SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

LESTER E. DUBINS
MICHEL ÉMERY
MARC YOR

A continuous martingale in the plane that may spiral away to infinity

Séminaire de probabilités (Strasbourg), tome 25 (1991), p. 284-290
<http://www.numdam.org/item?id=SPS_1991__25__284_0>

A CONTINUOUS MARTINGALE IN THE PLANE THAT MAY SPIRAL AWAY TO INFINITY

by L. E. Dubins[1], M. Emery and M. Yor

If $Z_t = \rho_t e^{i\theta_t}$ is a continuous, complex-valued martingale, is it possible that, with positive probability, both ρ_t and θ_t tend to infinity when $t \to \infty$? If Z is a conformal martingale, the answer is clearly no (for both $\operatorname{Log} \rho_t$ and θ_t are local martingales too). But if conformality is not required, such a behavior is possible. This note gives an example of a planar spiral curve σ and a continuous martingale that never hits σ but still has a non-zero probability of escaping to infinity.

The asymptotic behavior of a real continuous martingale is well known (and can easily be obtained by time-changing it into a Brownian motion): almost every path $t \mapsto M_t(\omega)$, either has a finite limit $M_\infty(\omega)$, or oscillates on the whole line ($\liminf_{t\to\infty} M_t(\omega) = -\infty$ and $\limsup_{t\to\infty} M_t(\omega) = +\infty$). As a consequence, if M takes its values in a proper subset of the line $\mathbb{R}$, it must converge a. s. to a finite limit M_∞.

In higher dimensions, things are much less clear: given a subset A of $\mathbb{R}^n$, what are the convergence or divergence properties of continuous martingales[2] taking their values in A? Some subsets *allow explosions*, in the sense that there exists a A-valued continuous martingale tending to infinity a. s. in $\mathbb{R}^n$ (that is, it eventually leaves every compact of $\mathbb{R}^n$); other *force convergence*, and every A-valued continuous martingale has an almost surely finite limit (in $\overline{A}$). Are these stochastic properties of A related to its geometry, in other words, is it possible to characterize geometrically sets that allow explosions or force convergence? We don't know; the aim of this note is to help clarify these matters by studying a few examples.

---

1. Research supported in part by N.S.F. grant MCS80-02535.

2. Or continuous local martingales: this is equivalent by a change of time.

We will start with sets that force convergence, simply called *convergence sets* in the sequel. Obviously, all bounded sets are convergence sets and convergence sets are stable by taking products, subsets, and images by affine transformations.

For convex sets, purely geometric characterizations of being a convergence set are easy to obtain. In the next statement, n half-spaces of $\mathbb{R}^n$ are said to be independent if they can be written $f_i > a_i$ or $f_i \geq a_i$ where the f_i are n linearly independent linear forms on $\mathbb{R}^n$; equivalently, the hyperplanes $f_i = a_i$ limiting those half-spaces do not contain a common direction of line.

PROPOSITION 1. — *Let A be a subset of $\mathbb{R}^n$. Each of the following conditions implies the next one:*

(i) *A is included in the intersection of n linearly independent half-spaces;*

(ii) *A is a convergence set;*

(iii) *A does not contain a whole straight line.*

If furthermore the set A is convex, these three conditions are equivalent.

PROOF. (i) $\Rightarrow$ (ii). Since $\mathbb{R}_+$ is a convergence set in $\mathbb{R}$, the product $\mathbb{R}_+^n$ is a convergence set in $\mathbb{R}^n$. The intersection of n linearly independent half-spaces, obtained from $\mathbb{R}_+^n$ by an affine transformation, is a convergence set too, and so is each of its subsets.

(ii) $\Rightarrow$ (iii). A set containing a line carries a martingale with no limit, for instance a Brownian motion on this line, so it cannot be a convergence set.

The proposition will be established by showing that (iii) and the supplementary hypothesis that A is convex imply (i).

Let A be a convex subset of $\mathbb{R}^n$ not containing any line; we claim that neither does its closure $\overline{A}$. For suppose $\overline{A}$ contains a line L. Let E be the smallest affine sub-space of $\mathbb{R}^n$ containing A (and $\overline{A}$); A contains $1+\dim E$ affinely independent points, so by convexity it contains also a whole simplex in E, and by replacing if necessary the reference space $\mathbb{R}^n$ with E, we may suppose that A has a non-empty interior. Let p be an interior point of A; we are going to show that A contains the line L' parallel to L passing by p, thus establishing the claim. Let indeed q be another point on L'. Since $L \subset \overline{A}$, there exists a sequence (x_n) of points of A such that $\text{dist}(x_n, L) \to 0$ and that x_n tends to infinity in the direction going from p to q. Since the line qx_n tends to L', the projection p_n of p on this line tends to p, and for some n large enough, p_n is in A and q is between p_n and x_n; by convexity, q is in A as claimed. So if $A \subset E = \mathbb{R}^n$ does not contain any line, $\overline{A}$ does not either.

To prove (i) it suffices to verify that the whole dual E' of $\mathbb{R}^n$ is linearly spanned by the set $A' = \{f \in E' : \exists a \in \mathbb{R}\ \forall x \in A\ f(x) \geq a\}$ of all linear forms that are bounded below on A. If A is empty, the result is trivial; else, let x be an element of A. Consider an arbitrary non-zero linear form ϕ on E'; since $E'' = E$, there is a non-zero vector $y \in \mathbb{R}^n$ such that $\phi(f) = f(y)$ for all f in E'. As $\overline{A}$

contains no line, there exists a real λ such that $x+\lambda y$ does not belong to $\overline{A}$, hence, by the Hahn-Banach theorem, there exists a $g \in E'$ separating the point $x+\lambda y$ from the closed convex set $\overline{A}$: $g(x+\lambda y) < \inf_{z\in\overline{A}} g(z)$. In particular, g is bounded below on $\overline{A}$, so g is in A', and $g(x+\lambda y) < g(x)$, so $\phi(g) = g(y) \neq 0$. This shows that ϕ does not vanish identically on A' and, ϕ being arbitrary, A' is not contained in any hyperplane. ∎

But if the convexness assumption is dropped, we don't know any geometric characterization of all convergence sets. There exist sets containing no straight line that are not convergence sets, for instance the subset of $\mathbb{C} = \mathbb{R}^2$ consisting of 0 and of all complex numbers with argument 0, $2\pi/3$ or $4\pi/3$; a non convergent martingale on this set is the Walsh martingale, whose modulus is that of a real Brownian motion, the argument of each excursion being chosen at random among 0, $2\pi/3$ and $4\pi/3$ (see [4], page 44). On the other hand, there are convergence sets that are not contained in any half-space.

PROPOSITION 2. — *Let $f : [0,\infty) \to (0,\infty)$ be continuous, increasing, unbounded and such that $f(\theta+2\pi) \leq c\, f(\theta)$ for a constant c and all $\theta \geq 0$. Denote by σ the spiral with equation $\rho = f(\theta)$ in polar coordinates. Every continuous planar martingale that never hits σ is convergent; in other words, the complementary $\mathbb{R}^2 - \sigma$ is a convergence set.*

Examples of such curves are the logarithmic spirals $\rho = e^{a\theta}$ and all the spirals with a sub-exponential growth, for instance the Archimedes spirals $\rho = a\theta + b$.

PROOF. Define a continuous function $\bar\theta$ on the complementary set $A = \mathbb{R}^2 - \sigma$ by $\bar\theta(z) = 0$ if the segment $[0, z]$ does not meet σ and $\bar\theta$ is the determination of the argument θ such that $2(n-1)\pi \leq \bar\theta < 2n\pi$ if $[0, z] \cap \sigma$ has $n \neq 0$ points. The inequality $\rho < f(\bar\theta + 2\pi)$ holds identically on A; if $\bar\theta > 0$ one has also $f(\bar\theta) < \rho$.

Let X be a continuous martingale with values in A; denote by $R = \rho \circ X$ its modulus and set $\Theta = \bar\theta \circ X$. Define an increasing sequence of stopping times by

$$T_0 = \inf\{t : \Theta_t > 0\} \quad ; \qquad T_{n+1} = \inf\{t : \Theta_t = \Theta_{T_n} + \pi\} \,.$$

For the probability $\mathbb{P}_n(\Gamma) = \mathbb{P}[\Gamma | T_n < \infty]$ and the filtration $\mathcal{G}^n_t = \mathcal{F}_{T_n + t}$, the stopped process $Y^n_t = X_{(T_n+t)\wedge T_{n+1}}$ is a martingale, with modulus bounded by the $\mathcal{G}^n_0$-measurable random variable $f(\Theta_{T_n} + 3\pi)$. So Y^n_∞ exists and is finite, a. s. for $\mathbb{P}_n$, and verifies $\mathbb{E}[Y^n_\infty | \mathcal{F}_{T_n}] = X_{T_n}$ on $\{T_n < \infty\}$. Identifying $\mathbb{R}^2$ with the complex field $\mathbb{C}$, this can be rewritten $\mathbb{E}[X_{T_{n+1}}/X_{T_n} | \mathcal{F}_{T_n}] = 1$. Now $\mathrm{Re}\big(X_{T_{n+1}}/X_{T_n}\big)$ is bounded above by

$$\frac{|X_{T_{n+1}}|}{|X_{T_n}|} \leq \frac{f(\Theta_{T_n} + 3\pi)}{f(\Theta_{T_n})} \leq \frac{f(\Theta_{T_n} + 4\pi)}{f(\Theta_{T_n})} \leq c^2 \,;$$

and, on the event $\{T_{n+1} < \infty\}$, $X_{T_{n+1}}/X_{T_n}$ is real and negative. So, letting $p = \mathbb{P}[T_{n+1} < \infty | \mathcal{F}_{T_n}]$, one has on $\{T_n < \infty\}$

$$1 = \mathrm{Re}\,\mathbb{E}\big[X_{T_{n+1}}/X_{T_n} \,\big|\, \mathcal{F}_{T_n}\big] \leq c^2(1-p) + 0p \,,$$

giving $p \leq 1 - c^{-2} = \varepsilon < 1$ and, by integration, $\mathbb{P}[T_{n+1} < \infty | T_n < \infty] \leq \varepsilon$. This implies $\mathbb{P}[T_n < \infty] \leq \varepsilon^n$, and T_n must be infinite for some a. s. finite value of n. Consequently, Θ is almost surely bounded; as $R \leq f(\Theta + 2\pi)$, almost every path of X is bounded, and hence convergent. ∎

If the requirement that f increases at most exponentially is dropped, the result is no longer true; when the spiral grows fast enough, its complementary set is no longer a convergence set.

PROPOSITION 3. — *Consider in the plane a spiral* γ *with equation* $\rho = f(\theta)$, *where* $f : [\theta_0, \infty) \to (0, \infty)$ *is* C^2*, strictly increasing, unbounded and such that* $f^2 + 2f'^2 - ff'' > 0$ *(locally,* γ *is between its tangent and the origin).*

Suppose that f'/f *is bounded away from zero and that for some* $\alpha < \pi$

$$\int_{\theta_0}^{\infty} \frac{f'(\theta)}{f(\theta+\alpha) - f(\theta)} \, d\theta < \infty .$$

Then the set of all points having some polar coordinates ρ *and* θ *verifying* $\theta \geq \theta_0$, $f(\theta) \leq \rho \leq f(\theta + \alpha + \pi)$ *is not a convergence set.*

An example of a function meeting these requirements is $f(\theta) = e^{a\theta^{1+\varepsilon}}$ with a and ε strictly positive.

Notice that, for $\alpha + \pi < \beta < 2\pi$, this set is included in the complementary of the spiral $\rho = f(\theta + \beta)$; so the latter cannot be a convergence set either. Remark also that this does not leave much hope of finding a purely geometric characterization of convergence sets as in the convex case, for such a characterization should be able to discriminate between a logarithmic and a faster growing spiral.

The proof will use a real-valued Brownian motion $(B_t)_{t \geq 0}$ starting from 0 and its current maximum $S_t = \sup_{s \leq t} B_s$. We start with a lemma, borrowed from [1], 2', page 92.

LEMMA 1. — *Let* $\gamma : [0, \infty) \to \mathbb{R}^n$ *be a curve of class* C^2*, or, more generally, an absolutely continuous curve with locally bounded Lebesgue derivative* $\dot{\gamma}$*. The* $\mathbb{R}^n$*-valued process*

$$Z_t = \gamma(S_t) - (S_t - B_t)\dot{\gamma}(S_t)$$

is a continuous local martingale, verifying $dZ_t = \dot{\gamma}(S_t)\, dB_t$*. In particular, if the speed* $\|\dot{\gamma}\|$ *is bounded,* Z *is a martingale.*

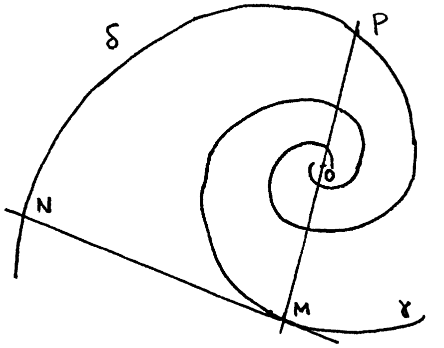
PROOF. Suppose first that γ is C^2. Since $\gamma(S_t)$ and $\dot{\gamma}(S_t)$ have finite variation, and since the increasing process $\int (S - B)\, dS$ vanishes identically,

$$Z_t = \gamma(0) + \int_0^t \dot{\gamma}(S_u)\, dS_u - \int_0^t \dot{\gamma}(S_u)\, d(S-B)_u - \int_0^t (S_u - B_u)\ddot{\gamma}(S_u)\, dS_u ,$$

yielding $Z_t = \gamma(0) + \int_0^t \dot{\gamma}(S_u)\, dB_u$. By a monotone class argument, this formula is still valid when $\dot{\gamma}$ is only locally bounded; it implies that Z is continuous. ∎

Notice that the point Z_t lives on the tangent to γ at $\gamma(S_t)$; when $B = S$, $Z = \gamma(S)$ is on the curve itself; during an "excursion" of B away from S, S and $\gamma(S)$ are constant, and Z performs a similar excursion away from $\gamma(S)$ on the tangent line to the curve, in the backward direction.

PROOF OF PROPOSITION 3. Denote by A the set of all points having some polar coordinates ρ and θ verifying $\theta \geq \theta_0$, $f(\theta) \leq \rho \leq f(\theta+\alpha)$. Parametrize γ by its arc-length s, with $s = 0$ corresponding to $\theta = \theta_0 + \pi$. Lemma 1 provides us with a martingale Z that has no limit at infinity; the proposition will be proved if we show that Z has a positive probability of never leaving the set A, since the stopped process $Z^{|T}$ (with T the hitting time of A^c) will be a continuous martingale in A with no a. s. limit at infinity. Let δ be the spiral $\rho = f(\theta + \alpha + \pi)$, obtained from γ by the rotation with angle $-(\alpha+\pi)$. Denote by M the generic point of γ, and by N the intersection of the tangent to γ at M with δ, more precisely the intersection point closest to M in the direction such that $\overrightarrow{\mathrm{MN}}$ is a negative multiple of $\dot\gamma$; if $\theta \geq \theta_0 + \pi$, the whole segment MN is included in A. Now, the distance between M and N depends on the position of M on γ, so it can be considered as a function $g(s)$ of the arc-length parameter s(M) on γ. Since, when $\gamma(S_t) = s(\mathrm{M})$, the point Z_t is on the line MN at a distance $S_t - B_t$ from M, it suffices to verify that $\mathbb{P}\big[\forall t \geq 0\,,\ S_t - B_t \leq g(S_t)\big]$ is not zero; this will prove that Z has a positive probability of never leaving A, and the proposition will be established.



LEMMA 2. — *If* $g : [0,\infty) \to [0,\infty]$ *is a Borel function,*

$$\mathbb{P}\big[\forall t \geq 0\,,\ S_t - B_t \leq g(S_t)\big] = \exp\Big(-\int_0^\infty \frac{ds}{g(s)}\Big)\,.$$

In particular, $\mathbb{P}\big[\forall t \geq 0\,,\ S_t - B_t \leq g(S_t)\big] > 0$ *if and only if* $\displaystyle\int_0^\infty \frac{ds}{g(s)}$ *is finite.*

This result is due to Knight ([3], Corollary 1.3). It implies the proposition because $g(s) = \mathrm{dist}(\mathrm{M},\mathrm{N})$ satisfies $\int_0^\infty ds/g(s) < \infty$. Indeed, the estimate $g(s) = \mathrm{MN} \geq \mathrm{ON} - \mathrm{OM} \geq \mathrm{OP} - \mathrm{OM} = f(\theta+\alpha) - f(\theta)$ yields

$$\int_0^\infty \frac{ds}{g(s)} = \int_{\theta_0+\pi}^\infty \frac{ds}{d\theta}\,\frac{d\theta}{g(s)} \leq \int_{\theta_0+\pi}^\infty \sqrt{f^2(\theta) + f'^2(\theta)}\,\frac{d\theta}{f(\theta+\alpha) - f(\theta)}$$

and this is finite owing to the ad hoc hypotheses on f. ∎

Knight states his result as the following corollary, obtained when applying Lemma 2 to the function equal to g on $[0,x]$ and identically infinite after x.

COROLLARY OF LEMMA 2. — *For $x \geq 0$, let T_x denote the stopping time $\inf\{t : S_t > x\} = \inf\{t : B_t > x\}$. If $g : [0,x] \to [0,\infty]$ is a Borel function,*

$$\mathbb{P}\left[\forall t \in [0,T_x]\,,\ S_t - B_t \leq g(S_t)\right] = \exp\left(-\int_0^x \frac{ds}{g(s)}\right).$$

He obtains it in [3] as a by-product of the explicit value of the Laplace transform of the law of the total amount of time such that $S_t - B_t \leq g(S_t)$. But if one is interested in Lemma 2 only, it is possible to reach it more shortly; here are two direct proofs of it.

FIRST PROOF OF LEMMA 2. Denote by Γ_g the event $\{\forall t \geq 0\,,\ S_t - B_t \leq g(S_t)\}$. If g_n is a decreasing sequence of functions with limit g, the events Γ_{g_n} are decreasing with intersection Γ_g, and their probabilities tend to that of Γ_g; so by approximating g from above, we may suppose $1/g$ bounded and integrable.

Define $h(x) = \exp\left[-\int_x^\infty ds/g(s)\right] > 0$ and $M_t = h(S_t)\left[1 - (S_t - B_t)/g(S_t)\right]$, so that Γ_g is the event $\{\forall t,\ M_t \geq 0\}$. The function $h'(x) = h(x)/g(x)$ is bounded and is a Lebesgue derivative of the increasing function h; by lemma 1, $M = h(S) - (S-B)h'(S)$ is a continuous martingale. As it verifies $M \leq h{\circ}S \leq 1$, it has an a. s. limit M_∞; on the random set $\{t : B_t = S_t\}$, it verifies $M = h{\circ}S$, so $M_\infty = h(\infty) = 1$. Consequently, by stopping M at the first time when it becomes strictly negative, one gets

$$\mathbb{P}[\Gamma_g] = \mathbb{P}[\forall t\,,\ M_t \geq 0] = M_0 = h(0) = \exp\left(-\int_0^\infty \frac{ds}{g(s)}\right). \qquad \blacksquare$$

SECOND PROOF OF LEMMA 2. As above, we may suppose $\phi = 1/g$ finite and integrable. According to Lévy's equivalence, if L is the local time of B at the origin, the $\mathbb{R}^2$-valued processes $(S_t - B_t, S_t)_{t\geq 0}$ and $(|B_t|, L_t)_{t\geq 0}$ have the same law; the probability $\mathbb{P}[\Gamma_g]$ we are interested in is equal to that of the event $\{\forall t \geq 0\,,\ |B_t| \leq g(L_t)\} = \{\forall t \geq 0\,,\ |\phi(L_t)B_t| \leq 1\}$.

As in lemma 1, $N_t = \phi(L_t)B_t$ is a continuous local martingale, equal to $\int_0^t \phi(L_s)\,dB_s$ (see for instance proposition 5 of [2]). It has quadratic variation $\langle N,N\rangle_t = \int_0^t \phi^2(L_s)\,ds$ and local time $\Phi(L_t)$, where $\Phi(x) = \int_0^x \phi(s)\,ds$. Now, when read in its own time-scale, N becomes Brownian, that is, there exists a Brownian motion β such that $N_t = \beta_{\langle N,N\rangle_t}$. Furthermore, if ℓ is the local time of β at 0, we get $\Phi(L_t) = \ell_{\langle N,N\rangle_t}$, whence $\langle N,N\rangle_\infty = \inf\{t : \ell_t \geq \Phi(\infty)\}$. Calling $T_{\Phi(\infty)}$ this quantity, we have

$$\mathbb{P}[\Gamma_g] = \mathbb{P}\left[\forall t \geq 0\,,\ |\phi(L_t)B_t| \leq 1\right] = \mathbb{P}\left[\sup_{t\geq 0}|N_t| \leq 1\right] = \mathbb{P}\left[\sup_{t\leq T_{\Phi(\infty)}}|\beta_t| \leq 1\right]$$

and it remains to prove that $\mathbb{P}\left[\sup_{t\leq T_x}|\beta_t| \leq 1\right] = e^{-x}$.

Letting $\tau = \inf\{t : |\beta_t| = 1\}$, this just says that $\mathbb{P}[L_\tau \geq x] = e^{-x}$; now the Markov property of β at time T_x implies that the law of L_τ is exponential, and $\mathbb{E}[L_\tau] = \mathbb{E}[|\beta|_\tau] = 1$ gives the result. $\blacksquare$

We now turn to sets that allow explosions: they carry a martingale whose distance to the origin tends almost surely to infinity. We will restrict ourselves to the two-dimensional case and deal with subsets of the plane.

PROPOSITION 4. — *Let A be a subset of $\mathbb{R}^2$. Each of the following conditions implies the next one:*

(i) *Its complementary A^c is included in an angle strictly less than π;*

(ii) *A allows explosions;*

(iii) *A^c does not contain a whole straight line.*

If furthermore A^c is convex, these three conditions are equivalent.

PROOF. (i) $\Rightarrow$ (ii). If A^c is included in an angle strictly less than π, choose affine coordinates (x, y) in the plane such A contains all points with $x \geq 0$ or $y \geq 0$. Construct a martingale $Z_t = (X_t, Y_t)$ in the following way: $Z_0 = (0,0)$; then, keeping Y frozen at 0, move X Brownianly until it reaches the value 1; keep it at this value and move Y Brownianly until it reaches the value 1 too; then freeze Y again and let X wander until it reaches 2, etc. Clearly, this yields a martingale in A escaping away to infinity. [Remark that this is an instance of Lemma 1, with a stair-like curve γ made of segments parallel to the axes; but we no longer need an estimate such as Lemma 2 since we have an infinite length avilable on the tangents.]

(ii) $\Rightarrow$ (iii). If A^c contains a line (for instance the line $x = 0$), no continuous martingale in A can tend a. s. to infinity, for its x-component is a real martingale avoiding a point, hence convergent, and its y-component is not allowed to converge a. s. to $+\infty$ or $-\infty$.

When A^c is convex, (iii) $\Rightarrow$ (i) has been seen in Proposition 1. ∎

This proposition applies, for instance, to sets obtained as the image of a half-plane by a homeomorphism of the whole plane onto itself. But even when considering only such subsets, we don't see how to characterize geometrically those that allow explosions.

REFERENCES

[1] J. Azéma et M. Yor. Une solution simple au problème de Skorokhod. *Séminaire de Probabilités XIII*, Lecture Notes in Mathematics 721, Springer 1979.

[2] J. Azéma et M. Yor. En guise d'introduction. Temps locaux, *Astérisque* 52–53, Société Mathématique de France, 1978.

[3] F. B. Knight. On the Sojourn Times of Killed Brownian Motion. *Séminaire de Probabilités XII*, Lecture Notes in Mathematics 649, Springer 1978.

[4] J. B. Walsh. A Diffusion with a Discontinuous Local Time. Temps locaux, *Astérisque* 52–53, Société Mathématique de France, 1978.