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On Filtrations of Brownian polynomials

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In [3] Lane considered, among other things, the filtrations of the processes $\int_0^t h(B_s)dB_s$ for a certain class of functions h on the real line. He showed that in many instances the filtration of such a process is either that of the Brownian motion itself or that of an appropriate reflected Brownian motion. In this note we make a rather curious observation regarding the filtrations of the processes $H_n(B_t,t)$. This also helps us to describe the filtrations of a large class of polynomials in (B_t,t). We conclude with an extremal property of these martingales.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(B_t)_{t\geq 0}$ be a standard Brownian motion defined on the space. For each $n \geq 1$ recall that the nth Hermite polynomial in (x,t) is defined by

$$H_n(x,t) = \frac{(-t)^n}{n!} e^{x^2/2t} D_x^n(e^{-x^2/2t})$$

It is wellknown and easily verifiable that $D_x H_n = H_{n-1}$ and $(D_t + \frac{1}{2} D_{xx}) H_n = 0$ for each $n \geq 1$.

Let $Y_n(t) = H_n(B_t,t)$ and $(\mathcal{G}_t^n)$ its canonical filtration. Here and in what follows canonical filtration of a process means the right continuous modification of the natural filtration of the process augmented by P-null sets of the process. In particular $(\mathcal{G}_t^1)$ is the Brownian filtration and $(\mathcal{G}_t^2)$ is the filtration of the reflected Brownian motion $|B|$.

<u>Theorem 1</u> : 1. For each $n \geq 1$, $(\mathcal{G}_t^n) = (\mathcal{G}_t^1)$ or $(\mathcal{G}_t^2)$ according as n is odd or even.

2. Let $P(x,t)$ be any nonconstant polynomial in (x,t) satisfying $(D_t + \frac{1}{2} D_{xx}) P = 0$. Assume that the coefficient of x^{n-1} is zero where n is the degree of P in x. Then the canonical filtration of $P(B_t,t)$ is either $(\mathcal{G}_t^1)$ or $(\mathcal{G}_t^2)$ according as an odd power of x is present in P or not.

The following simple lemma will be used repeatedly in the proof of the theorem.

<u>Lemma</u>. Let (M_t) be a continuous martingale. Then, (i) $\langle M \rangle$ is adapted to the canonical filtration of $|M|$. (ii) if moreover $M_t = \int_0^t h_s dB_s$ then $|h|$ is $|M|$ adapted.

<u>Proof of the Lemma</u>. (i) is a direct consequence of the fact that in the Doob-Meyer decomposition of a submartingale, the increasing process is adapted to the canonical filtration of the submartingale. To prove (ii) note that by (i) $(\int_0^t h_s^2 ds)$ is adapted to $|M|$ and hence so also is (h_t^2).

<u>Proof of the Theorem</u> : 1. By Ito's formula $Y_n(t) = \int_0^t Y_{n-1}(s)dB_s$. By the lemma it follows that $|Y_{n-1}|$ is $(\mathcal{G}_t^n)$ adapted. In turn $Y_{n-1}(t) = \int_0^t Y_{n-2}(s)dB_s$, so that $|Y_{n-2}|$ is $|Y_{n-1}|$ adapted and hence $(\mathcal{G}_t^n)$ adapted. Proceeding in this way we observe that $|B|$ is $(\mathcal{G}_t^n)$ adapted. In other words $(\mathcal{G}_t^2) \subset (\mathcal{G}_t^n)$. In case n is even the proof is complete since $H_n(x,t)$ involves only even powers of x. In case n is odd, $Y_n(t) = B_t.Q(B_t,t)$ where Q is a polynomial in (x,t) involving only even powers of x. $(Q(B_t,t))$ is adapted to $(\mathcal{G}_t^2)$ and hence to $(\mathcal{G}_t^n)$. But for fixed t, $Q(B_t,t)$ is nonzero almost surely so that B is $(\mathcal{G}_t^n)$ adapted. The proof is complete.

2. Denote by $(\mathcal{G}_t)$ the canonical filtration of the process $(P(B_t,t))$. Let the degree of P in x be n. Denoting derivative w.r.t. x by $'$, we see that $P', P'', \ldots$ are all solutions of $(D_t + \frac{1}{2} D_{xx})\, u = 0$, so that $P^{(k)}(B_t,t) = \int_0^t P^{(k+1)}(B_s,s)dB_s$. Further $P^{(n-1)}(x,t) = c.x$ by the assumption on P where c is a nonzero constant. Now proceeding as in 1 above we get that $(\mathcal{G}_t^2) \subset (\mathcal{G}_t)$. In case P has only even powers of x, the proof is complete. Otherwise, write $P = Q_1 + x.Q_2$ where both Q_1 and Q_2 involve only even powers of x to complete the proof.

Remark 1. For each $n \geq 1$, if $(\tilde{\mathcal{G}}_t^n)$ is the natural filtration of the (Y_n) process augmented by P-null sets of $\tilde{\mathcal{G}}_\infty^n$ then $(\tilde{\mathcal{G}}_t^n) = (\mathcal{G}_t^n)$. In other words $(\tilde{\mathcal{G}}_t^n)$ is itself right continuous. This is because the same is known to be true for n=1 and 2. A routine argument now completes the proof.

Remark 2 . The assumption in the second part of the theorem – namely, that the coefficient of x^{n-1} be zero – is essential. To see this let $P(x,t) = x^2 - x - t$. Clearly the canonical filtration of $(P(B_t,t))$ can not be $(\mathcal{G}_t^2)$. It is not $(\mathcal{G}_t^1)$ either. The quickest way to see this is to take Ω to be $C[0,\infty)$ and B the coordinate process. If τ is the hitting time of $\frac{1}{2}$ by B then $P(B_t,t)$ does not distinguish between the paths ω and ω^* where ω^* is the usual reflection of ω at τ. The measure preserving nature of the map $\omega \mapsto \omega^*$ can now be used to complete the proof.

Remark 3. It is curious to note that the theorem is not valid for arbitrary nonconstant solutions u of $(D_t + \frac{1}{2} D_{xx})u = 0$. The function $u(x,t) = e^{t/2} \sin x$ is such a function and it also has a series expansion in terms of Hermite polynomials given by $u = \Sigma(-1)^k H_{2k+1}$. This is verified by using the generating function for Hermite polynomials [1']. Of course, the filtration of $(u(B_t,t))$ is same as that of the process sin B, which is neither $(\mathcal{G}_t^1)$ nor $(\mathcal{G}_t^2)$. However it is of interest to note that its canonical filtration is a Brownian filtration. Indeed if $M_t = e^{t/2}\sin B_t$ then $\int_0^t \frac{e^{-s/2}}{\sqrt{1-e^{-s}M_s^2}}\,dM_s$ is a Brownian motion and its canonical filtration is same as that of M .

<u>Remark 4</u>. The theorem is a truely infinite time dimensional theorem. That is to say, for n odd (resp. even) the σ-field $\sigma(B_{t_1},\dots,B_{t_k})$ (resp. $\sigma(|B_{t_1}|,\dots,|B_{t_k}|)$) is strictly larger than $\sigma(Y_n(t_1),\dots,Y_n(t_k))$ for any finite set of time points $t_1 < t_2 < \dots < t_k$ and for any $n \geq 3$.

As a consequence of Theorem 1, we have the following result which is perhaps known, but we have not found in the literature.

<u>Theorem 2</u>. 1. Y_n has martingale representation property. That is, every $(\mathcal{G}_t^n)$ martingale is a stochastic integral w.r.t. Y_n .

2. Y_n is an extremal martingale. That is, the law μ_n of Y_n is an extreme point of the convex set of all probabilities on $C[0,\infty)$ making the coordinate process a martingale.

3. For $n \neq m$, $\mu_n \perp \mu_m$.

<u>Proof</u>. 1. Let n be odd. Then Y_{n-1} is $(\mathcal{G}_t^n)$ adapted and $dB = \frac{1}{Y_{n-1}}\,dY_n$ so that any $(\mathcal{G}_t^n)$ martingale, being an integral w.r.t. B is also an integral w.r.t. Y_n. Let n be even. Then, $dY_n = Y_{n-1}dB = Z_{n-1}dY_2$ where $Z_{n-1}(s) = P(B_s,s)$ and $P(x,t) = \frac{1}{x}H_{n-1}(x,t)$. Since H_{n-1} involves only odd powers of x, P is a polynomial involving only even powers of x. Z_{n-1}

being $(\mathcal{G}_t^n)$ adapted, we deduce that $dY_2 = \frac{1}{Z_{n-1}} dY_n$. Now, any $(\mathcal{G}_t^n)$ martingale is a $(\mathcal{G}_t^2)$ martingale and hence an integral w.r.t. Y_2 and so in turn is an integral w.r.t. Y_n. Incidentally, the fact that any $(\mathcal{G}_t^2)$ martingale is an Y_2 integral follows from observing that $M_t = \int_0^t \operatorname{sgn}(B_s)dB_s$ is a Brownian motion, its canonical filtration is $(\mathcal{G}_t^2)$ and $dM = \frac{1}{|B|} dY_2$.

2. can be deduced using Theorem 11.2, p.338 and
3. using Theorem 11.4, p.340 of Jacod [2].

<u>References</u>

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2. Jacod, J. (1979): <u>Calcul Stochastique et Problemes de Martingales</u>. Springer LNM 714.

3. Lane, D.A.(1978): On the fields of some Brownian martingales, <u>Ann. Prob</u>. 6 p.499-508.