Thomas J. Ransford

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PREDICTABLE SETS AND SET-VALUED PROCESSES

by T.J.Ransford

Introduction.

Throughout this article, we suppose that we are given a complete probability space (Ω, Σ, P), together with a filtration $(\mathcal{F}_0, \{\mathcal{F}_t\}_{0\leq t\leq\infty})$ satisfying the usual conditions of right continuity, completeness, and left continuity at ∞. Denote by $\mathcal{P}$ the *predictable σ-field*, namely the σ-field on $[0,\infty] \times \Omega$ generated by all sets of the form

$$\{0\} \times A \quad (A \in \mathcal{F}_{0-}) \qquad \text{and} \qquad (t,\infty] \times B \quad (B \in \mathcal{F}_t, t \geq 0)$$

together with the evanescent sets (which are always to be treated as negligible). Our purpose is to establish analogues of the classical 'analytic implies measurable' and projection theorems for $\mathcal{P}$, even though $\mathcal{P}$ is *not* complete relative to any probability measure. The last section explores some connections with set-valued processes.

We follow the notation of [**3**] throughout, except for the minor change that out time interval is $[0,\infty]$ rather than $[0,\infty)$ (however, see [**3**, IV.61(b)]). Finally, we remark that, with obvious modifications to the proofs, all the results below remain valid if $\mathcal{P}$ is replaced throughout by $\mathcal{O}$, the optional σ-field.

1. A Measurability Theorem.

Given a measurable space $(E, \mathcal{E})$, denote by $\mathcal{A}(\mathcal{E})$ the class of $\mathcal{E}$-analytic sets (see [**3**, III.7]). Then $\mathcal{E} \subset \mathcal{A}(\mathcal{E})$, with equality if $(E, \mathcal{E})$ is complete relative to some probability measure ([**3**, III.33(a)]), though not however in general. In particular, it is *never* true that $\mathcal{A}(\mathcal{P}) = \mathcal{P}$: for if Z is any analytic subset of $[0,\infty]$ which is not Borel, then $Z \times \Omega \in \mathcal{A}(\mathcal{P}) \setminus \mathcal{P}$. Instead, writing $\mathcal{B}$ for the Borel sets, we have the following theorem.

Theorem 1. *Let $H \subset [0,\infty] \times \Omega$. Then $H \in \mathcal{P}$ if and only if $H \in \mathcal{A}(\mathcal{P})$ and $H \in \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty$.*

Proof. The 'only if' is clear. For the 'if', suppose that $H \in \mathcal{A}(\mathcal{P}) \cap (\mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty)$. Then the set $H \cap (\{\infty\} \times \Omega)$ belongs to $\{\infty\} \times \mathcal{F}_\infty$, and hence to $\mathcal{P}$, so subtracting it off we may assume that $H \subset [0,\infty) \times \Omega$. Let $X = {}^p(1_H)$, the predictable projection of 1_H (see [**3**, VI.43]). As ${}^p(.)$ is order-preserving we certainly have $0 \leq X \leq 1$, and proving that $H \in \mathcal{P}$ is equivalent to showing that $X = 1_H$, which we now proceed to do.

First we show that $X \geq 1_H$. Suppose, if possible, that this is false. Then there exists $\delta > 0$ such that $H \cap \{X \leq 1-\delta\}$ is not evanescent. As this set belongs to $\mathcal{A}(\mathcal{P})$, the (proof of) the predictable section theorem ([**3**, IV.85]) shows that there exists a predictable time T, with $P(T < \infty) > 0$, such that

$$[\![T]\!] \quad \subset \quad (H \cap \{X \leq 1-\delta\}) \cup [\![\infty]\!].$$

By the defining property of predictable projections we have

$$E[1_H(T)1_{(T<\infty)}|\mathcal{F}_{T-}] = X(T)1_{(T<\infty)} \qquad \text{a.s.}$$

Therefore

$$P(T<\infty) = E[1_H(T)1_{(T<\infty)}] = E[X(T)1_{(T<\infty)}] \leq (1-\delta).P(T<\infty),$$

which gives the desired contradiction.

Now we show that $X \leq 1_H$. Again, suppose, if possible, that this is false. Then there exists $\delta > 0$ such that $\{X \geq \delta\} \setminus H$ is not evanescent. As this set belongs to $\mathcal{B}[0,\infty) \otimes \mathcal{F}_\infty$, the (ordinary) section theorem ([**3**, III.44]) shows that there exists a random time T, with $P(T < \infty) > 0$, such that

$$[\![T]\!] \quad \subset \quad (\{X \geq \delta\} \setminus H) \cup [\![\infty]\!].$$

Define a measure μ on $\mathcal{P}$ by

$$\mu(Q) = E[1_Q(T)1_{(T<\infty)}] \qquad (Q \in \mathcal{P}),$$

and then a $\mathcal{P}$-outer measure μ^* on $[0,\infty) \times \Omega$ by

$$\mu^*(R) = \inf\{\mu(Q) : Q \in \mathcal{P}, Q \supset R\} \qquad (R \subset [0,\infty) \times \Omega). \tag{1}$$

A standard argument shows that μ^* is a $\mathcal{P}$-capacity (see [**3**, III.32]). As $H \in \mathcal{A}(\mathcal{P})$, it follows by Choquet's theorem ([**3**, III.28]) that

$$\mu^*(H) = \sup\{\mu(Q) : Q \in \mathcal{P}, Q \subset H\}. \tag{2}$$

Now on the one hand, if $Q \in \mathcal{P}$ and $Q \supset H$, then $1_Q = {}^p(1_Q) \geq {}^p(1_H) = X$, so

$$\mu(Q) \geq E[X(T)1_{(T<\infty)}] \geq \delta.P(T<\infty),$$

and hence by (1),

$$\mu^*(H) \geq \delta.P(T<\infty) > 0.$$

On the other hand, if $Q \in \mathcal{P}$ and $Q \subset H$, then $1_Q \leq 1_H$, so

$$\mu(Q) \leq E[1_H(T)1_{(T<\infty)}] = 0,$$

and hence by (2),

$$\mu^*(H) = 0.$$

This gives the desired contradiction, and completes the proof. □

Remark. The proof of Theorem 1 was influenced by [**2**] and by [**3**, IV.76(c)].

2. A Projection Theorem.

To exploit Theorem 1 we use a little topology. Throughout this section, let C be a compact metrizable space. Denote by $\mathcal{P}(C)$ the collection of all subsets J of $C \times [0,\infty] \times \Omega$ such that
(i) J belongs to $\mathcal{B}(C) \otimes \mathcal{P}$, and
(ii) J_ω is compact almost surely, where

$$J_\omega = \{(x,t) \in C \times [0,\infty] : (x,t,\omega) \in J\} \qquad (\omega \in \Omega).$$

The class $\mathcal{P}(C)$ is stable under finite unions and countable intersections.

Theorem 2. *Let $J \in \mathcal{P}(C)$. If $\pi : C \times [0,\infty] \times \Omega \to [0,\infty] \times \Omega$ denotes the canonical projection map, then $\pi(J) \in \mathcal{P}$.*

Proof. As $J \in \mathcal{B}(C) \otimes \mathcal{P}$, it follows by [**3**, III.13] that $\pi(J) \in \mathcal{A}(\mathcal{P})$. We claim that also $\pi(J) \in \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty$. If so, then applying Theorem 1 yields the desired conclusion that $\pi(J) \in \mathcal{P}$.

To prove the claim, put $H = \pi(J)$. Given $B \in \mathcal{B}[0,\infty]$, set

$$\Omega_B = \pi'((B \times \Omega) \cap H),$$

where $\pi' : [0,\infty] \times \Omega \to \Omega$ is the canonical projection. Then

$$\Omega_B = \pi'\pi((C \times B \times \Omega) \cap J),$$

so since $(C \times B \times \Omega) \cap J \in \mathcal{B}(C \times [0,\infty]) \otimes \mathcal{F}_\infty$, it follows by [**3**, III.13] again that $\Omega_B \in \mathcal{A}(\mathcal{F}_\infty)$. As $\mathcal{F}_\infty$ is P-complete, we therefore have $\Omega_B \in \mathcal{F}_\infty$. In particular, taking $B_{k,n} = [k/n, (k+1)/n]$, we deduce that each of the sets

$$H_n = \bigcup_{k \geq 0} (B_{k,n} \times \Omega_{B_{k,n}}) \cup (\{\infty\} \times \Omega_{\{\infty\}}) \qquad (n \geq 1)$$

belongs to $\mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty$. Also, since almost every ω-section of J is compact, the same is true of H, and this easily implies that $\bigcap_{n \geq 1} H_n = H$. Hence $H \in \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty$, justifying the claim. □

We now give an application to the predictability of an uncountable supremum of processes. Note that by this is meant the *actual* supremum, not just an essential supremum in the sense of [**1**] for example.

Corollary. *Let $\Psi : C \times [0,\infty] \times \Omega \to [-\infty,\infty]$ be a map such that*
(i) Ψ is $\mathcal{B}(C) \otimes \mathcal{P}$-measurable, and
(ii) the map $(x,t) \mapsto \Psi(x,t,\omega)$ is upper semicontinuous almost surely.
Then the process $\Phi : [0,\infty] \times \Omega \to [-\infty,\infty]$ is predictable, where

$$\Phi(t,\omega) = \sup_{x \in C} \Psi(x,t,\omega) \qquad ((t,\omega) \in [0,\infty] \times \Omega).$$

Proof. By upper semicontinuity, the supremum in the definition of Φ is always attained. Hence, given $\alpha \in \mathbf{R}$, we have

$$\{\Phi \geq \alpha\} = \pi(\{\Psi \geq \alpha\}),$$

where π is as in Theorem 2. The hypotheses on Ψ guarantee that $\{\Psi \geq \alpha\} \in \mathcal{P}(C)$, so by Theorem 2 it follows that $\{\Phi \geq \alpha\} \in \mathcal{P}$. Thus Φ is predictable. □

3. Set-Valued Processes.

One way to extend the last corollary is to allow the supremum to be taken over a set which itself varies, namely, a set-valued stochastic process. Set-valued processes arise in a number of contexts, and in [4] at least, such suprema play a fundamental rôle.

As before, let C be a compact metrizable space, and now denote by $\mathcal{K}(C)$ the collection of all compact subsets of C. A set-valued map $K : [0,\infty] \times \Omega \to \mathcal{K}(C)$ is:
(i) *predictable* if for every $F \in \mathcal{K}(C)$

$$\{(t,\omega) \in [0,\infty] \times \Omega : K(t,\omega) \cap F \neq \varnothing\} \in \mathcal{P};$$

(ii) *upper semicontinuous* if, for almost all ω, for every $F \in \mathcal{K}(C)$

$$\{t \in [0,\infty] : K(t,\omega) \cap F \neq \varnothing\} \in \mathcal{K}[0,\infty].$$

These two properties can be characterized very simply in terms of the graph of K.

Theorem 3. *A map $K : [0,\infty] \times \Omega \to \mathcal{K}(C)$ is predictable and upper semicontinuous if and only if $\Gamma(K) \in \mathcal{P}(C)$, where*

$$\Gamma(K) = \{(x,t,\omega) \in C \times [0,\infty] \times \Omega : x \in K(t,\omega)\}.$$

Proof. First suppose that $\Gamma(K) \in \mathcal{P}(C)$. Then given $F \in \mathcal{K}(C)$, we have

$$\{(t,\omega) : K(t,\omega) \cap F \neq \varnothing\} \quad = \quad \pi(\Gamma(K) \cap (F \times [0,\infty] \times \Omega)),$$

so by Theorem 2 it follows that K is predictable. As almost every ω-section of $\Gamma(K)$ is compact, it is plain that K is upper semicontinuous.

Conversely, suppose that K is predictable and upper semicontinuous. In particular it then follows that for each $F \in \mathcal{K}(C)$ we have $J(F) \in \mathcal{P}(C)$, where

$$J(F) \quad = \quad ((C \setminus \mathrm{int}(F)) \times [0,\infty] \times \Omega) \quad \cup \quad (C \times \{(t,\omega) : K(t,\omega) \cap F \neq \varnothing\}).$$

Now as C is compact metrizable, we may choose a sequence (F_n) in $\mathcal{K}(C)$ with the following property: given $C' \in \mathcal{K}(C)$ and $x \in C \setminus C'$, there exists n such that $x \in \mathrm{int}(F_n)$ and $C' \cap F_n = \varnothing$. With this sequence it is then elementary to check that $\Gamma(K) = \cap_{n\geq 1} J(F_n)$. Hence $\Gamma(K) \in \mathcal{P}(C)$. □

Finally we can read off the result that was hinted at earlier.

Corollary. *Let $\Psi : C \times [0,\infty] \times \Omega \to [-\infty,\infty]$ be a map satisfying the same conditions as in the Corollary to Theorem 2. Let $K : [0,\infty]\times\Omega \to \mathcal{K}(C)$ be a predictable, upper semicontinuous process. Then $\Phi : [0,\infty] \times \Omega \to [-\infty,\infty]$ is a predictable process, where*

$$\Phi(t,\omega) = \sup_{x\in K(t,\omega)} \Psi(x,t,\omega) \qquad ((t,\omega) \in [0,\infty] \times \Omega).$$

Proof. This time, given $\alpha \in \mathbf{R}$, we have

$$\{\Phi \geq \alpha\} \quad = \quad \pi(\{\Psi \geq \alpha\} \cap \Gamma(K)).$$

Using Theorem 3, $(\{\Psi \geq \alpha\} \cap \Gamma(K)) \in \mathcal{P}(C)$, so as before $\{\Phi \geq \alpha\} \in \mathcal{P}$ and Φ is predictable. □

References.

[1] C.Dellacherie, 'Sur l'existence de certains ess. inf et ess. sup de familles de processus mesurables', *Séminaire de Probabilités XII*, pp.512–514, Lecture Notes in Mathematics 649, Springer, Berlin, 1978.

[2] C.Dellacherie, 'Supports optionels et prévisibles d'une P-mesure et applications', *Séminaire de Probabilités XII*, pp.515-522, Lecture Notes in Mathematics 649, Springer, Berlin, 1978.

[3] C.Dellacherie and P.A.Meyer, *Probabilities and Potential*, volumes A and B, North-Holland Mathematical Studies Nos. 29 and 72, Amsterdam, 1978 and 1982.

[4] T.J.Ransford, 'Holomorphic, subharmonic and subholomorphic processes', *to appear.*

Department of Pure Mathematics and Mathematical Statistics,
16 Mill Lane,
Cambridge CB2 1SB,
United Kingdom.