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An extension of Krein's inverse spectral theorem
to strings with nonreflecting left boundaries

by

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Abstract: Krein's inverse spectral theorem describes the spectral measures τ of the differential operators $D_m D_x$ with boundary condition $f'_-(0) = 0$, if m runs through all nondecreasing functions on $[0, \infty)$. This result will be extended to boundary conditions of the type $af'_-(0) - f(0) = 0$ ($a \in [0, \infty)$). Other conditions as in Krein's theorem appear.

Key words: gap-diffusions, quasidiffusions, generalized second order differential operator, spectral measures, local times, Krein's inverse spectral theorem, Krein's correspondence

60J35, 60J60, 34B20

1. Introduction

It is well-known that every nondecreasing function m on $[0, \infty)$ performed with appropriate boundary conditions at zero and at $1 := \sup \text{supp } m$ (a so-called string) generates a strong Markov process (X_t) on $\text{supp } m$, where $\text{supp } m$ denotes the set of points where m increases. This process has as its (selfadjoint) infinitesimal generator in $L_2(m)$ the generalized second order differential operator $D_m D_x$ together with the mentioned boundary conditions. (X_t) is called a quasi- (or gap-) diffusion with speed measure m . Examples are diffusions and birth- and death-processes. Several probabilistic quantities of (X_t) as e.g. transition densities, first hitting time densities, Lvy-measures of the inverse local time at zero, can be expressed in terms of spectral measures $\tau^{(m)}$ of $D_m D_x$ under different boundary conditions, see e.g. Ito, McKean [2], Kchler [7], [8], Kchler, Salminen [9].

An essential result concerning these spectral measures is M.G. Krein's inverse spectral theorem, in a more extended form known as Krein's correspondence theorem, see Kac, Krein [3], Kotani, Watanabe [6]. Roughly speaking it states that the mapping $m \rightarrow \tau^{(m)}$ is a one-to-one and onto correspondence between the strings m with the "reflecting" boundary condition $f^-(0) := f^+(0-) = 0$ and the set of all measures τ on $[0, \infty)$ that integrate $(1 + \mu)^{-1}$ thereon, see Theorem 2.2 below. What we are going to do is to study the situation for the boundary conditions

$$af^-(0) - f(0) = 0,$$

where $a \in [0, \infty)$ is fixed. (The case above corresponds to $a = \infty$.)

If $a \in (0, \infty)$ ("elastic killing boundary"), then there is still a one-to-one and into correspondence (Theorem 2.4). If $a = 0$, then $m \rightarrow \tau^{(m)}$ maps the strings m with the "killing" boundary condition $f(0) = 0$ onto the set of measures on $(0, \infty)$ that integrate $[\mu(1+\mu)]^{-1}$, but not one-to-one. In Theorem 3.2 we shall describe the preimages for every τ which form one-parametric families.

As an application we get the description of all measures ν that can appear as the Lévy-measure of the inverse local times at zero for quasidiffusions (see Remark 3.6). This result was proved by other (probabilistic) means in Knight [5]. Here we shall present an analytical approach.

Moreover, a generalization of Lemma 1 of Karlin, McGregors paper [4] concerning birth- and death-processes to strings is given (see Corollary 3.7).

2. Strings, spectral measures and Krein's theorem

Here we shall summarize some facts from the theory of generalized second order differential operators $D_m D_x$. For details the reader is referred to Kac, Krein [3] or Dym, McKean [1], the latter uses another terminology.

By R and K we denote the real axis and the complex plane, respectively. R_+ stands for $[0, \infty)$, K_- for $K \setminus R_+$. Put $\bar{R}_+ := [0, \infty]$ and $\frac{1}{0} := \infty$, $\frac{1}{\infty} := 0$. Let m be a nondecreasing right-continuous extended real-valued function on R with $m(x) \equiv 0$, $x < 0$. Define E_m to be the set of points where m increases and is finite:

$$E_m := \{x \in R_+ \mid \exists \varepsilon_0 > 0: m(x-\varepsilon) < m(x+\varepsilon) < \infty \quad \forall \varepsilon \in (0, \varepsilon_0)\}.$$

We shall assume $E_m \neq \emptyset$ and denote by the same letter m the measure generated by the function m . Such a measure m is called a speed measure.

Introduce c , l and r by

$$c := \inf E_m = \inf \{x \geq 0 \mid m(x) > 0\},$$

$$l := \sup E_m \leq \infty,$$

$$r := \sup \{x \geq 0 \mid m(x) < \infty\} \leq \infty.$$

We have $0 \leq c \leq l \leq r$ and put $h := r - l$ if $l < \infty$. Otherwise h is irrelevant and for convenience we put $h = 0$ in this case. Note that $h = 0$ if $l < \infty$ and $m(l) = \infty$. If $l + m(l) < \infty$ and $m(\{l\}) > 0$, then h must be greater than zero. The number $r = l + h$ is called the length of the string.

Sometimes we shall write $c_m, l_m, \dots$ to express that these numbers come from m .

By $\mathcal{V}$ we denote the set of all real functions f on R having a representation

$$f(x) = \bar{a} + \bar{b} \cdot x + \int_0^x (x-s)g(s)m(ds) \quad (2.1)$$

for some measurable g on R and some reals $\bar{a}, \bar{b}$.

Note that every $f \in \mathcal{V}$ is continuous and linear on the open intervals of $R \setminus E_m$.

On $\mathcal{V}$ we define a generalized second order differential operator $D_m D_x$ by $D_m D_x f = g$, details can also be found in Kchler [7], [8].

For every fixed $a \in [0, \infty]$ the restriction A_a of $D_m D_x$ to

$$\Delta_a := \{f \in \mathcal{V} \cap L_2(m) \mid D_m D_x f \in L_2(m), af^-(0) - f(0) = 0\} \quad (2.2)$$

(for $a = \infty$ we mean $f^-(0) = 0$) is a nonnegative selfadjoint operator in $L_2(m)$.

(By f^+ and f^- we denote the right- and left-hand-side derivative of f , respectively.)

Note that $f \in \mathcal{V} \cap L_2(m)$ implies $f(r) = 0$ if $r = l + h < \infty$.

Because of the linearity of f on the intervals of $R \setminus E_m$ this can be written as a boundary condition $hf^+(1) + f(1) = 0$ with $f^+(1) = 0$ if $h = \infty$. Otherwise, the boundary condition appearing in (2.2) can also be included in $f \in \mathcal{V} \cap L_2(m)$ if we change m to the left of $-a$ into $m(x) = -\infty, x < -a$.

In the following, m will be understood in this way.

This change of m charges $-a$ with infinite mass. The original

measure m on R_+ remains unchanged by this procedure if $a > 0$. In case $a = 0$, the value of $m(\{0\})$ is not reconstructable. But this does not disturb the corresponding spectral theory, as we will see below. Thus we suppose $m(0) = 0$ if we consider $a = 0$. Now m has infinite mass at $-a$ (and r if $r < \infty$) and thus $f \in \mathcal{V} \cap L_2(m)$ implies also $f(-a) = 0$, i.e. $af^-(0) - f(0) = 0$.

Therefore, the selfadjoint operators A_a are characterized by the (changed) function m , or by (m, a) . We call the pair (m, a) a string and denote it by $S_a(m)$. If the length $r = 1 + h$ is infinite, then we say that the string $S_a(m)$ is infinite. Depending on $1 + m(1-) < \infty$ or $= \infty$ the string $S_a(m)$ is called regular or singular. The resolvent operator $R_{\lambda, a} := (A_a - \lambda I)^{-1}$ exists for $\lambda \in (-\infty, 0)$, and it can be shown analogously to Dym, McKean [1] that $R_{\lambda, a}$ is given by

$$(R_{\lambda, a} f)(x) = \int_0^1 r_{\lambda, a}(x, y) f(y) m(dy), \quad f \in L_2(m),$$

where

$$r_{\lambda, a}(x, y) := \frac{\Phi_a^\uparrow(x \wedge y, \lambda) \Phi_a^\downarrow(x \vee y, \lambda)}{W}.$$

Here $\Phi_a^\uparrow$ and $\Phi_a^\downarrow$ denote the solutions $f \in \mathcal{V}$ of

$$D_m D_x f + \lambda f = 0$$

satisfying the boundary conditions

$$\Phi_a^\uparrow(0, \lambda) = 1, \quad a \in (0, \infty]; \quad \Phi_0^{\uparrow-}(0, \lambda) = 1, \quad (2.3)$$

$$a \Phi_a^{\uparrow-}(0, \lambda) - \Phi_a^\uparrow(0, \lambda) = 0, \quad a \in [0, \infty); \quad \Phi_\infty^{\uparrow-}(0, \lambda) = 0, \quad (2.4)$$

and

$$\Phi_a^{\downarrow-}(0, \lambda) = -1, \quad \text{and} \quad (2.5)$$

$$h \Phi_a^{\downarrow+}(1, \lambda) + \Phi_a^{\downarrow-}(1, \lambda) = 0. \quad (2.6)$$

Note that $\Phi_a^\uparrow(\cdot, \lambda)$ is increasing and $\Phi_a^\downarrow(\cdot, \lambda)$ is decreasing for fixed $\lambda < 0$.

W denotes the Wronskian:

$$W = W(\lambda) := \Phi_a^{\uparrow-} \Phi_a^\downarrow - \Phi_a^\uparrow \Phi_a^{\downarrow-}$$

Several times we will use that $\Phi_a^\uparrow(\cdot, \lambda)$ is the uniquely determined solution of

$$\Phi(x, \lambda) = 1 + \frac{x}{a} - \lambda \int_0^x (x-s) \Phi(s, \lambda) m(ds), \quad x \in [0, r) \quad (2.7)$$

for $a \in (0, \infty]$, and of

$$\Phi(x, \lambda) = x - \lambda \int_0^x (x-s) \Phi(s, \lambda) m(ds), \quad x \in [0, r) \quad (2.8)$$

for $a = 0$.

Similarly, $\Phi^\downarrow(\cdot, \lambda)$ is the unique solution of

$$\Phi(x, \lambda) = \Phi(0, \lambda) - x - \lambda \int_0^x (x-s) \Phi(s, \lambda) m(ds), \quad x \in [0, r). \quad (2.9)$$

DEFINITION 2.1: Assume $S_a(m)$ is a string with $a \in [0, \infty]$. Then a measure τ on $[0, \infty)$ is called a spectral measure of $S_a(m)$, if

$$r_{\lambda, a}(x, y) = \int_0^\infty \frac{\Phi_a^\uparrow(x, \mu) \Phi_a^\uparrow(y, \mu)}{\mu - \lambda} d\tau(\mu), \quad \lambda < 0; x, y \in E_m.$$

The set $\text{supp } \tau$ is called the spectrum of $S_a(m)$.

As for the case of $a = \infty$, treated in Kac, Krein [3] and Dym, McKean [1], one can show that for every string $S_a(m)$ a unique spectral measure τ exists (on $(0, \infty)$ if $a \neq 0$). It will often be denoted by $\tau_a^{(m)}$. (We shall identify measures τ on R_+ and their generating function $\mu \rightarrow \tau([0, \mu])$.)

Note that $\Phi_0^\uparrow(\cdot, \lambda)$, and therefore $\tau_0^{(m)}$ does not depend on the mass of m at zero. Thus, considering a string $S_0(m)$ we shall always suppose that $m(0) = 0$.

If the string $S_a(m)$ is regular, then $\tau_a^{(m)}$ is given by

$$\tau_a^{(m)}(\mu) = \sum_{k=0}^{\infty} \tau_a^{(m)}(\{\mu_k\}) \cdot \mathbb{1}_{[0, \mu_k]}(\mu) \quad (2.10)$$

where $(\mu_k)_{k \geq 0}$ denotes the sequence of solutions of

$$h \Phi_a^{\uparrow, +}(1, \mu) + \Phi_a^\uparrow(1, \mu) = 0$$

and

$$\tau_a^{(m)}(\{\mu_k\}) = \left[\int_0^1 [\Phi_a^\uparrow(x, \mu_k)]^2 dm \right]^{-1}.$$

We have

$$0 \leq \mu_0 < \mu_1 < \dots < \mu_n < \dots \quad \text{and} \quad \sum_{n \geq 1} \mu_n^{-1} < \infty.$$

The following theorem answers the question which measures may appear as spectral measures for strings $S_\infty(m)$. Its second part is M.G. Krein's inverse spectral theorem, (i) and (ii) together are known as Krein's correspondence (Kotani, Watanabe [6]).

THEOREM 2.2:

(i) For every string $S_\infty(m)$ its spectral measure $\tau = \tau_\infty^{(m)}$ satisfies

$$\int_{0-}^{\infty} \frac{d\tau(\mu)}{1+\mu} < \infty. \quad (2.11)$$

(ii) For every measure τ on R_+ with $\tau(R_+) > 0$ and (2.11) there exists one and only one string $S_\infty(m)$ with $c_m = 0$ having τ as its spectral measure.

Note that the condition $c_m = 0$ in (ii) ensures the unicity of m . Indeed, all "shifted" strings $S_\infty(m(\cdot - c))$ ($c > 0$) have the same spectral measure, compare Proposition 2.3 below.

For every string $S_\infty(m)$ its characteristic function $\Gamma_m(\cdot)$ is given by (see Chapter 4 below)

$$\Gamma_m(\lambda) := c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu - \lambda} = \lim_{x \uparrow r} \frac{\Phi_0^\uparrow(x, \lambda)}{\Phi_\infty^\uparrow(x, \lambda)}, \quad \lambda \in \kappa_-. \quad (2.12)$$

Because of the definition of the spectral measure we obtain

$$\Phi^\downarrow(0, \lambda) = r_{\lambda, \infty}(0, 0) = \Gamma_m(\lambda), \quad \lambda < 0. \quad (2.13)$$

Letting $\lambda \uparrow 0$ in (2.12) we get the formula

$$c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu} = r = 1 + h \quad (2.14)$$

with the understanding that $h = 0$ if $1 + m(1-) = \infty$.

Krein's theorem says that $\Gamma_m(\cdot)$ determines $S_\infty(m)$ uniquely.

In Chapter 4 below we shall see that it holds

$$-\frac{1}{\Gamma_m(\lambda)} = \lambda m(\{0\}) - r_m^{-1} - \int_0^\infty \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda} \right) \tau_0^{(m)}(d\mu), \quad \lambda \in \kappa_-. \quad (2.15)$$

For every string $S_{\infty(m)}$ with $c_m = 0$ we have

$$\tau_{\infty}^{(m)}(R_+) = [m(\{0\})]^{-1}. \quad (2.16)$$

Indeed, consider $\lambda \Gamma_m(\lambda)$ for $\lambda \downarrow -\infty$ and compare (2.12) and (2.15), then (2.16) is obvious.

Finally, note that $\tau_{\infty}^{(m)}(\{0\}) > 0$ if and only if the constant function $\Phi_{\infty}^{\uparrow}(\cdot, 0) \equiv 1$ is an eigenfunction of $D_m D_x$. This holds if and only if $r = 1 + h = \infty$ (or $l = \infty$) and $m(1-) < \infty$. Moreover, in this case we have

$$\tau_{\infty}^{(m)}(\{0\}) = [m(1)]^{-1}. \quad (2.17)$$

The next proposition shows how the spectral measure changes if m suffers certain transformations.

PROPOSITION 2.3: Let $S_a(m)$ be a string with $a \in [0, \infty]$ and assume $u, v \in (0, \infty)$, $w \in [0, \frac{a}{u}]$ and $w < \infty$. Define

$$\begin{aligned} \tilde{m}(x) &:= v \cdot m(u(x-w)), & x \in R, \\ \tilde{a} &:= \frac{a}{u} - w. \end{aligned}$$

Then, for the spectral measures $\tilde{\tau}_{\tilde{a}} := \tau_{\tilde{a}}^{(\tilde{m})}$ and $\tau_a := \tau_a^{(m)}$ of $S_{\tilde{a}}(\tilde{m})$ and $S_a(m)$, respectively, we have

(i) If $a = \infty$, then for all $w \in [0, \infty)$ we have $\tilde{a} = \infty$ and

$$\tilde{\tau}_{\infty}(\mu) = v^{-1} \tau_{\infty}(\frac{v}{u} \cdot \mu), \quad \mu \geq 0.$$

(ii) If $a \in (0, \infty)$, $0 < w < \frac{a}{u}$, then $\tilde{a} \in (0, \infty)$ and

$$\tilde{\tau}_{\tilde{a}}(\mu) = v^{-1} (1 - \frac{uw}{a})^2 \tau_a(\frac{v}{u} \cdot \mu), \quad \mu \geq 0.$$

(iii) If $a \in (0, \infty)$, $w = \frac{a}{u}$, then $\tilde{a} = 0$ and

$$\tilde{\tau}_0(\mu) = v^{-1} (\frac{u}{a})^2 \tau_a(\frac{v}{u} \cdot \mu), \quad \mu \geq 0.$$

(iv) If $a = w = 0$, then

$$\tilde{\tau}_0(\mu) = v^{-1} \cdot u^2 \tau_0(\frac{v}{u} \cdot \mu), \quad \mu \geq 0.$$

To prove this proposition one calculates the relevant $\tilde{\Phi}_{\tilde{a}}^{\uparrow}$ and $\tilde{\Phi}_{\tilde{a}}^{\downarrow}$ in terms of $\Phi_a^{\uparrow}$ and $\Phi_a^{\downarrow}$, respectively, using (2.7 - 2.9). This gives the relation between the resolvent kernels $\tilde{r}_{\tilde{\lambda}, \tilde{a}}$ and $r_{\lambda, a}$. Definition 2.1 leads to the assertion of Proposition 2.3.

3. Results

In this chapter we shall formulate correspondence theorems for strings $S_a(m)$ with $a \neq \infty$ which extend Krein's result. The proofs can be found in Chapter 4.

We shall start with the case of $a = 0$. For this purpose we still need a preparation.

Denote by Σ the set of all strings $S_0(m)$ with $m(0) = 0$. We introduce a relation $\sim$ in Σ by defining $S_0(m) \sim S_0(n)$ if there exists a real number $t \geq -\frac{1}{r_n}$ such that the transformation

$$x \longrightarrow T_t x := \frac{x}{1 - tx}, \quad x \in \mathbb{R}$$

maps $(0, r_m)$ onto $(0, r_n)$ and such that

$$m(x) = \int_{0+}^x (1-ts)^{-2} dn(T_t s), \quad x \in (0, r_m) \quad (3.1)$$

(indeed, $t = \frac{1}{r_m} - \frac{1}{r_n}$.) It is easy to see that $\sim$ forms an equivalence relation in Σ . Put $\hat{\Sigma} := \Sigma / \sim$ and for every string $S_0(m) \in \Sigma$ denote by $\hat{S}(m)$ the element of $\hat{\Sigma}$ generated by $S_0(m)$. For every string $S_0(m)$ and every $t \geq -\frac{1}{r_m}$ we define a new string $S_0(m_t)$ by

$$r_{m_t} := \frac{r_m}{1 + tr_m} \quad \text{and}$$

$$\left. \begin{aligned} m_t(x) &:= \int_{0+}^x (1-ts)^{-2} dm(T_t s), & x \in (0, r_{m_t}) \\ m_t(x) &:= \infty, & x \geq r_{m_t} \end{aligned} \right\} \quad (3.2)$$

Obviously, we have

$$c_{m_t} = \frac{c_m}{1 + tc_m}, \quad l_{m_t} = \frac{l_m}{1 + tl_m}, \quad t \geq -\frac{1}{r_m}, \quad (3.3)$$

and $S_0(m_t) \sim S_0(m)$ for every $t \geq -\frac{1}{r_m}$.

Otherwise, if $S_0(n) \sim S_0(m)$, then, by definition, there exists a real number $t \geq -\frac{1}{r_n}$ such that $m = n_t$. Observe $r_{m_t} = \infty$ if and only if $t = -\frac{1}{r_m}$.

Thus we have proved the following

LEMMA 3.1:

- (i) For every string $S_0(m)$ its equivalence class $\hat{S}(m)$ is equal to $\{S_0(m_t) \mid t \geq -\frac{1}{r_m}\}$.
- (ii) Every equivalence class $\hat{S} \in \hat{\Sigma}$ contains one and only one infinite string $S_0(m)$.

Now we are ready to formulate the analogue of Krein's correspondence for strings $S_0(m)$.

THEOREM 3.2:

- (i) For every string $S_0(m)$ its spectral measure $\tau = \tau_0^{(m)}$ is supported by $(0, \infty)$ and has the property

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu(1+\mu)} < \infty. \quad (3.4)$$

Moreover, it holds

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = c_m^{-1} - r_m^{-1}. \quad (3.5)$$

- (ii) If two strings $S_0(m)$ and $S_0(n)$ are equivalent (with respect to $\sim$), then $\tau_0^{(m)} = \tau_0^{(n)}$.
- (iii) For every measure τ on $(0, \infty)$ ($\tau((0, \infty)) > 0$) with (3.4) and every $r \in (0, \infty]$ there exists one and only one string $S_0(m)$ with length $r = r_m$ having τ as its spectral measure. If $S_0(m)$ and $S_0(m')$ are strings with the lengths r and r' , respectively, having the same spectral measure, then $S_0(m') = S_0(m_t)$ holds with $t = \frac{1}{r'} - \frac{1}{r}$. (m_t was defined in (3.2).)

This theorem can be reformulated in a shorter way as follows.

COROLLARY 3.3: There is a one-to-one and onto correspondence between the set $\hat{\Sigma}$ of equivalence classes $\hat{S}$ of strings $S_0(m)$ and the set of measures τ on $(0, \infty)$ satisfying (3.4), where τ is the spectral measure $\tau_0^{(m)}$ of every string $S_0(m)$ from $\hat{S}$.

Now let us turn to the case of $a \in (0, \infty)$.

THEOREM 3.4: Assume $a \in (0, \infty)$. Then it holds:

- (i) For every string $S_a(m)$ with $c_m = 0$ and $m(0) \geq 0$ its spectral measure $\tau = \tau_a^{(m)}$ is supported on $(0, \infty)$ and has the property

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = \left(\frac{1}{r_m} + \frac{1}{a} \right)^{-1} < \infty \quad (3.6)$$

- (ii) If τ is a measure on $(0, \infty)$ with nonzero mass, then there exists a string $S_a(m)$ with $c_m = 0$ having τ as its spectral measure if and only if

$$\xi(\tau) := \int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} \leq a. \quad (3.7)$$

In this case, $S_a(m)$ is uniquely determined.

Moreover, if $S_a(m)$ and $S_{a'}(m')$ with $a, a' \in (0, \infty)$, $c_m = c_{m'} = 0$, have the same spectral measure, then

$$m'(x - a') = m_t(x - a), \quad x \in \mathbb{R}_+$$

with $t := \frac{1}{a'} - \frac{1}{a}$, where m_t was defined in (3.2).

Consider a speed measure m on $[0, \infty)$ with $c_m = 0$, $m(0) \geq 0$ and form strings $S_{\infty}(m)$, $S_a(m)$ and $S_0(m)$ for some $a \in (0, \infty)$. (Note that $m(\{0\})$ disappears if we construct $S_0(m)$.) Then we have

PROPOSITION 3.5: Between the spectral measures τ_{∞} , τ_a and τ_0 of $S_{\infty}(m)$, $S_a(m)$ with $a \in (0, \infty)$ and $S_0(m)$, respectively, the following equation holds:

$$\begin{aligned} & [\lambda m(0) - r_m^{-1} - \int_{0+}^{\infty} (\frac{1}{\mu} - \frac{1}{\mu - \lambda}) d\tau_0(\mu)] \cdot \\ & \cdot \left[\int_{0+}^{\infty} \frac{d\tau_a(\mu)}{\mu - \lambda} \right] \cdot \left[a + \int_{0-}^{\infty} \frac{d\tau_{\infty}(\mu)}{\mu - \lambda} \right] = -a, \quad \lambda \in K_-. \end{aligned} \quad (3.8)$$

This generalizes a formula which was used by Knight [5], p. 60. Consider a string $S_{\infty}(m)$ and add to m some point mass $m_0 > 0$ at zero if necessary, i.e. if $c_m > 0$. As we know, this does not touch the spectral measure $\tau_0^{(m)}$ of $S_0(m)$. Now, let $l(t, 0)$, $t \geq 0$, be the local time at zero of the quasidiffusion generated by $S_{\infty}(m)$. Since $0 \in E_m$, this notion makes sense. Then $(l^{-1}(t, 0), t \geq 0)$ is an increasing process with independent stationary increments and it holds

$$E_0 \exp(\lambda l^{-1}(t, 0)) = \exp(-\frac{t}{r_m(\lambda)}) , \quad \lambda < 0, t \geq 0.$$

(See e.g. Knight [5] or Küchler [8].)

For $\lambda < 0$, (2.15) implies

$$-\frac{1}{r_m(\lambda)} = \lambda m(\{0\}) - \frac{1}{r_m} - \int_0^\infty (1 - e^{\lambda y}) \left[\int_{0+}^\infty e^{-\mu y} \tau_0^{(m)}(d\mu) \right] dy.$$

Thus, by Theorem 3.2(ii) and Lemma 3.1(i) the Lévy-measure n of $l^{-1}(\cdot, 0)$, given by

$$dn(y) := \int_{0+}^\infty e^{-\mu y} \tau_0^{(m)}(d\mu) dy , \quad y \in \mathbb{R}_+, \quad (3.9)$$

is the same for all $S_\infty(m_t)$, $t \geq -\frac{1}{r_m}$.

This means that the inverse local times at zero of the quasidiffusions corresponding to $S_\infty(m_t)$ differ in their killing rate

$$k = \frac{1}{r_m} + t \text{ only.}$$

Now Theorem 3.2 implies

COROLLARY 3.6: For every nontrivial measure τ on $(0, \infty)$ with (3.4), every $m(\{0\}) > 0$ and every constant $k \geq 0$ there exists a quasidiffusion with speed measure m , a reflecting boundary at zero and length $\frac{1}{k}$ of the string $S_\infty(m)$ such that $l^{-1}(\cdot, 0)$ has the Lévy-measure (3.9).

This result was proved by other means in Knight [5].

As an example consider a birth- and death-process on the set of non-negative integers with the intensities $\mu_0 \geq 0$, $\lambda_i > 0$, $\mu_{i+1} > 0$, $i \geq 0$. Then

$$m(x) := \sum_{i=0}^{\infty} m_i \cdot \mathbb{1}_{[0, x]}(x_i)$$

$$\text{with } x_0 := 0, \quad x_i := \sum_{j=0}^{i-1} \frac{1}{\lambda_j m_j},$$

$$m_0 := 1, \quad m_i := \frac{1}{\prod_{j=1}^i \frac{\lambda_{j-1}}{\mu_j}}, \quad i \geq 1$$

and $a := \mu_0^{-1}$, $h \geq 0$ define a string $S_a(m)$. (Necessarily, $h = 0$ if m is singular.) We have

$$D_m D_x f(x_i) = \left[\frac{\Delta f(x_i)}{\Delta x_i} - \frac{\Delta f(x_{i-1})}{\Delta x_{i-1}} \right] m_i^{-1} =$$

$$= \lambda_i f(x_{i+1}) - (\lambda_i + \mu_i) f(x_i) + \mu_i f(x_{i-1}), \quad i \geq 1$$

with $\Delta u(x_j) := u(x_{j+1}) - u(x_j)$.

Moreover,

$$D_m D_x f(x_0) = \frac{\frac{\Delta f(x_0)}{x_1} - f^-(x_0)}{m_0} \quad \text{and}$$

the boundary condition

$$a f^-(x_0) - f(x_0) = 0$$

is equivalent to

$$D_m D_x f(x_0) = -(\lambda_0 + \mu_0) f(x_0) + \lambda_0 f(x_1).$$

Thus, we have

$$\Phi_a^\uparrow(x_i, \lambda) = Q_i(\lambda), \quad i \geq 0, \quad \lambda \in \mathbb{R}$$

in the terminology of Karlin, McGregor [4].

The spectral measure $\tau_a^{(m)}$ of $S_a(m)$ is a solution of the Stieltjes moment problem connected with the Jacobi-matrix (a_{ij}) with

$$a_{ij} := \lambda_i \mathbb{1}_1(j-i) + \mu_i \mathbb{1}_1(i-j) - (\lambda_i + \mu_i) \mathbb{1}_0(i-j) \quad (i, j \geq 0).$$

Indeed, for $\lambda \rightarrow -\infty$ we have

$$\| -\lambda R_{\lambda, a} f - f \|_{L_2(m)} \rightarrow 0, \quad f \in L_2(m).$$

Consequently,

$$\langle -\lambda R_{\lambda, a} f, g \rangle_{L_2(m)} \rightarrow \langle f, g \rangle_{L_2(m)}, \quad f, g \in L_2(m).$$

Choosing $f = \mathbb{1}_{\{x_i\}}$, $g = \mathbb{1}_{\{x_j\}}$ we obtain

$$\lim_{\lambda \rightarrow -\infty} -\lambda r_{\lambda, a}(x_i, x_j) =$$

$$\int_0^\infty \Phi_a^\uparrow(x_i, \mu) \Phi_a^\uparrow(x_j, \mu) d\tau_a^{(m)}(\mu) = \frac{\delta_{ij}}{m_i}, \quad i, j \geq 0.$$

Compare this equation with Theorem 1 of Karlin, McGregor [4], p. 494 to get the assertion.

Now, Lemma 1 of Karlin, McGregor [4] can be generalized to strings as follows.

COROLLARY 3.7: Given a string $S_\infty(m)$ with $c_m = 0$ and with the spectral measure τ and assume $a > 0$. Then there exists a string $S_a(m')$ with $c_m = 0$ having the same spectral measure τ if and only if

$$r_m = l_m + h_m \leq a. \quad (3.10)$$

Proof: If $\tau(\{0\}) > 0$, then there does not exist such a string $S_a(m')$ because, for $a \neq \infty$, the spectral measure is concentrated on $(0, \infty)$. Otherwise, $r_m = \infty$, see the remarks before (2.17).

Assume $\tau(\{0\}) = 0$. From (2.14) we know $r_m = \int_0^\infty \frac{d\tau(\mu)}{\mu}$. Now apply Theorem 3.4(ii).

4. Proofs

At first we shall collect some results of the spectral theory of $D_m D_x$. For details see e.g. Kac, Krein [3]. Let us given a string $S_\infty(m)$. The characteristic function $\Gamma(\cdot)$ of $S_\infty(m)$ is given by the limit (see (2.12))

$$\Gamma(\lambda) = \lim_{x \uparrow r} \frac{\Phi_0^\uparrow(x, \lambda)}{\Phi_\infty^\uparrow(x, \lambda)}, \quad \lambda \in K_-. \quad (4.1)$$

In the regular case we have for $h < \infty$

$$\Gamma(\lambda) = \frac{\Phi_0^\uparrow(r, \lambda)}{\Phi_\infty^\uparrow(r, \lambda)} = \frac{\Phi_0^{\uparrow,+}(1, \lambda) \cdot h + \Phi_0^\uparrow(1, \lambda)}{\Phi_\infty^{\uparrow,+}(1, \lambda) \cdot h + \Phi_\infty^\uparrow(1, \lambda)}, \quad (4.2)$$

and for $h = \infty$ it holds

$$\Gamma(\lambda) = \frac{\Phi_0^{\uparrow,+}(1, \lambda)}{\Phi_\infty^{\uparrow,+}(1, \lambda)}. \quad (4.3)$$

If $S_\infty(m)$ is singular, then besides of (4.1) it holds

$$\Gamma(\lambda) = \lim_{x \uparrow r} \frac{\Phi_0^{\uparrow,+}(x, \lambda)}{\Phi_\infty^{\uparrow,+}(x, \lambda)}, \quad \lambda \in K_-. \quad (4.4)$$

Moreover, we have the representation (see (2.12))

$$\Gamma(\lambda) = c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu - \lambda}, \quad \lambda \in K_-. \quad (4.5)$$

In particular, by Krein's Theorem 2.2 and the remarks after this theorem, the string $S_\infty(m)$ is uniquely determined by Γ .

Assume $S_a(m)$ is a string ($a = 0$ or $= \infty$). Consider the right-continuous inverse function m^d of m . Then, by definition of $S_a(m)$, we have $m^d(x) \equiv 0$, $x < 0$, if $a = 0$, and $m^d(x) \equiv -\infty$, $x < 0$, if $a = \infty$. Therefore, as the dual string $S_0^d(m)$ of $S_0(m)$ ($S_\infty^d(m)$ of $S_\infty(m)$) we define $S_0^d(m) := S_\infty(m^d)$ ($S_\infty^d(m) := S_0(m^d)$, respectively). All quantities connected with the dual string are superscripted by d . Note that it holds

$$l^d = m(1) \quad , \quad h^d = \infty \quad , \quad \text{if } m(1-) + 1 < \infty \quad , \quad h \in [0, \infty) \quad , \quad (4.6)$$

$$l^d = m(1-) \quad , \quad h^d = m(\{1\}) < \infty \quad \text{if } m(1-) + 1 < \infty \quad , \quad h = \infty \quad , \quad (4.7)$$

$$l^d = m(1-) \quad , \quad \text{if } m(1-) + 1 = \infty \quad . \quad (4.8)$$

Moreover, we have

$$(S_0^d(m))^d = S_\infty^d(m^d) = S_0(m) \quad \text{and}$$

$$(S_\infty^d(m))^d = S_0^d(m^d) = S_\infty(m) \quad .$$

LEMMA 4.1: For all $x \in [0, 1)$ and all $\lambda \in K_-$ it holds with the notation $x_+ := \inf(E_m \cap (x, \infty))$

$$\Phi_0^{\uparrow, d}(m(x), \lambda) = -\lambda^{-1} \Phi_\infty^{\uparrow, +}(x, \lambda) = -\lambda^{-1} \Phi_\infty^{\uparrow, -}(x_+, \lambda) \quad ,$$

$$\Phi_0^{\uparrow, d, +}(m(x), \lambda) = \Phi_\infty^{\uparrow}(x, \lambda) + (x_+ - x) \Phi_\infty^{\uparrow, +}(x, \lambda) = \Phi_\infty^{\uparrow}(x_+, \lambda) \quad ,$$

$$\Phi_\infty^{\uparrow, d}(m(x), \lambda) = \Phi_0^{\uparrow, +}(x, \lambda) = \Phi_0^{\uparrow, -}(x_+, \lambda)$$

$$\Phi_\infty^{\uparrow, d, +}(m(x), \lambda) = -\lambda \Phi_0^{\uparrow}(x, \lambda) - \lambda(x_+ - x) \Phi_0^{\uparrow, +}(x, \lambda) = -\lambda \cdot \Phi_0^{\uparrow}(x_+, \lambda)$$

The equations remain valid for $x = 1$ with $l_+ := 1 + h$ in the case $1 + m(1-) < \infty$, $h \in [0, \infty)$.

The proof is similar to those of Proposition 2.3. Indeed we have to show that the right-hand side of the first and third equation under consideration satisfy the equations (2.8), (2.7) for $\Phi_0^{\uparrow, d}(m(x), \lambda)$ and $\Phi_\infty^{\uparrow, d}(m(x), \lambda)$, respectively.

The corresponding equations for the derivatives $\Phi_a^{\uparrow, d, +}(m(x), \lambda)$, $a = 0, \infty$ follow from (2.7), (2.8) by differentiation (the details are given in Neumann [10]).

COROLLARY 4.2: For every string $S_\infty(m)$ the characteristic functions $\Gamma(\lambda)$ and $\Gamma^d(\lambda)$ of $S_\infty(m)$ and $S_\infty(m^d)$, respectively, are connected by

$$\Gamma^d(\lambda) = \frac{-1}{\lambda \Gamma(\lambda)} \quad \lambda \in \kappa_- \quad (4.9)$$

Proof: If $S_\infty(m)$ is regular and $h \in [0, \infty)$, then $1^d < \infty$ and $h^d = \infty$. Thus

$$\Gamma^d(\lambda) = \frac{\Phi_0^{\uparrow, d, +}(1^d, \lambda)}{\Phi_\infty^{\uparrow, d, +}(1^d, \lambda)} = - \frac{\Phi_\infty^{\uparrow}(1+h, \lambda)}{\lambda \Phi_0^{\uparrow}(1+h, \lambda)} = - \frac{1}{\lambda \Gamma(\lambda)}$$

If $h = \infty$, then $1^d + h^d < \infty$ and

$$\Gamma^d(\lambda) = \frac{\Phi_0^{\uparrow, d}(1^d+h^d, \lambda)}{\Phi_\infty^{\uparrow, d}(1^d+h^d, \lambda)} = - \frac{\Phi_\infty^{\uparrow, +}(1, \lambda)}{\lambda \Phi_0^{\uparrow, +}(1, \lambda)} = - \frac{1}{\lambda \Gamma(\lambda)}.$$

In the singular case the proof is obvious by $r = 1$, (4.4) and Lemma 4.1.

(For the singular case, (4.9) is well known from Kac, Krein [3].)

For singular strings $S_\infty(m)$ the following lemma is known (Kac, Krein [3], p. 83):

LEMMA 4.3: For the spectral measures $\tau_0^{(m)}$ and $\tau_\infty^{(m^d)}$ of $S_0(m)$ and $S_\infty(m^d)$, respectively, it holds

$$\tau_0^{(m)}(d\mu) = \mu \cdot \tau_\infty^{(m^d)}(d\mu) \quad \text{on } \mathbb{R}_+. \quad (4.10)$$

Proof: We sketch the proof for the regular case $1 + m(1-) < \infty$ only. Obviously, in this case we have $1^d + m^d(1^d-) < \infty$ also.

The spectrum of $D_m D_x$ with left boundary condition $af^-(0) - f(0) = 0$ consists of the zeros $\{\mu_k: k \geq 0\}$ of

$$\begin{aligned} \Phi_a^{\uparrow}(1+h, \cdot) &= 0 & \text{if } h < \infty & \text{ and} \\ \Phi_a^{\uparrow, +}(1, \cdot) &= 0 & \text{if } h = \infty. \end{aligned}$$

(See (2.10) above.)

Moreover, we have

$$\tau_a^{(m)}(\{\mu_k\}) = \left[\int_0^1 [\Phi_a^{\uparrow}(x, \mu_k)]^2 m(dx) \right]^{-1}, \quad k \geq 0 \quad (4.11)$$

($a = 0$ or $a = \infty$).

Firstly, let us assume $h < \infty$. Then $1^d = m(1)$ and $h^d = \infty$ (see (4.6)) and by Lemma 4.1 it holds

$$\Phi_{\infty}^{\uparrow, d, +}(1^d, \lambda) = -\lambda \Phi_0^{\uparrow}(r, \lambda). \quad (4.12)$$

If $h = \infty$, then it follows also from (4.7) that $1^d = m(1-)$, $h^d < \infty$ and from Lemma 4.1 we get

$$\Phi_{\infty}^{\uparrow, d}(1^d + h^d, \lambda) = \Phi_0^{\uparrow, +}(1, \lambda). \quad (4.13)$$

Thus we get that the spectra of $S_0(m)$ and $S_{\infty}(m^d)$ outside of zero are the same.

Now, the assertion (4.10) follows from (4.11) and the formula

$$\lambda \int_0^x [\Phi_0^{\uparrow}(\gamma, \lambda)]^2 m(d\gamma) = \int_0^{m(x)} [\Phi_{\infty}^{\uparrow, d}(\gamma, \lambda)]^2 m^d(d\gamma), \quad \lambda \in \mathbb{K}_-. \quad (4.14)$$

(Use Lemma 4.1.)

Now we are ready to prove Theorem 3.2.

The property (3.4) immediately follows from (4.10) and (2.11). We have

$$c_m = m^d(0) \text{ and } m^d(0) = [\tau_{\infty}^{(m^d)}([0, \infty))]^{-1} \text{ (see (2.16)).}$$

It is known that $\tau_{\infty}^{(m^d)}(\{0\}) > 0$ implies $1^d = \infty$ with $m^d(1^d-) < \infty$ or $1^d + m^d(1^d) < \infty$ with $h^d = \infty$. In both cases (2.17) implies

$$\tau_{\infty}^{(m^d)}(\{0\}) = (m^d(1^d))^{-1} = (1+h)^{-1} = r_m^{-1}.$$

(Put $h = 0$ if $m(1-) + 1 = \infty$.)

Thus we get

$$c_m^{-1} = r_m^{-1} + \int_{0+}^{\infty} \frac{d\tau_0^{(m)}(\mu)}{\mu},$$

i.e., (3.5) holds. Therefore (i) is proved.

The crucial point to show (ii) and (iii) is (4.10). Indeed, introduce for $s \geq 0$ measures σ_s on $[0, \infty)$ by

$$\sigma_s(d\mu) := s \cdot \varepsilon_0(d\mu) + \tau_{\infty}^{(m^d)}(d\mu) \mathbb{1}_{(0, \infty)}(\mu), \quad \mu \geq 0,$$

where ε_0 denotes the measure concentrated with unit mass at zero.

Note that $\tau_{\infty}^{(m^d)}(\cdot) = \sigma_{r_m^{-1}}(\cdot)$ and $\tau_{\infty}^{(m^d)}(\{0\}) = r_m^{-1}$. Then by Krein's

Theorem 2.2 for every $s \geq 0$ there exists a string $S_{\infty}(n_s)$ with $n_s(x) > 0$ for $x > 0$, i.e. $c_{n_s} = 0$, having σ_s as its spectral measure.

From (2.17) it follows for $s \geq 0$ that $n_s(l_{n_s}) = s^{-1}$ with $s^{-1} = \infty$ if $s = 0$.

Put $q_s := n_s^d$, $s \geq 0$. Then the original m is included for $s = r_m^{-1}$ and from (4.10) we get that the spectral measures $\tau_o^{(q_s)}$ do not depend on $s \geq 0$ and are equal to $\tau_o^{(m)}$. If $s > 0$ then

$$s^{-1} = \zeta_s(\{0\})^{-1} = (n_s(l_s)) = r_{q_s} < \infty, \quad (4.15)$$

and if $s = 0$ we get $n_o(l_o) = \infty$, i.e. $l_{q_o} = \infty$.

Thus, among all q_s , $s \geq 0$ we find exactly one infinite string, namely m_o . Note that $q_s(0) = c_{n_s} \equiv 0$.

To finish the proof of Theorem 3.2 it suffices to identify the equivalence class $\hat{S}(m)$ introduced in Chapter 3 with $\{q_s \mid s \geq 0\}$.

We remark that the characteristic function Γ_s of q_s satisfies (see (4.9), (2.17))

$$\begin{aligned} \frac{1}{\Gamma_s(\lambda)} &= -\lambda \Gamma_{n_s}(\lambda) = -\lambda \left(-\frac{s}{\lambda} + \int_{0-}^{\infty} \frac{d\tau_o^{(m^d)}(\mu)}{\mu - \lambda} + \frac{1}{r_m \lambda} \right) \\ &= \left(s - \frac{1}{r_m} \right) - \lambda \Gamma_d(\lambda) = \left(s - \frac{1}{r_m} \right) + \frac{1}{\Gamma_m(\lambda)}, \quad \lambda \in K_-. \end{aligned} \quad (4.16)$$

Let us calculate the characteristic function of $S_{\infty}(m_t)$ with $m_t \in \hat{S}$, where m_t was defined in Lemma 3.1.

LEMMA 4.4: For every $t \geq -\frac{1}{r_m}$ the corresponding to m_t functions $\Phi_{o,t}^{\uparrow}$, $\Phi_{\infty,t}^{\uparrow}$ are given by

$$\Phi_{o,t}^{\uparrow}(x, \lambda) = (1-tx) \Phi_o^{\uparrow}\left(\frac{x}{1-tx}, \lambda\right) \quad (4.17)$$

$$\Phi_{\infty,t}^{\uparrow}(x, \lambda) = (1-tx) \Phi_{\infty}^{\uparrow}\left(\frac{1}{1-tx}, \lambda\right) + t(1-tx) \Phi_o^{\uparrow}\left(\frac{1}{1-tx}, \lambda\right) \quad (4.18)$$

Proof: The left hand sides of (4.17) and (4.18) are the unique solutions of (2.7) and (2.8) with m replaced by m_t , respectively. After scale transformations and some calculations it is seen that the right-hand sides of (4.17) and (4.18) satisfy these equations. This proves the lemma.

COROLLARY 4.5: We have

$$\frac{1}{\Gamma_{m_t}(\lambda)} = \lim_{x \uparrow r_{m_t}} \frac{\Phi_{\infty,t}^{\uparrow}(x, \lambda)}{\Phi_{o,t}^{\uparrow}(x, \lambda)} = \frac{1}{\Gamma_m(\lambda)} + t, \quad \lambda \in K_-. \quad (4.19)$$

The proof follows immediately from (4.1), (4.17) and (4.18).

Now, compare (4.19) with (4.16). From Krein's inverse spectral theorem we get $m_t = q_s$ for $t = s - r_m^{-1}$. Thus Theorem 3.2 is proved.

As a consequence of (4.9), (4.10) we get the formula (2.15):

$$\begin{aligned} -\frac{1}{\Gamma_m(\lambda)} &= \lambda \Gamma_m^d(\lambda) = \lambda \int_{0-}^{\infty} \frac{d\tau_{\infty}^{(m^d)}(\mu)}{\mu - \lambda} \\ &= -\tau_{\infty}^{(m^d)}(\{0\}) - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_0^{(m)}(\mu) \\ &= -r_m^{-1} - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_0^{(m)}(d\mu), \quad \lambda \in \kappa_-. \end{aligned} \quad (4.20)$$

Note, that we have supposed $m(0) = 0$. If some $m(\{0\}) > 0$ is added to m at zero, the term $\lambda m(\{0\})$ is added on the right-hand side of (4.20).

The Corollary 3.3 follows immediately from the Theorem 3.2.

Proof of Theorem 3.4:

Let $S_a(m)$ be a string with $a \in (0, \infty)$ and $c_m = 0$. Put $w := a$ and define $\tilde{m}(x) := m(x-a)$, $x \in \mathbb{R}$. Obviously, it holds $c_{\tilde{m}} = a$ and

$$r_{\tilde{m}} = r_m + a.$$

If τ_a and $\tilde{\tau}_0$ denote the spectral measures of $S_a(m)$ and $S_0(\tilde{m})$, respectively, then we have by Proposition 2.3.(iii)

$$d\tau_a(\mu) = a^2 d\tilde{\tau}_0(\mu), \quad \mu > 0$$

From (3.5) it follows

$$\int_{0+}^{\infty} \frac{d\tau_a(\mu)}{\mu} = a^2 \int_{0+}^{\infty} \frac{d\tilde{\tau}_0(\mu)}{\mu} = a^2(a^{-1} - (r_m + a)^{-1}) = a(1 - \frac{a}{a+r_m}),$$

i.e. (3.6) and (3.7) hold.

Conversely, if $a \in (0, \infty)$ is fixed and τ is a measure on $(0, \infty)$ with $\tau((0, \infty)) > 0$ and (3.7) then choose a number $u \in (0, \infty]$ with

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = a(1 - \frac{a}{a+u}).$$

Put

$$\sigma(d\mu) := a^{-2}\tau(d\mu), \quad \mu \in (0, \infty),$$

and choose the string $S_0(m)$ with $m(0) = 0$ and $l_m = \infty$ having σ as its spectral measure (see Theorem 3.2.(iii)).

By the same theorem, for every $s \in [0, \infty)$ the string $S_0(m_s)$ with

$$\begin{aligned} m_s(x) &:= (1 - sx)^2 m\left(\frac{x}{1 - sx}\right), & x \in [0, s^{-1}], \\ &= \infty & x > s^{-1} \end{aligned}$$

has the same spectral measure σ as $S_0(m)$.

It holds by (3.5)

$$c_{m_s}^{-1} = \int_{0+}^{\infty} \frac{d\sigma(\mu)}{\mu} + r_{m_s}^{-1} = \int_{0+}^{\infty} \frac{d\sigma(\mu)}{\mu} + s = a^{-1} \left(1 - \frac{a}{a+u}\right) + s.$$

Now choose s in such a way that $c_{m_s} = a$ holds, i.e. put $s = \frac{1}{a+u}$.

By shifting m_s to the left

$$\tilde{m}_s(x) := m_s(x + a)$$

we get a string $S_a(\tilde{m}_s)$ with $c_{\tilde{m}_s} = 0$ having τ as its spectral measure. The uniqueness follows from the uniqueness of $S_0(m)$ with $l_m = \infty$.

For the last part of Theorem 3.4.(ii) note that the strings

$$S_0\left(\frac{m'(\cdot - a')}{(a')^2}\right) \text{ and } S_0\left(\frac{m(\cdot - a)}{a^2}\right)$$
 have the common spectral measure τ

(see Proposition 2.3.(iii)).

From Theorem 3.2.(iii) it follows

$$\begin{aligned} S_0\left(\frac{m'(\cdot - a')}{(a')^2}\right) &= S_0\left(\left(\frac{m(\cdot - a)}{a^2}\right)_t\right) \quad \text{with} \\ t &= \frac{1}{r' - a'} - \frac{1}{r - a}. \end{aligned}$$

Proof of Proposition 3.5:

Choose $a' \in (0, \infty]$ and consider a string $S_{a'}(m)$. Then it holds (see the definition of $r_{\lambda, a'}(x, y)$)

$$r_{\lambda, a'}(0, 0) = \frac{\Phi^\downarrow(0, \lambda)}{\frac{1}{a'} \Phi^\downarrow(0, \lambda) + 1} = \frac{1}{\frac{1}{a'} + \frac{1}{r_m(\lambda)}} \quad (4.21)$$

and, by definition of the spectral measure $\tau_{a'}^{(m)}$,

$$r_{\lambda, a'}(0, 0) = \int_0^\infty \frac{d\tau_{a'}^{(m)}(\mu)}{\mu - \lambda} \quad (4.22)$$

Now let be $a \in (0, \infty)$. Then (3.8) is a consequence of

$$-\frac{1}{\Gamma_m(\lambda)} \frac{1}{\frac{1}{a} + \frac{1}{\Gamma_m(\lambda)}} (a + \Gamma_m(\lambda)) = -a \quad (4.23)$$

(2.15), (4.21), (4.22) for $a' = a$ and $a' = \infty$.

Letting $a \downarrow 0$ in (4.23) divided by a we get Knight's formula.

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