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# On Filtrations of Brownian polynomials

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In [3] Lane considered, among other things, the filtrations of the processes  $\int_0^t h(B_s) dB_s$  for a certain class of functions  $h$  on the real line. He showed that in many instances the filtration of such a process is either that of the Brownian motion itself or that of an appropriate reflected Brownian motion. In this note we make a rather curious observation regarding the filtrations of the processes  $H_n(B_t, t)$ . This also helps us to describe the filtrations of a large class of polynomials in  $(B_t, t)$ . We conclude with an extremal property of these martingales.

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $(B_t)_{t \geq 0}$  be a standard Brownian motion defined on the space. For each  $n \geq 1$  recall that the  $n$ th Hermite polynomial in  $(x, t)$  is defined by

$$H_n(x, t) = \frac{(-t)^n}{n!} e^{x^2/2t} D_x^n(e^{-x^2/2t})$$

It is wellknown and easily verifiable that  $D_x H_n = H_{n-1}$  and  $(D_t + \frac{1}{2} D_{xx}) H_n = 0$  for each  $n \geq 1$ .

Let  $Y_n(t) = H_n(B_t, t)$  and  $(\mathcal{G}_t^n)$  its canonical filtration. Here and in what follows canonical filtration of a process means the right continuous modification of the natural filtration of the process augmented by  $P$ -null sets of the process. In particular  $(\mathcal{G}_t^1)$  is the Brownian filtration and  $(\mathcal{G}_t^2)$  is the filtration of the reflected Brownian motion  $|B|$ .

**Theorem 1** : 1. For each  $n \geq 1$ ,  $(\mathcal{G}_t^n) = (\mathcal{G}_t^1)$  or  $(\mathcal{G}_t^2)$  according as  $n$  is odd or even.

2. Let  $P(x, t)$  be any nonconstant polynomial in  $(x, t)$  satisfying  $(D_t + \frac{1}{2} D_{xx}) P = 0$ . Assume that the coefficient of  $x^{n-1}$  is zero where  $n$  is the degree of  $P$  in  $x$ . Then the canonical filtration of  $P(B_t, t)$  is either  $(\mathcal{G}_t^1)$  or  $(\mathcal{G}_t^2)$  according as an odd power of  $x$  is present in  $P$  or not.

The following simple lemma will be used repeatedly in the proof of the theorem.

**Lemma.** Let  $(M_t)$  be a continuous martingale. Then, (i)  $\langle M \rangle$  is adapted to the canonical filtration of  $|M|$ . (ii) if moreover  $M_t = \int_0^t h_s dB_s$  then  $|h|$  is  $|M|$  adapted.

**Proof of the Lemma.** (i) is a direct consequence of the fact that in the Doob-Meyer decomposition of a submartingale, the increasing process is adapted to the canonical filtration of the submartingale. To prove (ii) note that by (i)  $(\int_0^t h_s^2 ds)$  is adapted to  $|M|$  and hence so also is  $(h_t^2)$ .

**Proof of the Theorem** : 1. By Ito's formula  $Y_n(t) = \int_0^t Y_{n-1}(s) dB_s$ . By the lemma it follows that  $|Y_{n-1}|$  is  $(\mathcal{G}_t^n)$  adapted. In turn  $Y_{n-1}(t) = \int_0^t Y_{n-2}(s) dB_s$ , so that  $|Y_{n-2}|$  is  $|Y_{n-1}|$  adapted and hence  $(\mathcal{G}_t^n)$  adapted. Proceeding in this way we observe that  $|B|$  is  $(\mathcal{G}_t^n)$  adapted. In other words  $(\mathcal{G}_t^2) \subset (\mathcal{G}_t^n)$ . In case  $n$  is even the proof is complete since  $H_n(x, t)$  involves only even powers of  $x$ . In case  $n$  is odd,  $Y_n(t) = B_t \cdot Q(B_t, t)$  where  $Q$  is a polynomial in  $(x, t)$  involving only even powers of  $x$ .  $(Q(B_t, t))$  is adapted to  $(\mathcal{G}_t^2)$  and hence to  $(\mathcal{G}_t^n)$ . But for fixed  $t$ ,  $Q(B_t, t)$  is nonzero almost surely so that  $B$  is  $(\mathcal{G}_t^n)$  adapted. The proof is complete.

2. Denote by  $(\mathcal{G}_t)$  the canonical filtration of the process  $(P(B_t, t))$ . Let the degree of  $P$  in  $x$  be  $n$ . Denoting derivative w.r.t.  $x$  by  $'$ , we see that  $P', P'', \dots$  are all solutions of  $(D_t + \frac{1}{2} D_{xx}) u = 0$ , so that  $P^{(k)}(B_t, t) = \int_0^t P^{(k+1)}(B_s, s) dB_s$ . Further  $P^{(n-1)}(x, t) = c \cdot x$  by the assumption on  $P$  where  $c$  is a nonzero constant. Now proceeding as in 1 above we get that  $(\mathcal{G}_t^2) \subset (\mathcal{G}_t)$ . In case  $P$  has only even powers of  $x$ , the proof is complete. Otherwise, write  $P = Q_1 + x \cdot Q_2$  where both  $Q_1$  and  $Q_2$  involve only even powers of  $x$  to complete the proof.

Remark 1. For each  $n \geq 1$ , if  $(\tilde{\mathcal{G}}_t^n)$  is the natural filtration of the  $(Y_n)$  process augmented by  $P$ -null sets of  $\tilde{\mathcal{G}}_\infty^n$  then  $(\tilde{\mathcal{G}}_t^n) = (\mathcal{G}_t^n)$ . In other words  $(\tilde{\mathcal{G}}_t^n)$  is itself right continuous. This is because the same is known to be true for  $n=1$  and 2. A routine argument now completes the proof.

Remark 2. The assumption in the second part of the theorem - namely, that the coefficient of  $x^{n-1}$  be zero - is essential. To see this let  $P(x, t) = x^2 - x - t$ . Clearly the canonical filtration of  $(P(B_t, t))$  can not be  $(\mathcal{G}_t^2)$ . It is not  $(\mathcal{G}_t^1)$  either. The quickest way to see this is to take  $\Omega$  to be  $C[0, \infty)$  and  $B$  the coordinate process. If  $\tau$  is the hitting time of  $\frac{1}{2}$  by  $B$  then  $P(B_t, t)$  does not distinguish between the paths  $\omega$  and  $\omega^*$  where  $\omega^*$  is the usual reflection of  $\omega$  at  $\tau$ . The measure preserving nature of the map  $\omega \mapsto \omega^*$  can now be used to complete the proof.

Remark 3. It is curious to note that the theorem is not valid for arbitrary nonconstant solutions  $u$  of  $(D_t + \frac{1}{2} D_{xx})u = 0$ . The function  $u(x, t) = e^{t/2} \sin x$  is such a function and it also has a series expansion in terms of Hermite polynomials

given by  $u = \sum (-1)^k H_{2k+1}$ . This is verified by using the generating function for Hermite polynomials [1']. Of course, the filtration of  $(u(B_t, t))$  is same as that of the process  $\sin B$ , which is neither  $(\mathcal{F}_t^1)$  nor  $(\mathcal{F}_t^2)$ . However it is of interest to note that its canonical filtration is a Brownian filtration. Indeed if  $M_t = e^{t/2} \sin B_t$  then  $\int_0^t \frac{e^{-s/2}}{\sqrt{1-e^{-s} M_s^2}} dM_s$  is a Brownian motion and its canonical filtration is same as that of  $M$ .

**Remark 4.** The theorem is a truly infinite time dimensional theorem. That is to say, for  $n$  odd (resp. even) the  $\sigma$ -field  $\sigma(B_{t_1}, \dots, B_{t_k})$  (resp.  $\sigma(|B_{t_1}|, \dots, |B_{t_k}|)$ ) is strictly larger than  $\sigma(Y_n(t_1), \dots, Y_n(t_k))$  for any finite set of time points  $t_1 < t_2 < \dots < t_k$  and for any  $n \geq 3$ .

As a consequence of Theorem 1, we have the following result which is perhaps known, but we have not found in the literature.

**Theorem 2.** 1.  $Y_n$  has martingale representation property. That is, every  $(\mathcal{F}_t^n)$  martingale is a stochastic integral w.r.t.  $Y_n$ .

2.  $Y_n$  is an extremal martingale. That is, the law  $\mu_n$  of  $Y_n$  is an extreme point of the convex set of all probabilities on  $C[0, \infty)$  making the coordinate process a martingale.

3. For  $n \neq m$ ,  $\mu_n \perp \mu_m$ .

**Proof.** 1. Let  $n$  be odd. Then  $Y_{n-1}$  is  $(\mathcal{F}_t^n)$  adapted and  $dB = \frac{1}{Y_{n-1}} dY_n$  so that any  $(\mathcal{F}_t^n)$  martingale, being an integral w.r.t.  $B$  is also an integral w.r.t.  $Y_n$ . Let  $n$  be even. Then,  $dY_n = Y_{n-1} dB = Z_{n-1} dY_2$  where  $Z_{n-1}(s) = P(B_s, s)$  and  $P(x, t) = \frac{1}{x} H_{n-1}(x, t)$ . Since  $H_{n-1}$  involves only odd powers of  $x$ ,  $P$  is a polynomial involving only even powers of  $x$ .  $Z_{n-1}$

being  $(\mathcal{G}_t^n)$  adapted, we deduce that  $dY_2 = \frac{1}{Z_{n-1}} dY_n$ . Now, any  $(\mathcal{G}_t^n)$  martingale is a  $(\mathcal{G}_t^2)$  martingale and hence an integral w.r.t.  $Y_2$  and so in turn is an integral w.r.t.  $Y_n$ . Incidentally, the fact that any  $(\mathcal{G}_t^2)$  martingale is an  $Y_2$  integral follows from observing that  $M_t = \int_0^t \text{sgn}(B_s) dB_s$  is a Brownian motion, its canonical filtration is  $(\mathcal{G}_t^2)$  and  $dM = \frac{1}{|B|} dY_2$ .

2. can be deduced using Theorem 11.2, p.338 and
3. using Theorem 11.4, p.340 of Jacod [2].

### References

1. Hida, T. (1979) : Brownian motion, Springer-Verlag.
2. Jacod, J. (1979): Calcul Stochastique et Problemes de Martingales. Springer LNM 714.
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