m(\{0\}) - r_m^{-1} - \int_0^\infty \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) \tau_0^{(m)}(d\mu), \quad \lambda \in K_-. \quad (2.15)$$

For every string  $S_\infty(m)$  with  $c_m = 0$  we have

$$\tau_\infty^{(m)}(R_+) = [m(\{0\})]^{-1}. \quad (2.16)$$

Indeed, consider  $\lambda \Gamma_m(\lambda)$  for  $\lambda \downarrow -\infty$  and compare (2.12) and (2.15), then (2.16) is obvious.

Finally, note that  $\tau_\infty^{(m)}(\{0\}) > 0$  if and only if the constant function  $\Phi_\infty^\uparrow(\cdot, 0) \equiv 1$  is an eigenfunction of  $D_m D_x$ . This holds if and only if  $r = 1 + h = \infty$  (or  $1 = \infty$ ) and  $m(1-) < \infty$ . Moreover, in this case we have

$$\tau_\infty^{(m)}(\{0\}) = [m(1)]^{-1}. \quad (2.17)$$

The next proposition shows how the spectral measure changes if  $m$  suffers certain transformations.

**PROPOSITION 2.3:** Let  $S_a(m)$  be a string with  $a \in [0, \infty]$  and assume  $u, v \in (0, \infty)$ ,  $w \in [0, \frac{a}{u}]$  and  $w < \infty$ . Define

$$\begin{aligned} \tilde{m}(x) &:= v \cdot m(u(x-w)), & x \in R, \\ \tilde{a} &:= \frac{a}{u} - w. \end{aligned}$$

Then, for the spectral measures  $\tilde{\tau}_{\tilde{a}} := \tau_{\tilde{a}}^{(\tilde{m})}$  and  $\tau_a := \tau_a^{(m)}$  of  $S_{\tilde{a}}(\tilde{m})$  and  $S_a(m)$ , respectively, we have

(i) If  $a = \infty$ , then for all  $w \in [0, \infty)$  we have  $\tilde{a} = \infty$  and

$$\tilde{\tau}_\infty(\mu) = v^{-1} \tau_\infty\left(\frac{v}{u} \cdot \mu\right), \quad \mu \geq 0.$$

(ii) If  $a \in (0, \infty)$ ,  $0 < w < \frac{a}{u}$ , then  $\tilde{a} \in (0, \infty)$  and

$$\tilde{\tau}_{\tilde{a}}(\mu) = v^{-1} \left(1 - \frac{uw}{a}\right)^2 \tau_a\left(\frac{v}{u} \cdot \mu\right), \quad \mu \geq 0.$$

(iii) If  $a \in (0, \infty)$ ,  $w = \frac{a}{u}$ , then  $\tilde{a} = 0$  and

$$\tilde{\tau}_0(\mu) = v^{-1} \left(\frac{u}{a}\right)^2 \tau_a\left(\frac{v}{u} \cdot \mu\right), \quad \mu \geq 0.$$

(iv) If  $a = w = 0$ , then

$$\tilde{\tau}_0(\mu) = v^{-1} \cdot u^2 \tau_0\left(\frac{v}{u} \cdot \mu\right), \quad \mu \geq 0.$$

To prove this proposition one calculates the relevant  $\tilde{\Phi}_{\tilde{a}}^\uparrow$  and  $\tilde{\Phi}_{\tilde{a}}^\downarrow$  in terms of  $\Phi_a^\uparrow$  and  $\Phi_a^\downarrow$ , respectively, using (2.7 - 2.9). This gives the relation between the resolvent kernels  $\tilde{r}_{\tilde{a}, \tilde{a}}$  and  $r_{\lambda, a}$ . Definition 2.1 leads to the assertion of Proposition 2.3.



### 3. Results

In this chapter we shall formulate correspondence theorems for strings  $S_a(m)$  with  $a \neq \infty$  which extend Krein's result. The proofs can be found in Chapter 4.

We shall start with the case of  $a = 0$ . For this purpose we still need a preparation.

Denote by  $\Sigma$  the set of all strings  $S_0(m)$  with  $m(0) = 0$ . We introduce a relation  $\sim$  in  $\Sigma$  by defining  $S_0(m) \sim S_0(n)$  if there exists a real number  $t \geq -\frac{1}{r_n}$  such that the transformation

$$x \longrightarrow T_t x := \frac{x}{1 - tx}, \quad x \in \mathbb{R}$$

maps  $(0, r_m)$  onto  $(0, r_n)$  and such that

$$m(x) = \int_{0+}^x (1-ts)^{-2} dn(T_t s), \quad x \in (0, r_m) \quad (3.1)$$

(indeed,  $t = \frac{1}{r_m} - \frac{1}{r_n}$ .) It is easy to see that  $\sim$  forms an equivalence relation in  $\Sigma$ . Put  $\hat{\Sigma} := \Sigma / \sim$  and for every string  $S_0(m) \in \Sigma$  denote by  $\hat{S}(m)$  the element of  $\hat{\Sigma}$  generated by  $S_0(m)$ . For every string  $S_0(m)$  and every  $t \geq -\frac{1}{r_m}$  we define a new string  $S_0(m_t)$  by  $r_{m_t} := \frac{r_m}{1 + tr_m}$  and

$$\left. \begin{aligned} m_t(x) &:= \int_{0+}^x (1-ts)^{-2} dm(T_t s), & x \in (0, r_{m_t}) \\ m_t(x) &:= \infty, & x \geq r_{m_t} \end{aligned} \right\} \quad (3.2)$$

Obviously, we have

$$c_{m_t} = \frac{c_m}{1 + tc_m}, \quad l_{m_t} = \frac{l_m}{1 + tl_m}, \quad t \geq -\frac{1}{r_m}, \quad (3.3)$$

and  $S_0(m_t) \sim S_0(m)$  for every  $t \geq -\frac{1}{r_m}$ .

Otherwise, if  $S_0(n) \sim S_0(m)$ , then, by definition, there exists a real number  $t \geq -\frac{1}{r_n}$  such that  $m = n_t$ . Observe  $r_{m_t} = \infty$  if and only if  $t = -\frac{1}{r_m}$ .

Thus we have proved the following

**LEMMA 3.1:**

- (i) For every string  $S_0(m)$  its equivalence class  $\hat{S}(m)$  is equal to  $\{S_0(m_t) \mid t \geq -\frac{1}{r_m}\}$ .
- (ii) Every equivalence class  $\hat{S} \in \hat{\Sigma}$  contains one and only one infinite string  $S_0(m)$ .

Now we are ready to formulate the analogue of Krein's correspondence for strings  $S_0(m)$ .

**THEOREM 3.2:**

- (i) For every string  $S_0(m)$  its spectral measure  $\tau = \tau_0^{(m)}$  is supported by  $(0, \infty)$  and has the property

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu(1+\mu)} < \infty. \quad (3.4)$$

Moreover, it holds

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = c_m^{-1} - r_m^{-1}. \quad (3.5)$$

- (ii) If two strings  $S_0(m)$  and  $S_0(n)$  are equivalent (with respect to  $\sim$ ), then  $\tau_0^{(m)} = \tau_0^{(n)}$ .
- (iii) For every measure  $\tau$  on  $(0, \infty)$  ( $\tau((0, \infty)) > 0$ ) with (3.4) and every  $r \in (0, \infty]$  there exists one and only one string  $S_0(m)$  with length  $r = r_m$  having  $\tau$  as its spectral measure. If  $S_0(m)$  and  $S_0(m')$  are strings with the lengths  $r$  and  $r'$ , respectively, having the same spectral measure, then  $S_0(m') = S_0(m_t)$  holds with  $t = \frac{1}{r'} - \frac{1}{r}$ . ( $m_t$  was defined in (3.2).)

This theorem can be reformulated in a shorter way as follows.

**COROLLARY 3.3:** There is a one-to-one and onto correspondence between the set  $\hat{\Sigma}$  of equivalence classes  $\hat{S}$  of strings  $S_0(m)$  and the set of measures  $\tau$  on  $(0, \infty)$  satisfying (3.4), where  $\tau$  is the spectral measure  $\tau_0^{(m)}$  of every string  $S_0(m)$  from  $\hat{S}$ .

Now let us turn to the case of  $a \in (0, \infty)$ .

**THEOREM 3.4:** Assume  $a \in (0, \infty)$ . Then it holds:

- (i) For every string  $S_a(m)$  with  $c_m = 0$  and  $m(0) \geq 0$  its spectral measure  $\tau = \tau_a^{(m)}$  is supported on  $(0, \infty)$  and has the property

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = \left( \frac{1}{r_m} + \frac{1}{a} \right)^{-1} < \infty \quad (3.6)$$

- (ii) If  $\tau$  is a measure on  $(0, \infty)$  with nonzero mass, then there exists a string  $S_a(m)$  with  $c_m = 0$  having  $\tau$  as its spectral measure if and only if

$$g(\tau) := \int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} \leq a. \quad (3.7)$$

In this case,  $S_a(m)$  is uniquely determined.

Moreover, if  $S_a(m)$  and  $S_{a'}(m')$  with  $a, a' \in (0, \infty)$ ,  $c_m = c_{m'} = 0$ , have the same spectral measure, then

$$m'(x - a') = m_t(x - a), \quad x \in \mathbb{R}_+$$

with  $t := \frac{1}{a'} - \frac{1}{a}$ , where  $m_t$  was defined in (3.2).

Consider a speed measure  $m$  on  $[0, \infty)$  with  $c_m = 0$ ,  $m(0) \geq 0$  and form strings  $S_{\infty}(m)$ ,  $S_a(m)$  and  $S_0(m)$  for some  $a \in (0, \infty)$ .

(Note that  $m(\{0\})$  disappears if we construct  $S_0(m)$ .)

Then we have

**PROPOSITION 3.5:** Between the spectral measures  $\tau_{\infty}$ ,  $\tau_a$  and  $\tau_0$  of  $S_{\infty}(m)$ ,  $S_a(m)$  with  $a \in (0, \infty)$  and  $S_0(m)$ , respectively, the following equation holds:

$$\begin{aligned} & [\lambda m(0) - r_m^{-1} - \int_{0+}^{\infty} \left( \frac{1}{\mu} - \frac{1}{\mu - \lambda} \right) d\tau_0(\mu)] \cdot \\ & \cdot \left[ \int_{0+}^{\infty} \frac{d\tau_a(\mu)}{\mu - \lambda} \right] \cdot \left[ a + \int_{0-}^{\infty} \frac{d\tau_{\infty}(\mu)}{\mu - \lambda} \right] = -a, \quad \lambda \in K_-. \end{aligned} \quad (3.8)$$

This generalizes a formula which was used by Knight [5], p. 60.

Consider a string  $S_{\infty}(m)$  and add to  $m$  some point mass  $m_0 > 0$  at zero if necessary, i.e. if  $c_m > 0$ . As we know, this does not touch the spectral measure  $\tau_0^{(m)}$  of  $S_0(m)$ . Now, let  $l(t, 0)$ ,  $t \geq 0$ , be the local time at zero of the quasidiffusion generated by  $S_{\infty}(m)$ . Since  $0 \in E_m$ , this notion makes sense. Then  $(l^{-1}(t, 0), t \geq 0)$  is an increasing process with independent stationary increments and it holds

$$E_0 \exp(\lambda l^{-1}(t, 0)) = \exp(-\frac{t}{r_m(\lambda)}) , \quad \lambda < 0, t \geq 0.$$

(See e.g. Knight [5] or Küchler [8].)

For  $\lambda < 0$ , (2.15) implies

$$-\frac{1}{r_m(\lambda)} = \lambda m(\{0\}) - \frac{1}{r_m} - \int_0^\infty (1 - e^{\lambda y}) \left[ \int_{0+}^\infty e^{-\mu y} \tau_0^{(m)}(d\mu) \right] dy.$$

Thus, by Theorem 3.2(ii) and Lemma 3.1(i) the Lévy-measure  $n$  of  $l^{-1}(\cdot, 0)$ , given by

$$dn(y) := \int_{0+}^\infty e^{-\mu y} \tau_0^{(m)}(d\mu) dy , \quad y \in \mathbb{R}_+, \quad (3.9)$$

is the same for all  $S_\infty(m_t)$ ,  $t \geq -\frac{1}{r_m}$ .

This means that the inverse local times at zero of the quasidiffusions corresponding to  $S_\infty(m_t)$  differ in their killing rate

$k = \frac{1}{r_m} + t$  only.

Now Theorem 3.2 implies

**COROLLARY 3.6:** For every nontrivial measure  $\tau$  on  $(0, \infty)$  with (3.4), every  $m(\{0\}) > 0$  and every constant  $k \geq 0$  there exists a quasidiffusion with speed measure  $m$ , a reflecting boundary at zero and length  $\frac{1}{k}$  of the string  $S_\infty(m)$  such that  $l^{-1}(\cdot, 0)$  has the Lévy-measure (3.9).

This result was proved by other means in Knight [5].

As an example consider a birth- and death-process on the set of non-negative integers with the intensities  $\mu_0 \geq 0$ ,  $\lambda_i > 0$ ,  $\mu_{i+1} > 0$ ,  $i \geq 0$ . Then

$$m(x) := \sum_{i=0}^{\infty} m_i \cdot \mathbb{1}_{[0, x]}(x_i)$$

$$\text{with } x_0 := 0, \quad x_i := \sum_{j=0}^{i-1} \frac{1}{\lambda_j m_j},$$

$$m_0 := 1, \quad m_i := \prod_{j=1}^i \frac{\lambda_{j-1}}{\mu_j}, \quad i \geq 1$$

and  $a := \mu_0^{-1}$ ,  $h \geq 0$  define a string  $S_a(m)$ . (Necessarily,  $h = 0$  if  $m$  is singular.) We have

$$D_m D_x f(x_i) = \left[ \frac{\Delta f(x_i)}{\Delta x_i} - \frac{\Delta f(x_{i-1})}{\Delta x_{i-1}} \right] m_i^{-1} =$$

$$= \lambda_i f(x_{i+1}) - (\lambda_i + \mu_i) f(x_i) + \mu_i f(x_{i-1}), \quad i \geq 1$$

with  $\Delta u(x_j) := u(x_{j+1}) - u(x_j)$ .

Moreover,

$$D_m D_x f(x_0) = \frac{\frac{\Delta f(x_0)}{x_1} - f^-(x_0)}{m_0} \quad \text{and}$$

the boundary condition

$$a f^-(x_0) - f(x_0) = 0$$

is equivalent to

$$D_m D_x f(x_0) = -(\lambda_0 + \mu_0) f(x_0) + \lambda_0 f(x_1).$$

Thus, we have

$$\bar{\Phi}_a^\dagger(x_i, \lambda) = Q_i(\lambda), \quad i \geq 0, \lambda \in \mathbb{R}$$

in the terminology of Karlin, McGregor [4].

The spectral measure  $\tau_a^{(m)}$  of  $S_a(m)$  is a solution of the Stieltjes moment problem connected with the Jacobi-matrix  $(a_{ij})$  with

$$a_{ij} := \lambda_i \mathbb{1}_{1(j-i)} + \mu_i \mathbb{1}_{1(i-j)} - (\lambda_i + \mu_i) \mathbb{1}_0(i-j) \quad (i, j \geq 0).$$

Indeed, for  $\lambda \rightarrow -\infty$  we have

$$\| -\lambda R_{\lambda, a} f - f \|_{L_2(m)} \rightarrow 0, \quad f \in L_2(m).$$

Consequently,

$$\langle -\lambda R_{\lambda, a} f, g \rangle_{L_2(m)} \rightarrow \langle f, g \rangle_{L_2(m)}, \quad f, g \in L_2(m).$$

Choosing  $f = \mathbb{1}_{\{x_i\}}$ ,  $g = \mathbb{1}_{\{x_j\}}$  we obtain

$$\lim_{\lambda \rightarrow -\infty} -\lambda r_{\lambda, a}(x_i, x_j) =$$

$$\int_0^\infty \bar{\Phi}_a^\dagger(x_i, \mu) \bar{\Phi}_a^\dagger(x_j, \mu) d\tau_a^{(m)}(\mu) = \frac{\delta_{ij}}{m_i}, \quad i, j \geq 0.$$

Compare this equation with Theorem 1 of Karlin, McGregor [4], p. 494 to get the assertion.

Now, Lemma 1 of Karlin, McGregor [4] can be generalized to strings as follows.

**COROLLARY 3.7:** Given a string  $S_\infty(m)$  with  $c_m = 0$  and with the spectral measure  $\tau$  and assume  $a > 0$ . Then there exists a string  $S_a(m')$  with  $c_{m'} = 0$  having the same spectral measure  $\tau$  if and only if

$$r_m = l_m + h_m \leq a. \quad (3.10)$$

**Proof:** If  $\tau(\{0\}) > 0$ , then there does not exist such a string  $S_a(m')$  because, for  $a \neq \infty$ , the spectral measure is concentrated on  $(0, \infty)$ . Otherwise,  $r_m = \infty$ , see the remarks before (2.17).

Assume  $\tau(\{0\}) = 0$ . From (2.14) we know  $r_m = \int_0^\infty \frac{d\tau(\mu)}{\mu}$ . Now apply Theorem 3.4(ii).

#### 4. Proofs

At first we shall collect some results of the spectral theory of  $D_m D_x$ . For details see e.g. Kac, Krein [3]. Let us given a string  $S_\infty(m)$ . The characteristic function  $\Gamma(\cdot)$  of  $S_\infty(m)$  is given by the limit (see (2.12))

$$\Gamma(\lambda) = \lim_{x \uparrow r} \frac{\Phi_0^\uparrow(x, \lambda)}{\Phi_\infty^\uparrow(x, \lambda)}, \quad \lambda \in K_-. \quad (4.1)$$

In the regular case we have for  $h < \infty$

$$\Gamma(\lambda) = \frac{\Phi_0^\uparrow(r, \lambda)}{\Phi_\infty^\uparrow(r, \lambda)} = \frac{\Phi_0^{\uparrow,+}(1, \lambda) \cdot h + \Phi_0^\uparrow(1, \lambda)}{\Phi_\infty^{\uparrow,+}(1, \lambda) \cdot h + \Phi_\infty^\uparrow(1, \lambda)}, \quad (4.2)$$

and for  $h = \infty$  it holds

$$\Gamma(\lambda) = \frac{\Phi_0^{\uparrow,+}(1, \lambda)}{\Phi_\infty^{\uparrow,+}(1, \lambda)}. \quad (4.3)$$

If  $S_\infty(m)$  is singular, then besides of (4.1) it holds

$$\Gamma(\lambda) = \lim_{x \uparrow r} \frac{\Phi_0^{\uparrow,+}(x, \lambda)}{\Phi_\infty^{\uparrow,+}(x, \lambda)}, \quad \lambda \in K_-. \quad (4.4)$$

Moreover, we have the representation (see (2.12))

$$\Gamma(\lambda) = c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu - \lambda}, \quad \lambda \in K_-. \quad (4.5)$$

In particular, by Krein's Theorem 2.2 and the remarks after this theorem, the string  $S_\infty(m)$  is uniquely determined by  $\Gamma$ .

Assume  $S_a(m)$  is a string ( $a = 0$  or  $= \infty$ ). Consider the right-continuous inverse function  $m^d$  of  $m$ . Then, by definition of  $S_a(m)$ , we have  $m^d(x) \equiv 0$ ,  $x < 0$ , if  $a = 0$ , and  $m^d(x) \equiv -\infty$ ,  $x < 0$ , if  $a = \infty$ . Therefore, as the dual string  $S_0^d(m)$  of  $S_0(m)$  ( $S_\infty^d(m)$  of  $S_\infty(m)$ ) we define  $S_0^d(m) := S_\infty(m^d)$  ( $S_\infty^d(m) := S_0(m^d)$ , respectively).

All quantities connected with the dual string are superscripted by  $d$ . Note that it holds

$$l^d = m(1) \quad , \quad h^d = \infty \quad , \quad \text{if } m(1-) + 1 < \infty \quad , \quad h \in [0, \infty) \quad , \quad (4.6)$$

$$l^d = m(1-) \quad , \quad h^d = m(\{1\}) < \infty \quad \text{if } m(1-) + 1 < \infty \quad , \quad h = \infty \quad , \quad (4.7)$$

$$l^d = m(1-) \quad , \quad \text{if } m(1-) + 1 = \infty \quad . \quad (4.8)$$

Moreover, we have

$$(S_0^d(m))^d = S_\infty^d(m^d) = S_0(m) \quad \text{and}$$

$$(S_\infty^d(m))^d = S_0^d(m^d) = S_\infty(m) \quad .$$

**LEMMA 4.1:** For all  $x \in [0, 1)$  and all  $\lambda \in K_-$  it holds with the notation  $x_+ := \inf(E_m \cap (x, \infty))$

$$\Phi_0^{\uparrow, d}(m(x), \lambda) = -\lambda^{-1} \Phi_\infty^{\uparrow, +}(x, \lambda) = -\lambda^{-1} \Phi_\infty^{\uparrow, -}(x_+, \lambda) \quad ,$$

$$\Phi_0^{\uparrow, d, +}(m(x), \lambda) = \Phi_\infty^{\uparrow}(x, \lambda) + (x_+ - x) \Phi_\infty^{\uparrow, +}(x, \lambda) = \Phi_\infty^{\uparrow}(x_+, \lambda) \quad ,$$

$$\Phi_\infty^{\uparrow, d}(m(x), \lambda) = \Phi_0^{\uparrow, +}(x, \lambda) = \Phi_0^{\uparrow, -}(x_+, \lambda) \quad ,$$

$$\Phi_\infty^{\uparrow, d, +}(m(x), \lambda) = -\lambda \Phi_0^{\uparrow}(x, \lambda) - \lambda(x_+ - x) \Phi_0^{\uparrow, +}(x, \lambda) = -\lambda \cdot \Phi_0^{\uparrow}(x_+, \lambda) \quad ,$$

The equations remain valid for  $x = 1$  with  $l_+ := 1 + h$  in the case  $1 + m(1-) < \infty$ ,  $h \in [0, \infty)$ .

The proof is similar to those of Proposition 2.3. Indeed we have to show that the right-hand side of the first and third equation under consideration satisfy the equations (2.8), (2.7) for  $\Phi_0^{\uparrow, d}(m(x), \lambda)$  and  $\Phi_\infty^{\uparrow, d}(m(x), \lambda)$ , respectively.

The corresponding equations for the derivatives  $\Phi_a^{\uparrow, d, +}(m(x), \lambda)$ ,  $a = 0, \infty$  follow from (2.7), (2.8) by differentiation (the details are given in Neumann [10]).

**COROLLARY 4.2:** For every string  $S_\infty(m)$  the characteristic functions  $\Gamma(\lambda)$  and  $\Gamma^d(\lambda)$  of  $S_\infty(m)$  and  $S_\infty(m^d)$ , respectively, are connected by

$$\Gamma^d(\lambda) = \frac{-1}{\lambda \Gamma(\lambda)} \quad \lambda \in K_- \quad (4.9)$$

**Proof:** If  $S_\infty(m)$  is regular and  $h \in [0, \infty)$ , then  $1^d < \infty$  and  $h^d = \infty$ . Thus

$$\Gamma^d(\lambda) = \frac{\Phi_0^{\uparrow, d, +}(1^d, \lambda)}{\Phi_\infty^{\uparrow, d, +}(1^d, \lambda)} = - \frac{\Phi_\infty^{\uparrow}(1+h, \lambda)}{\lambda \Phi_0^{\uparrow}(1+h, \lambda)} = - \frac{1}{\lambda \Gamma(\lambda)}$$

If  $h = \infty$ , then  $1^d + h^d < \infty$  and

$$\Gamma^d(\lambda) = \frac{\Phi_0^{\uparrow, d}(1^d + h^d, \lambda)}{\Phi_\infty^{\uparrow, d}(1^d + h^d, \lambda)} = - \frac{\Phi_\infty^{\uparrow, +}(1, \lambda)}{\lambda \Phi_0^{\uparrow, +}(1, \lambda)} = - \frac{1}{\lambda \Gamma(\lambda)}.$$

In the singular case the proof is obvious by  $r = 1$ , (4.4) and Lemma 4.1.

(For the singular case, (4.9) is well known from Kac, Krein [3].)

For singular strings  $S_\infty(m)$  the following lemma is known (Kac, Krein [3], p. 83):

**LEMMA 4.3:** For the spectral measures  $\tau_0^{(m)}$  and  $\tau_\infty^{(m^d)}$  of  $S_0(m)$  and  $S_\infty(m^d)$ , respectively, it holds

$$\tau_0^{(m)}(d\mu) = \mu \cdot \tau_\infty^{(m^d)}(d\mu) \quad \text{on } R_+. \quad (4.10)$$

**Proof:** We sketch the proof for the regular case  $1 + m(1^-) < \infty$  only. Obviously, in this case we have  $1^d + m^d(1^d-) < \infty$  also.

The spectrum of  $D_m D_x$  with left boundary condition  $af^-(0) - f(0) = 0$  consists of the zeros  $\{\mu_k: k \geq 0\}$  of

$$\Phi_a^{\uparrow}(1+h, \cdot) = 0 \quad \text{if } h < \infty \quad \text{and}$$

$$\Phi_a^{\uparrow, +}(1, \cdot) = 0 \quad \text{if } h = \infty.$$

(See (2.10) above.)

Moreover, we have

$$\tau_a^{(m)}(\{\mu_k\}) = \left[ \int_0^1 [\Phi_a^{\uparrow}(x, \mu_k)]^2 m(dx) \right]^{-1}, \quad k \geq 0 \quad (4.11)$$

( $a = 0$  or  $a = \infty$ ).



Firstly, let us assume  $h < \infty$ . Then  $l^d = m(1)$  and  $h^d = \infty$  (see (4.6)) and by Lemma 4.1 it holds

$$\Phi_{\infty}^{\uparrow, d, +}(l^d, \lambda) = -\lambda \Phi_0^{\uparrow}(r, \lambda). \quad (4.12)$$

If  $h = \infty$ , then it follows also from (4.7) that  $l^d = m(1-)$ ,  $h^d < \infty$  and from Lemma 4.1 we get

$$\Phi_{\infty}^{\uparrow, d}(l^d + h^d, \lambda) = \Phi_0^{\uparrow, +}(1, \lambda). \quad (4.13)$$

Thus we get that the spectra of  $S_0(m)$  and  $S_{\infty}(m^d)$  outside of zero are the same.

Now, the assertion (4.10) follows from (4.11) and the formula

$$\lambda \int_0^x [\Phi_0^{\uparrow}(\gamma, \lambda)]^2 m(d\gamma) = \int_0^{m(x)} [\Phi_{\infty}^{\uparrow, d}(\gamma, \lambda)]^2 m^d(d\gamma), \quad \lambda \in \mathbb{K}_+. \quad (4.14)$$

(Use Lemma 4.1.)

Now we are ready to prove Theorem 3.2.

The property (3.4) immediately follows from (4.10) and (2.11). We have

$$c_m = m^d(0) \quad \text{and} \quad m^d(0) = [\tau_{\infty}^{(m^d)}([0, \infty))^{-1} \quad (\text{see (2.16)}).$$

It is known that  $\tau_{\infty}^{(m^d)}(\{0\}) > 0$  implies  $l^d = \infty$  with  $m^d(l^d-) < \infty$  or  $l^d + m^d(l^d) < \infty$  with  $h^d = \infty$ . In both cases (2.17) implies

$$\tau_{\infty}^{(m^d)}(\{0\}) = (m^d(l^d))^{-1} = (1+h)^{-1} = r_m^{-1}.$$

(Put  $h = 0$  if  $m(1-) + 1 = \infty$ .)

Thus we get

$$c_m^{-1} = r_m^{-1} + \int_{0+}^{\infty} \frac{d\tau_0^{(m)}(\mu)}{\mu},$$

i.e., (3.5) holds. Therefore (i) is proved.

The crucial point to show (ii) and (iii) is (4.10). Indeed, introduce for  $s \geq 0$  measures  $\sigma_s$  on  $[0, \infty)$  by

$$\sigma_s(d\mu) := s \cdot \varepsilon_0(d\mu) + \tau_{\infty}^{(m^d)}(d\mu) \mathbb{1}_{(0, \infty)}(\mu), \quad \mu \geq 0,$$

where  $\varepsilon_0$  denotes the measure concentrated with unit mass at zero.

Note that  $\tau_{\infty}^{(m^d)}(\cdot) = \tau_{r_m^{-1}}^{(m^d)}(\cdot)$  and  $\tau_{\infty}^{(m^d)}(\{0\}) = r_m^{-1}$ . Then by Krein's

Theorem 2.2 for every  $s \geq 0$  there exists a string  $S_{\infty}(n_s)$  with  $n_s(x) > 0$  for  $x > 0$ , i.e.  $c_{n_s} = 0$ , having  $\sigma_s$  as its spectral measure.

From (2.17) it follows for  $s \geq 0$  that  $n_s(1_{n_s}) = s^{-1}$  with  $s^{-1} = \infty$  if  $s = 0$ .

Put  $q_s := n_s^d$ ,  $s \geq 0$ . Then the original  $m$  is included for  $s = r_m^{-1}$  and from (4.10) we get that the spectral measures  $\tau_o^{(q_s)}$  do not depend on  $s \geq 0$  and are equal to  $\tau_o^{(m)}$ . If  $s > 0$  then

$$s^{-1} = \zeta_s(\{0\})^{-1} = (n_s(1_s)) = r_{q_s} < \infty, \quad (4.15)$$

and if  $s = 0$  we get  $n_o(1_o-) = \infty$ , i.e.  $1_{q_o} = \infty$ .

Thus, among all  $q_s$ ,  $s \geq 0$  we find exactly one infinite string, namely  $m_o$ . Note that  $q_s(0) = c_{n_s} \equiv 0$ .

To finish the proof of Theorem 3.2 it suffices to identify the equivalence class  $\hat{S}(m)$  introduced in Chapter 3 with  $\{q_s \mid s \geq 0\}$ .

We remark that the characteristic function  $\Gamma_s$  of  $q_s$  satisfies (see (4.9), (2.17))

$$\begin{aligned} \frac{1}{\Gamma_s(\lambda)} &= -\lambda \Gamma_{n_s}(\lambda) = -\lambda \left( -\frac{s}{\lambda} + \int_{0-}^{\infty} \frac{d\tau_o^{(m^d)}(\mu)}{\mu - \lambda} + \frac{1}{r_m \lambda} \right) \\ &= \left( s - \frac{1}{r_m} \right) - \lambda \Gamma_{m^d}(\lambda) = \left( s - \frac{1}{r_m} \right) + \frac{1}{\Gamma_m(\lambda)}, \quad \lambda \in K_-. \end{aligned} \quad (4.16)$$

Let us calculate the characteristic function of  $S_{\infty}(m_t)$  with  $m_t \in \hat{S}$ , where  $m_t$  was defined in Lemma 3.1.

**LEMMA 4.4:** For every  $t \geq -\frac{1}{r_m}$  the corresponding to  $m_t$  functions  $\Phi_{o,t}^{\uparrow}$ ,  $\Phi_{\infty,t}^{\uparrow}$  are given by

$$\Phi_{o,t}^{\uparrow}(x, \lambda) = (1-tx) \Phi_o^{\uparrow}\left(\frac{x}{1-tx}, \lambda\right) \quad (4.17)$$

$$\Phi_{\infty,t}^{\uparrow}(x, \lambda) = (1-tx) \Phi_{\infty}^{\uparrow}\left(\frac{1}{1-tx}, \lambda\right) + t(1-tx) \Phi_o^{\uparrow}\left(\frac{1}{1-tx}, \lambda\right) \quad (4.18)$$

**Proof:** The left hand sides of (4.17) and (4.18) are the unique solutions of (2.7) and (2.8) with  $m$  replaced by  $m_t$ , respectively. After scale transformations and some calculations it is seen that the right-hand sides of (4.17) and (4.18) satisfy these equations. This proves the lemma.

**COROLLARY 4.5:** We have

$$\frac{1}{\Gamma_{m_t}(\lambda)} = \lim_{x \uparrow r_{m_t}} \frac{\Phi_{\infty,t}^{\uparrow}(x, \lambda)}{\Phi_{o,t}^{\uparrow}(x, \lambda)} = \frac{1}{\Gamma_m(\lambda)} + t, \quad \lambda \in K_-. \quad (4.19)$$

The proof follows immediately from (4.1), (4.17) and (4.18).

Now, compare (4.19) with (4.16). From Krein's inverse spectral theorem we get  $m_t = q_s$  for  $t = s - r_m^{-1}$ .

Thus Theorem 3.2 is proved.

As a consequence of (4.9), (4.10) we get the formula (2.15):

$$\begin{aligned} -\frac{1}{\Gamma_m(\lambda)} &= \lambda \Gamma_m^d(\lambda) = \lambda \int_{0-}^{\infty} \frac{d\tau_{\infty}^{(m^d)}(\mu)}{\mu - \lambda} \\ &= -\tau_{\infty}^{(m^d)}(\{0\}) - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_0^{(m)}(\mu) \\ &= -r_m^{-1} - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_0^{(m)}(d\mu), \quad \lambda \in K_-. \end{aligned} \quad (4.20)$$

Note, that we have supposed  $m(0) = 0$ . If some  $m(\{0\}) > 0$  is added to  $m$  at zero, the term  $\lambda m(\{0\})$  is added on the right-hand side of (4.20).

The Corollary 3.3 follows immediately from the Theorem 3.2.

#### Proof of Theorem 3.4:

Let  $S_a(m)$  be a string with  $a \in (0, \infty)$  and  $c_m = 0$ . Put  $w := a$  and define  $\tilde{m}(x) := m(x-a)$ ,  $x \in \mathbb{R}$ . Obviously, it holds  $c_{\tilde{m}} = a$  and

$$r_{\tilde{m}} = r_m + a.$$

If  $\tau_a$  and  $\tilde{\tau}_0$  denote the spectral measures of  $S_a(m)$  and  $S_0(\tilde{m})$ , respectively, then we have by Proposition 2.3.(iii)

$$d\tau_a(\mu) = a^2 d\tilde{\tau}_0(\mu), \quad \mu > 0$$

From (3.5) it follows

$$\int_{0+}^{\infty} \frac{d\tau_a(\mu)}{\mu} = a^2 \int_{0+}^{\infty} \frac{d\tilde{\tau}_0(\mu)}{\mu} = a^2(a^{-1} - (r_m + a)^{-1}) = a(1 - \frac{a}{a+r_m}),$$

i.e. (3.6) and (3.7) hold.

Conversely, if  $a \in (0, \infty)$  is fixed and  $\tau$  is a measure on  $(0, \infty)$  with  $\tau((0, \infty)) > 0$  and (3.7) then choose a number  $u \in (0, \infty]$  with

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = a(1 - \frac{a}{a+u}).$$

Put

$$\sigma(d\mu) := a^{-2}\tau(d\mu), \quad \mu \in (0, \infty),$$

and choose the string  $S_0(m)$  with  $m(0) = 0$  and  $l_m = \infty$  having  $\sigma$  as its spectral measure (see Theorem 3.2.(iii)).

By the same theorem, for every  $s \in [0, \infty)$  the string  $S_0(m_s)$  with

$$\begin{aligned} m_s(x) &:= (1 - sx)^2 m\left(\frac{x}{1 - sx}\right), & x \in [0, s^{-1}], \\ &= \infty & x > s^{-1} \end{aligned}$$

has the same spectral measure  $\sigma$  as  $S_0(m)$ .

It holds by (3.5)

$$c_{m_s}^{-1} = \int_{0+}^{\infty} \frac{d\sigma(\mu)}{\mu} + r_{m_s}^{-1} = \int_{0+}^{\infty} \frac{d\sigma(\mu)}{\mu} + s = a^{-1} \left(1 - \frac{a}{a+u}\right) + s.$$

Now choose  $s$  in such a way that  $c_{m_s} = a$  holds, i.e. put  $s = \frac{1}{a+u}$ .

By shifting  $m_s$  to the left

$$\tilde{m}_s(x) := m_s(x + a)$$

we get a string  $S_a(\tilde{m}_s)$  with  $c_{\tilde{m}_s} = 0$  having  $\tau$  as its spectral

measure. The uniqueness follows from the uniqueness of  $S_0(m)$  with  $l_m = \infty$ .

For the last part of Theorem 3.4.(ii) note that the strings

$S_0\left(\frac{m'(\cdot - a')}{(a')^2}\right)$  and  $S_0\left(\frac{m(\cdot - a)}{a^2}\right)$  have the common spectral measure  $\tau$

(see Proposition 2.3.(iii)).

From Theorem 3.2.(iii) it follows

$$\begin{aligned} S_0\left(\frac{m'(\cdot - a')}{(a')^2}\right) &= S_0\left(\left(\frac{m(\cdot - a)}{a^2}\right)_t\right) \quad \text{with} \\ t &= \frac{1}{r' - a'} - \frac{1}{r - a}. \end{aligned}$$

#### Proof of Proposition 3.5:

Choose  $a' \in (0, \infty]$  and consider a string  $S_{a'}(m)$ . Then it holds (see the definition of  $r_{\lambda, a'}(x, y)$ )

$$r_{\lambda, a'}(0, 0) = \frac{\Phi^\downarrow(0, \lambda)}{\frac{1}{a'} \Phi^\downarrow(0, \lambda) + 1} = \frac{1}{\frac{1}{a'} + \frac{1}{r_m(\lambda)}} \quad (4.21)$$

and, by definition of the spectral measure  $\tau_a^{(m)}$ ,

$$r_{\lambda, a}(0, 0) = \int_0^\infty \frac{d\tau_a^{(m)}(\mu)}{\mu - \lambda} \quad (4.22)$$

Now let be  $a \in (0, \infty)$ . Then (3.8) is a consequence of

$$-\frac{1}{\Gamma_m(\lambda)} - \frac{1}{\frac{1}{a} + \frac{1}{\Gamma_m(\lambda)}} (a + \Gamma_m(\lambda)) = -a \quad (4.23)$$

(2.15), (4.21), (4.22) for  $a' = a$  and  $a' = \infty$ .

Letting  $a \downarrow 0$  in (4.23) divided by  $a$  we get Knight's formula.

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