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The Azéma Martingales as Components of Quantum Independent Increment Processes

Michael Schürmann

Inspired by the work of J. Azéma [3], M. Emery and P.A. Meyer, K.R. Parthasarathy investigated the quantum stochastic differential equation

$$dX = (c - 1)Xd\Lambda + dA^\dagger + dA$$

for a real number c ; see [7]. The solution of such an equation is called an Azéma martingale. We demonstrate how an Azéma martingale can be regarded as a component of a quantum independent stationary increment process in the sense of [2].

A classical stochastic process (X_{st}) taking values in a semi-group G and indexed by pairs $(s, t) \in \mathbb{R}_+^2$, $s \leq t$, is an increment process if

$$\begin{aligned} X_{rs}X_{st} &= X_{rt}, \quad r \leq s \leq t, \\ X_{tt} &= e, \quad e \text{ the unit element of } G. \end{aligned}$$

To give sense to increments in the non-commutative case, we replace the group by a *-bialgebra. This object is defined as follows. A coalgebra \mathcal{C} is a (complex) vector space on which two linear mappings

$$\begin{aligned} \Delta : \mathcal{C} &\rightarrow \mathcal{C} \otimes \mathcal{C} && \text{(comultiplication)} \\ \delta : \mathcal{C} &\rightarrow \mathbb{C} && \text{(counit)} \end{aligned}$$

are given such that

$$\begin{aligned} (\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta && \text{(coassociativity law)} \\ (\delta \otimes \text{id}) \circ \Delta &= \text{id} = (\text{id} \otimes \delta) \circ \Delta && \text{(counit property)}. \end{aligned}$$

A *-bialgebra is a *-algebra which is also a coalgebra in such a way that Δ and δ are *-algebra homomorphisms.

The vector space $L(\mathcal{C}, \mathcal{A})$ formed by the linear mappings from a coalgebra \mathcal{C} to a (complex, unital) algebra \mathcal{A} is an algebra with the multiplication

$$R \star S = M \circ (R \otimes S) \circ \Delta$$

where $M : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ denotes multiplication in \mathcal{A} . The unit of $L(\mathcal{C}, \mathcal{A})$ is given by $b \mapsto \delta(b)\mathbf{1}$. Especially, the algebraic dual space $\mathcal{C}^* = L(\mathcal{C}, \mathbb{C})$ of a coalgebra \mathcal{C} is an algebra (with unit δ).

If the *-bialgebra \mathcal{B} has an antipode, that is a linear operator S on \mathcal{B} such that $S \star \text{id} = \text{id} \star S = \delta\mathbf{1}$ (i.e. S is the inverse of the identity with respect to \star), then we call \mathcal{B} a *-Hopf algebra.

EXAMPLES:

1) Let G be a semi-group. The semi-group algebra $\mathbb{C}G$ is a $*$ -bialgebra if we define $*$ by antilinear extension of $x^* = x^{-1}$ and Δ and δ by linear extension of $\Delta x = x \otimes x$, $\delta x = 1$, $x \in G$. If G is a group $\mathbb{C}G$ is a $*$ -Hopf algebra with $S(x) = x^{-1}$.

2) Let G be a sub-semi-group of the semi-group $M_{\mathbb{C},d}$ of complex $d \times d$ -matrices. Then we denote by $G[d]$ the $*$ -algebra of complex-valued functions on G generated by the functions ξ_{kl} , $k, l = 1, \dots, d$, which map an element $(\alpha_{mn})_{m,n=1,\dots,d}$ of G to α_{kl} . If we set

$$\Delta \xi_{kl} = \sum_{n=1}^d \xi_{kn} \otimes \xi_{nl}$$

$$\delta(\xi_{kl}) = \delta_{kl}$$

we can extend Δ and δ to $*$ -algebra homomorphisms in a unique way. $G[d]$ becomes a $*$ -bialgebra. We call $G[d]$ the coefficient algebra of G .

3) Denote by $M_{\mathbb{C}}\langle d \rangle$ the free algebra generated by indeterminates x_{kl} and x_{kl}^* , $k, l = 1, \dots, d$. The mappings $*$, Δ and δ are given by extending

$$(x_{kl})^* = x_{kl}^*$$

$$\Delta x_{kl} = \sum_{n=1}^d x_{kn} \otimes x_{nl} \quad (1)$$

$$\delta x_{kl} = \delta_{kl} \quad (2)$$

in the unique way which makes $*$ an involution and Δ and δ $*$ -algebra homomorphisms. Similarly, $M_{\mathbb{R}}\langle d \rangle$ is defined as the free algebra generated by x_{kl} , $k, l = 1, \dots, d$, with the involution given by $(x_{kl})^* = x_{kl}$ and Δ and δ again defined by (1) and (2). $M_{\mathbb{R}}\langle d \rangle$ is a quotient (i.e. a homomorphic image) of $M_{\mathbb{C}}\langle d \rangle$ (it has the additional relations $x_{kl}^* = x_{kl}$). If we make $M_{\mathbb{K}}\langle d \rangle$ commutative we obtain the coefficient algebra $M_{\mathbb{K}}[d]$ of $M_{\mathbb{K},d}$, $\mathbb{K} = \mathbb{C}$ or \mathbb{R} . Any $G[d]$ of Example 2 is a quotient of $M_{\mathbb{R}}[d]$ or at least of $M_{\mathbb{C}}[d]$.

4) Denote by $\mathbb{C}\langle x_1, \dots, x_d \rangle = \mathbb{C}\langle d \rangle$ the free algebra generated by indeterminates x_1, \dots, x_d . We extend the mappings $*$, Δ and δ with

$$(x_l)^* = x_l$$

$$\Delta x_l = x_l \otimes \mathbf{1} + \mathbf{1} \otimes x_l$$

$$\delta x_l = 0$$

to obtain a $*$ -bialgebra which is a quotient of $M_{\mathbb{R}}\langle 2d \rangle$. The $*$ -bialgebra $\mathbb{C}\langle d \rangle$ is a $*$ -Hopf algebra with antipode $S(x_{l_1} \dots x_{l_n}) = (-1)^n x_{l_n} \dots x_{l_1}$.

5) Divide $M_{\mathbb{C}}\langle d \rangle$ by the ideal J_U generated by the elements

$$\sum_{n=1}^d x_{kn} x_{ln}^* - \delta_{kl} \mathbf{1},$$

$$\sum_{n=1}^d x_{nk}^* x_{nl} - \delta_{kl} \mathbf{1}.$$

Then J_U is a $*$ -biideal. We denote the $*$ -bialgebra $M_{\mathbb{C}}\langle d \rangle / J_U$ by $U\langle d \rangle$. It can be shown that $U\langle d \rangle$ has no antipode.

6) By making $U\langle d \rangle$ commutative one obtains the coefficient algebra $U[d]$ of the group U_d of unitary $d \times d$ -matrices; see [5] where $U\langle d \rangle$ was called the non-commutative analogue of the coefficient algebra of U_d and where a structure theorem for $U\langle d \rangle$ was proved. $U[d]$ is a $*$ -Hopf algebra with the $*$ -algebra homomorphism $S(x_{kl}) = x_{lk}^*$ as the antipode.

7) Consider in $M_{\mathbb{R}}\langle 2 \rangle$ the ideal generated by the elements $x_{11} - \mathbf{1}$ and x_{21} . This is a $*$ -biideal. We denote the quotient $*$ -bialgebra by $H_0\langle 2 \rangle$. It is equal to the free algebra $\mathbb{C}\langle x, y \rangle$ generated by two indeterminates x and y with the involution $x^* = x$, $y^* = y$, and Δ and δ given by

$$\Delta x = x \otimes y + \mathbf{1} \otimes x, \quad \delta x = 0$$

$$\Delta y = y \otimes y, \quad \delta y = \mathbf{1}.$$

8) By making $H_0\langle 2 \rangle$ commutative one obtains the coefficient algebra $H_0[2]$ of the semi-group

$$H_0 = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

The set of complex-valued $*$ -algebra homomorphisms on $H_0[2]$ equipped with \star as the multiplication is isomorphic to H_0 .

9) A $*$ -Hopf algebra $H\langle 2 \rangle$ containing $H_0\langle 2 \rangle$ as a sub- $*$ -bialgebra is obtained if we divide the $*$ -bialgebra $\mathbb{C}\langle x, y, y^{-1} \rangle$ with

$$\Delta x = x \otimes y + \mathbf{1} \otimes x, \quad \delta x = 0$$

$$\Delta y = y \otimes y, \quad \delta y = \mathbf{1}$$

$$\Delta y^{-1} = y^{-1} \otimes y^{-1}, \quad \delta y^{-1} = \mathbf{1}$$

$$x^* = x, \quad y^* = y, \quad (y^{-1})^* = y^{-1}$$

by the $*$ -biideal generated by the elements $yy^{-1} - \mathbf{1}$ and $y^{-1}y - \mathbf{1}$. An antipode is given by extending $S(x) = xy^{-1}$, $S(y) = y^{-1}$, $S(y^{-1}) = y$, to a linear anti-homomorphism; see [12].

10) We can make $H\langle 2 \rangle$ commutative to obtain the $*$ -Hopf algebra H_2 . The set of complex-valued $*$ -algebra homomorphisms on H_2 is isomorphic to the group

$$H = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}, \beta \neq 0 \right\},$$

but H_2 is not equal to $H[2] = H_0[2]$.

GENERAL THEORY:

Let (j_{st}) be a quantum stochastic process in the sense of Accardi, Frigerio and Lewis [1], indexed by pairs $(s, t) \in \mathbb{R}_+^2$, $s \leq t$. The j_{st} are $*$ -algebra homomorphisms from a $*$ -algebra \mathcal{B} to a $*$ -algebra \mathcal{A} where there is also given a state Φ on \mathcal{A} . Let \mathcal{B} be a $*$ -bialgebra. We call (j_{st}) a quantum independent stationary increment process if the following conditions are fulfilled (see [2])

$$(a) \quad j_{rs} \star j_{st} = j_{rt}, \quad r \leq s \leq t; \quad j_{tt} = \delta \mathbf{1}$$

- (b1) The algebras $j_{st}(\mathcal{B})$ and $j_{s't'}(\mathcal{B})$ commute for disjoint intervals (s, t) and (s', t') .
 (b2) The state Φ factorizes on the sub-algebras $j_{t_1 t_2}(\mathcal{B}), \dots, j_{t_n t_{n+1}}(\mathcal{B})$ of \mathcal{A} for $n \in \mathbb{N}$,
 $t_1 < \dots < t_{n+1}$.
 (c) The states $\Phi \circ j_{st}$ only depend on the difference $t - s$, i.e. $\Phi \circ j_{st} = \varphi_{t-s}$.
 (d) $\lim_{t \downarrow 0} \varphi_t(b) = \delta(b)$ for all $b \in \mathcal{B}$.

Two independent stationary increment processes are called equivalent if the numbers $\Phi(j_{s_1 t_1}(b_1) \dots j_{s_n t_n}(b_n))$ are the same for both processes.

Let \mathcal{B} be a *-Hopf algebra and let $(j_t)_{t \in \mathbb{R}_+}$ be a quantum stochastic process over \mathcal{B} in the sense of Accardi, Frigerio and Lewis. Then $j_{st} = (j_s \circ S) \star j_t$ satisfies (a), and (j_t) is called a process with independent and stationary increments if (j_{st}) is an independent stationary increment process.

An independent stationary increment process (j_{st}) is, up to equivalence, determined by its (infinitesimal) generator ψ which is the linear functional on \mathcal{B} given by

$$\psi(b) = \frac{d}{dt} \varphi_t(b)|_{t=0}.$$

The set of generators coincides with the elements in \mathcal{B} satisfying

$$\begin{aligned} \psi(\mathbf{1}) &= 0 \\ \psi[\text{Kern } \delta] &\text{ is positive} \\ \psi(b^*) &= \overline{\psi(b)}. \end{aligned}$$

Given ψ satisfying these properties, one can make the following construction (see [9], cf. [8, 6]). Divide \mathcal{B} by the null space of the positive semi-definite sesquilinear form

$$(b, c) = \psi((b - \delta(b)\mathbf{1})^*(c - \delta(c)\mathbf{1}))$$

on \mathcal{B} to obtain the pre-Hilbert space D . Denote by $\eta : \mathcal{B} \rightarrow D$ the canonical mapping and define the *-representation ρ of \mathcal{B} on D by

$$\rho(b)\eta(c) = \eta(bc) - \eta(b)\delta(c).$$

We can write down the quantum stochastic integral equations

$$j_{st}(b) = \delta(b) + \int_s^t (j_{sr} \star dI_r^\psi)(b) \quad (3)$$

on the Bose Fockspace \mathcal{F} over $L^2(\mathbb{R}_+, H)$, H the completion of D , where $b \in \mathcal{B}$, $s \leq t$, and

$$I_t^\psi(b) = A_t^\dagger(\eta(b)) + \Lambda_t(\rho(b) - \delta(b)\mathbf{1}) + A_t(\eta(b^*)) + \psi(b)t.$$

In short-hand differential notation

$$dj_{st} = j_{st} \star dI_t^\psi, \quad j_{tt} = \delta\mathbf{1}.$$

The operators $j_{st}(b)$ are defined on a dense linear sub-space of \mathcal{F} which is the span of certain exponential vectors; see [4]. In a formal algebraic sense, the j_{st} constitute a

version of an independent stationary increment process with generator ψ . We believe that this statement can be made rigorous for an arbitrary *-bialgebra by showing that the linear span of

$$\{j_{s_1 t_1}(b_1) \dots j_{s_n t_n}(b_n)\Omega : n \in \mathbb{N}, (s_l, t_l) \in \mathbb{R}_+^2, s_l \leq t_l, b_1, \dots, b_n \in \mathcal{B}\}$$

is in the domain of the closure of the operator $j_{st}(b)$. Only the restriction of j_{st} to this linear subspace of the Fock space can be the independent stationary increment process in question, so that the representation (3) is an embedding theorem. For $\mathbb{C}G$, $\mathbb{C}\langle d \rangle$ and $U(d)$ a rigorous treatment of equation (3) can be found in [4, 10], [11] and [9]. For $\mathbb{C}G$, G a group, the operators $j_{st}(x)$, $x \in G$, are unitary and are representations of G of type S (cf. [6]). For $\mathbb{C}\langle d \rangle$ the operators $j_{st}(x_l)$ are sums of creation, preservation, annihilation and scalar processes [11]. For $U(d)$ the operators $(j_{st}(x_{kl}))_{k,l=1,\dots,d}$ are increments $(U_s)^\dagger U_t$ of a solution U_t of a linear quantum stochastic differential equation on $\mathbb{C}^d \otimes \mathcal{F}$ with constant coefficients [9].

APPLICATION TO $H\langle 2 \rangle$:

We concentrate on Example 7. A generator ψ on $H_0\langle 2 \rangle$ can always be constructed by the following procedure. Assume that we are given a pre-Hilbert space D , two hermitian operators ρ_x and ρ_y on D , two vectors η_x and η_y in D and two real numbers ψ_x and ψ_y . We then define the *-representation ρ of $H_0\langle 2 \rangle$ by extending $\rho(x) = \rho_x$, $\rho(y) = \rho_y$. Next we define the linear mapping $\eta : H_0\langle 2 \rangle \rightarrow D$ by the equations

$$\begin{aligned} \eta(x) &= \eta_x \\ \eta(y) &= \eta_y \\ \eta(bc) &= \rho(b)\eta(c) + \eta(b)\delta(c). \end{aligned}$$

Finally, we define $\psi \in H_0\langle 2 \rangle^*$ by

$$\begin{aligned} \psi(x) &= \psi_x \\ \psi(y) &= \psi_y \\ \psi(bc) &= \psi(b)\delta(c) + \delta(b)\psi(c) + \langle \eta(a^*), \eta(b) \rangle. \end{aligned}$$

Then ψ is a generator, and the associated equations (3) for $b = x$ and $b = y$ are

$$\begin{aligned} dX_{st} &= X_{st}(dA_t^\dagger(\eta_y) + d\Lambda_t(\rho_y - \mathbf{1}) + dA_t(\eta_y) + \psi_y dt) \\ &\quad + dA_t^\dagger(\eta_x) + d\Lambda_t(\rho_x) + dA_t(\eta_x) + \psi_x dt \\ X_{ss} &= 0, \end{aligned} \tag{4}$$

and

$$\begin{aligned} dY_{st} &= Y_{st}(dA_t^\dagger(\eta_y) + d\Lambda_t(\rho_y - \mathbf{1}) + dA_t(\eta_y) + \psi_y dt) \\ Y_{ss} &= \mathbf{1}, \end{aligned} \tag{5}$$

where we set $X_{st} = j_{st}(x)$ and $Y_{st} = j_{st}(y)$. By property (a) of an independent stationary increment process we obtain for $r \leq s \leq t$

$$X_{rt} = (j_{rs} * j_{st})(x) = X_{rs}Y_{st} + X_{st}$$

and

$$Y_{rt} = Y_{rs}Y_{st}.$$

Using this and property (b1) we have for $s \leq t$

$$\begin{aligned} X_{0s}X_{0t} &= X_{0s}(X_{0s}Y_{st} + X_{st}) \\ &= X_{0s}Y_{st}X_{0s} + X_{st}X_{0s} \\ &= X_{0t}X_{0s} \end{aligned}$$

and

$$\begin{aligned} Y_{0s}Y_{0t} &= Y_{0s}Y_{0s}Y_{st} \\ &= Y_{0s}Y_{st}Y_{0s} \\ &= Y_{0t}Y_{0s} \end{aligned}$$

showing that both $X_t = X_{0t}$ and $Y_t = Y_{0t}$ are commutative processes.

The equations for the Azéma martingales arise as the following special cases. Choose $D = \mathbb{C}$, $\rho_x = 0$, $\rho_y = c \in \mathbb{R}$, $\eta_x = 1$, $\eta_y = 0$ and $\psi_x = \psi_y = 0$. This determines a generator $\psi^{(c)}$ on $H_0\langle 2 \rangle$. Equation (4) and (5) become

$$dX_{st} = (c-1)X_{st}d\Lambda_t + dQ_t, \quad X_{ss} = 0 \quad (6)$$

(where we put $Q_t = A_t^\dagger + A_t$) and

$$dY_{st} = (c-1)Y_{st}d\Lambda_t, \quad Y_{ss} = 1. \quad (7)$$

Equation (7) is the one for the second quantization operator

$$Y_{st} = \Gamma(\chi_{[0,s]} + c\chi_{[s,t]} + \chi_{[t,\infty)}),$$

equation (6) is solved by $X_{st} = X_t - X_sY_{st}$ and X_t satisfies the Azéma martingale equation

$$dX_t = (c-1)X_t d\Lambda_t + dQ_t, \quad X_0 = 0.$$

We have

$$\begin{aligned} \psi^{(c)}(xyx) &= \overline{\eta(x)}\eta(yx) \\ &= \overline{\eta(x)}(\rho(y)\eta(x) + \eta(y)\delta(x)) \\ &= c. \end{aligned}$$

But

$$\begin{aligned} \psi^{(c)}(x^2y) &= \overline{\eta(x)}\eta(xy) \\ &= \overline{\eta(x)}(\rho(x)\eta(y) + \eta(x)\delta(y)) \\ &= 1, \end{aligned}$$

which shows that for $c \neq 1$ the process (X_{st}, Y_{st}) cannot be reduced to an independent stationary increment process over $H_0[2]$.

In the case $c \neq 0$ we can extend the generator $\psi^{(c)}$ to a generator on $H\langle 2 \rangle$ in the only possible way by setting $\rho(y^{-1}) = c^{-1}$, $\eta(y^{-1}) = 0$, and $\psi^{(c)}(y^{-1}) = 0$. Then $(X_t, Y_t, (Y_t)^{-1})$ is a process with independent stationary increments over $H\langle 2 \rangle$.

REMARK: Nothing has been said about the domains of our processes. However, for $-1 \leq c \leq 1$ the Y_{st} are bounded and for $-1 \leq c < 1$ this is also true for X_{st} (see [7]). For $c = 1$ we have $X_{st} = Q_{st}$ and this is actually the case of Brownian motion and the *-bialgebra $C(1)$. Also from [7] we know that for $-1 \leq c \leq 1$ the process X_t has the chaos completeness property which means that the embedding of j_{st} into (X_{st}, Y_{st}) is an isomorphism.

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