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KALYANAPURAM RANGACHARI PARTHASARATHY

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REALISATION OF A CLASS OF MARKOV PROCESSES
THROUGH UNITARY EVOLUTIONS IN FOCK SPACE

by
K.R. Parthasarathy
Indian Statistical Institute
Delhi Centre, New Delhi-110016

1. Introduction: Pursuing the chain of ideas initiated in [1, 2, 3] and further discussed in [4] we modify the notations of quantum stochastic calculus in Fock space and demonstrate how a class of continuous as well as discrete state space Markov processes can be realised through unitary operator evolutions in the tensor product of an initial Hilbert space with a boson Fock space.

2. The basic results of quantum stochastic calculus in a new notation: Let

$$\tilde{H} = h_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes k) \quad (2.1)$$

where h_0 and k are complex separable Hilbert spaces and for any Hilbert space H $\Gamma(H)$ denotes the boson Fock space over H . Put

$$h = h_0 \otimes (\mathbb{C} e_{-\infty} \oplus k \oplus \mathbb{C} e_{\infty}) \quad (2.2)$$

where $e_{\pm\infty}$ are unit vectors and \oplus indicates Hilbert space direct sum. Fix an orthonormal basis $\{e_i | i \in S\}$ in k and put $\tilde{S} = S \cup \{-\infty\} \cup \{\infty\}$. The basic noise processes $\{\Lambda_j^i\}$ of boson stochastic calculus in \tilde{H} can be expressed as

$$\Lambda_i^j = \Lambda_{|e_i\rangle\langle e_j|}, \quad i, j \in S,$$

$$\Lambda_{-\infty}^j = \Lambda_{|e_{-\infty}\rangle\langle e_j|} = A_j, \quad j \in S,$$

$$\Lambda_i^{\infty} = \Lambda_{|e_i\rangle\langle e_{\infty}|} = A_i^{\dagger}, \quad i \in S,$$

$$\Lambda_{-\infty}^{\infty}(t) = tI, \quad t \geq 0$$

where Λ_i^j , $i, j \in S$ are the conservation (or exchange) processes, A_j , $j \in S$ are

the annihilation processes and A_i^\dagger , $i \in S$ are the creation processes. We adopt the convention that $\Lambda_i^{-\infty} = \Lambda_\infty^j = 0$.

Inspired by a conversation with V.P. Belavkin in Moscow in 1989 we introduce a subalgebra $I(h) \subset B(h)$ with a special involution as follows:

$$I(h) = \{L \mid L \in B(h), L f \otimes e_{-\infty} = L^* f \otimes e_\infty = 0 \text{ for all } f \in h_0\}, \quad (2.3)$$

$$L^b = F L^* F \quad (2.4)$$

where $B(h)$ is the algebra of all bounded operators on h and F is the unique unitary (flip) operator in h satisfying

$$F f \otimes e_{-\infty} = f \otimes e_\infty, F f \otimes e_\infty = f \otimes e_{-\infty}, F f \otimes u = f \otimes u$$

for all $f \in h_0$, $u \in k$. Then $I(h)$ is a subalgebra of $B(h)$ and the correspondence $L \rightarrow L^b$ is an involution under which $I(h)$ is closed. To any $L \in I(h)$ we associate the family $\{L_j^i \mid i, j \in \tilde{S}\}$ of operators in h_0 by putting

$$\langle f, L_j^i g \rangle = \langle f \otimes e_i, L g \otimes e_j \rangle, \quad i, j \in \tilde{S}, f, g \in h_0. \quad (2.5)$$

Then by (2.3)

$$L_j^\infty = L_{-\infty}^i = 0 \text{ for all } i, j \in \tilde{S},$$

$$\sum_{i \in \tilde{S}} \|L_j^i f\|^2 = \|L f \otimes e_j\|^2, \quad f \in h_0.$$

Hence by the basic results of quantum stochastic calculus (q.s.c.) there exists a unique adapted process Λ_L in \tilde{H} satisfying

$$\Lambda_L(0) = 0, d\Lambda_L = \sum_{i,j \in \tilde{S}} L_j^i d\Lambda_i^j, \quad L \in I(h). \quad (2.6)$$

(See, for example, Proposition 27.1 in [4]). The following two propositions are immediate from the methods of q.s.c. (Ch. III, [4]).

Proposition 2.1. The processes $\{\Lambda_L \mid L \in I(h)\}$ defined by (2.6) satisfy the following

$$(i) \quad \langle f e(u), \Lambda_L(t) g e(v) \rangle = \int_0^t \langle f \otimes (e_{-\infty} + u(s)), L g \otimes (v(s) + e_\infty) \rangle ds \langle e(u), e(v) \rangle,$$

(ii) If $\Lambda_L^\dagger(t) = \Lambda_{L^b}(t)$ then $\{\Lambda_L, \Lambda_L^\dagger\}$ is an adjoint pair;

(iii) $d\Lambda_L d\Lambda_M = d\Lambda_{LM}$.

In particular, Λ_L is independent of the orthonormal basis $\{e_i | i \in S\}$ employed in its definition.

Proposition 2.2. Let $L \in \mathcal{I}(h)$. Then there exists a unique unitary operator valued adapted process U_L satisfying the quantum stochastic differential equation (q.s.d.e.)

$$U_L(0) = 0, \quad dU_L = (d\Lambda_L) U_L$$

if and only if

$$L+L^b + L^b L = L+L^b + L L^b = 0. \quad (2.7)$$

If h_i , $i = 1, 2$ are Hilbert spaces and X is a bounded operator in h_1 we adopt the convention of denoting by the same symbol X , the operator $X \otimes 1$ in $h_1 \otimes h_2$ where 1 denotes the identity operator in h_2 . For any $L \in \mathcal{I}(h)$ and $X \in \mathcal{B}(h_0)$ the operators XL and LX belong to $\mathcal{I}(h)$. Furthermore $X d\Lambda_L = d\Lambda_{XL}$, $(d\Lambda_L)X = d\Lambda_{LX}$.

Proposition 2.3. Let $L \in \mathcal{I}(h)$. Suppose (2.7) holds and U_L is the unitary operator valued process defined by Proposition 2.2. Then

$$d U_L^* X U_L = U_L^* d\Lambda_{L^b X + XL + L^b XL} U_L \quad \text{for all } X \in \mathcal{B}(h_0).$$

If

$$T_t(X) = \mathbb{E}_0 U_L^*(t) X U_L(t)$$

where \mathbb{E}_0 denotes the boson vacuum conditional expectation map from $\tilde{\mathcal{B}}(H)$ onto $\mathcal{B}(h_0)$ then $\{T_t | t \geq 0\}$ is a uniformly continuous one parameter semigroup of operators on the Banach space $\mathcal{B}(h_0)$ whose infinitesimal generator L is given by

$$L(X) = \left. \frac{dT_t(X)}{dt} \right|_{t=0},$$

$$\langle f, L(X)g \rangle = \langle f \otimes e_{-\infty}, (L^b X + XL + L^b XL)g \otimes e_{\infty} \rangle \quad \text{for all } f, g \in h_0.$$

Proof: Propositions 1-3 are the basic results of q.s.c. and we refer to Chapter III, [4]. □

3. Construction of some classical Markov flows through unitary evolutions :

Let G be a locally compact second countable group acting on a separable σ -finite measure space (X, F, μ) with G -invariant measure μ . (Obvious generalizations can be worked out when μ is only quasi invariant). Define $h_0 = L^2(\mu)$ and $h = L^2(G)$ with respect to a left invariant Haar measure. Express any element $\underline{f} \in h = h_0 \otimes (\mathbb{C} e_{-\infty} \oplus h \oplus \mathbb{C} e_{\infty})$ as a column vector

$$\underline{f} = \begin{pmatrix} f_-(x) \\ f_0(x, g) \\ f_+(x) \end{pmatrix} \quad x \in X, \quad g \in G.$$

Let $\lambda(x, g)$ be any complex valued measurable function on $X \times G$ satisfying

$$\text{ess. sup}_{\mu} \int_G |\lambda(x, g)|^2 dg < \infty \quad (3.1)$$

where dg indicates integration with respect to the left invariant Haar measure.

Define the operator L_{λ} associated with λ in h by

$$L_{\lambda} \underline{f} = \begin{pmatrix} - \int_G \{ \overline{\lambda(x, g)} f_0(x, g) + \frac{1}{2} |\lambda(x, g)|^2 f_+(x) \} dg \\ f_0(g^{-1}x, g) - f_0(x, g) + \lambda(g^{-1}x, g) f_+(g^{-1}x) \\ 0 \end{pmatrix}$$

Then (3.1) implies that $L_{\lambda} \in \mathcal{B}(h)$. Furthermore the following holds:

(i) $L_{\lambda} \in (h)$;

$$L_{\lambda}^b \underline{f} = \begin{pmatrix} \int_G \{ \overline{\lambda(x, g)} f_0(gx, g) - \frac{1}{2} |\lambda(x, g)|^2 f_+(x) \} dg \\ f_0(gx, g) - f_0(x, g) - \lambda(x, g) f_+(x) \\ 0 \end{pmatrix};$$

(iii) $L_{\lambda}^b L_{\lambda} + L_{\lambda}^b + L_{\lambda} = L_{\lambda} L_{\lambda}^b + L_{\lambda} + L_{\lambda}^b = 0.$

Using Proposition 2.2 construct the unitary operator valued process $U_{\lambda} = U_{L_{\lambda}}$ in

\tilde{H} satisfying

$$U_\lambda(0) = 1, \quad dU_\lambda = (d\Lambda_{L_\lambda})U_\lambda.$$

Consider the Evans-Hudson flow $\{j_t | t \geq 0\}$ induced by U_λ :

$$j_t(x) = U_\lambda(t)^* x U_\lambda(t), \quad x \in B(h_0).$$

If $\{e_i | i \in S\}$ is any fixed orthonormal basis in $L^2(G)$ then the structure maps $\{\theta_j^i | i, j \in S\}$ of the flow $\{j_t\}$ are given by

$$\theta_j^i(x) = (L_\lambda^b x + x L_\lambda + L_\lambda^b x L_\lambda)_j^i$$

with the convention $\theta_j^\infty = \theta_\infty^i = 0$. Denote by A_0 the abelian von Neumann algebra $L^\infty(\mu)$ where any function $\phi \in L^\infty(\mu)$ is interpreted as the operator of multiplication by ϕ in $L^2(\mu) = h_0$. Then a routine computation yields the following: θ_j^i leaves A_0 invariant and

$$\theta_j^i(\phi)(x) = \int_G \phi(gx) \bar{e}_i(g) e_j(g) dg - \delta_j^i \phi(x), \quad i, j \in S,$$

$$\theta_j^{-\infty}(\phi)(x) = \int_G \overline{\lambda(x, g)} [\phi(gx) - \phi(x)] e_j(g) dg, \quad j \in S,$$

$$\theta_\infty^i(\phi)(x) = \int_G \lambda(x, g) \overline{e_i(g)} [\phi(gx) - \phi(x)] dg, \quad i \in S,$$

$$\theta_\infty^{-\infty}(\phi)(x) = \int_G |\lambda(x, g)|^2 [\phi(gx) - \phi(x)] dg.$$

It now follows from [2, 3] (and also Section 27, 28 in [4]) that

$$[j_s(\phi), j_t(\psi)] = 0 \quad \text{for all } s, t \geq 0, \quad \phi, \psi \in A_0.$$

In other words $\{j_t|_{A_0}, t \geq 0\}$ is a classical Markov flow in the Accardi-Frigerio-Lewis' formalism with infinitesimal generator L given by

$$L(\phi)(x) = \theta_\infty^{-\infty}(\phi)(x) = \int_G |\lambda(x, g)|^2 [\phi(gx) - \phi(x)] dg.$$

Thus $\lambda(x, g)$ can be interpreted as the rate of change of amplitude density from the state x to the state gx .

When G and X are finite this result reduces to the description in [1, 3]. If G and X are countable we obtain the picture of a Markov flow in [2].

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