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# PREDICTABLE SETS AND SET-VALUED PROCESSES

by T.J.Ransford

## Introduction.

Throughout this article, we suppose that we are given a complete probability space  $(\Omega, \Sigma, P)$ , together with a filtration  $(\mathcal{F}_0, \{\mathcal{F}_t\}_{0 \leq t \leq \infty})$  satisfying the usual conditions of right continuity, completeness, and left continuity at  $\infty$ . Denote by  $\mathcal{P}$  the *predictable*  $\sigma$ -field, namely the  $\sigma$ -field on  $[0, \infty] \times \Omega$  generated by all sets of the form

$$\{0\} \times A \quad (A \in \mathcal{F}_{0-}) \quad \text{and} \quad (t, \infty] \times B \quad (B \in \mathcal{F}_t, t \geq 0)$$

together with the evanescent sets (which are always to be treated as negligible). Our purpose is to establish analogues of the classical ‘analytic implies measurable’ and projection theorems for  $\mathcal{P}$ , even though  $\mathcal{P}$  is *not* complete relative to any probability measure. The last section explores some connections with set-valued processes.

We follow the notation of [3] throughout, except for the minor change that our time interval is  $[0, \infty]$  rather than  $[0, \infty)$  (however, see [3, IV.61(b)]). Finally, we remark that, with obvious modifications to the proofs, all the results below remain valid if  $\mathcal{P}$  is replaced throughout by  $\mathcal{O}$ , the optional  $\sigma$ -field.

## 1. A Measurability Theorem.

Given a measurable space  $(E, \mathcal{E})$ , denote by  $\mathcal{A}(\mathcal{E})$  the class of  $\mathcal{E}$ -analytic sets (see [3, III.7]). Then  $\mathcal{E} \subset \mathcal{A}(\mathcal{E})$ , with equality if  $(E, \mathcal{E})$  is complete relative to some probability measure ([3, III.33(a)]), though not however in general. In particular, it is *never* true that  $\mathcal{A}(\mathcal{P}) = \mathcal{P}$ : for if  $Z$  is any analytic subset of  $[0, \infty]$  which is not Borel, then  $Z \times \Omega \in \mathcal{A}(\mathcal{P}) \setminus \mathcal{P}$ . Instead, writing  $\mathcal{B}$  for the Borel sets, we have the following theorem.

**Theorem 1.** *Let  $H \subset [0, \infty] \times \Omega$ . Then  $H \in \mathcal{P}$  if and only if  $H \in \mathcal{A}(\mathcal{P})$  and  $H \in \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ .*

*Proof.* The ‘only if’ is clear. For the ‘if’, suppose that  $H \in \mathcal{A}(\mathcal{P}) \cap (\mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty)$ . Then the set  $H \cap (\{\infty\} \times \Omega)$  belongs to  $\{\infty\} \times \mathcal{F}_\infty$ , and hence to  $\mathcal{P}$ , so subtracting it off we may assume that  $H \subset [0, \infty) \times \Omega$ . Let  $X = {}^p(1_H)$ , the predictable projection of  $1_H$  (see [3, VI.43]). As  ${}^p(\cdot)$  is order-preserving we certainly have  $0 \leq X \leq 1$ , and proving that  $H \in \mathcal{P}$  is equivalent to showing that  $X = 1_H$ , which we now proceed to do.

First we show that  $X \geq 1_H$ . Suppose, if possible, that this is false. Then there exists  $\delta > 0$  such that  $H \cap \{X \leq 1 - \delta\}$  is not evanescent. As this set belongs to  $\mathcal{A}(\mathcal{P})$ , the (proof of) the predictable section theorem ([3, IV.85]) shows that there exists a predictable time  $T$ , with  $P(T < \infty) > 0$ , such that

$$[T] \subset (H \cap \{X \leq 1 - \delta\}) \cup [\infty].$$

By the defining property of predictable projections we have

$$E[1_H(T)1_{(T<\infty)}|\mathcal{F}_{T-}] = X(T)1_{(T<\infty)} \quad \text{a.s.}$$

Therefore

$$P(T < \infty) = E[1_H(T)1_{(T<\infty)}] = E[X(T)1_{(T<\infty)}] \leq (1 - \delta) \cdot P(T < \infty),$$

which gives the desired contradiction.

Now we show that  $X \leq 1_H$ . Again, suppose, if possible, that this is false. Then there exists  $\delta > 0$  such that  $\{X \geq \delta\} \setminus H$  is not evanescent. As this set belongs to  $\mathcal{B}[0, \infty) \otimes \mathcal{F}_\infty$ , the (ordinary) section theorem ([3, III.44]) shows that there exists a random time  $T$ , with  $P(T < \infty) > 0$ , such that

$$[T] \subset (\{X \geq \delta\} \setminus H) \cup [\infty].$$

Define a measure  $\mu$  on  $\mathcal{P}$  by

$$\mu(Q) = E[1_Q(T)1_{(T<\infty)}] \quad (Q \in \mathcal{P}),$$

and then a  $\mathcal{P}$ -outer measure  $\mu^*$  on  $[0, \infty) \times \Omega$  by

$$(1) \quad \mu^*(R) = \inf\{\mu(Q) : Q \in \mathcal{P}, Q \supset R\} \quad (R \subset [0, \infty) \times \Omega).$$

A standard argument shows that  $\mu^*$  is a  $\mathcal{P}$ -capacity (see [3, III.32]). As  $H \in \mathcal{A}(\mathcal{P})$ , it follows by Choquet's theorem ([3, III.28]) that

$$(2) \quad \mu^*(H) = \sup\{\mu(Q) : Q \in \mathcal{P}, Q \subset H\}.$$

Now on the one hand, if  $Q \in \mathcal{P}$  and  $Q \supset H$ , then  $1_Q = {}^p(1_Q) \geq {}^p(1_H) = X$ , so

$$\mu(Q) \geq E[X(T)1_{(T<\infty)}] \geq \delta \cdot P(T < \infty),$$

and hence by (1),

$$\mu^*(H) \geq \delta \cdot P(T < \infty) > 0.$$

On the other hand, if  $Q \in \mathcal{P}$  and  $Q \subset H$ , then  $1_Q \leq 1_H$ , so

$$\mu(Q) \leq E[1_H(T)1_{(T<\infty)}] = 0,$$

and hence by (2),

$$\mu^*(H) = 0.$$

This gives the desired contradiction, and completes the proof.  $\square$

**Remark.** The proof of Theorem 1 was influenced by [2] and by [3, IV.76(c)].

## 2. A Projection Theorem.

To exploit Theorem 1 we use a little topology. Throughout this section, let  $C$  be a compact metrizable space. Denote by  $\mathcal{P}(C)$  the collection of all subsets  $J$  of  $C \times [0, \infty] \times \Omega$  such that

- (i)  $J$  belongs to  $\mathcal{B}(C) \otimes \mathcal{P}$ , and
- (ii)  $J_\omega$  is compact almost surely, where

$$J_\omega = \{(x, t) \in C \times [0, \infty] : (x, t, \omega) \in J\} \quad (\omega \in \Omega).$$

The class  $\mathcal{P}(C)$  is stable under finite unions and countable intersections.

**Theorem 2.** *Let  $J \in \mathcal{P}(C)$ . If  $\pi : C \times [0, \infty] \times \Omega \rightarrow [0, \infty] \times \Omega$  denotes the canonical projection map, then  $\pi(J) \in \mathcal{P}$ .*

*Proof.* As  $J \in \mathcal{B}(C) \otimes \mathcal{P}$ , it follows by [3, III.13] that  $\pi(J) \in \mathcal{A}(\mathcal{P})$ . We claim that also  $\pi(J) \in \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ . If so, then applying Theorem 1 yields the desired conclusion that  $\pi(J) \in \mathcal{P}$ .

To prove the claim, put  $H = \pi(J)$ . Given  $B \in \mathcal{B}[0, \infty]$ , set

$$\Omega_B = \pi'((B \times \Omega) \cap H),$$

where  $\pi' : [0, \infty] \times \Omega \rightarrow \Omega$  is the canonical projection. Then

$$\Omega_B = \pi' \pi((C \times B \times \Omega) \cap J),$$

so since  $(C \times B \times \Omega) \cap J \in \mathcal{B}(C \times [0, \infty]) \otimes \mathcal{F}_\infty$ , it follows by [3, III.13] again that  $\Omega_B \in \mathcal{A}(\mathcal{F}_\infty)$ . As  $\mathcal{F}_\infty$  is  $P$ -complete, we therefore have  $\Omega_B \in \mathcal{F}_\infty$ . In particular, taking  $B_{k,n} = [k/n, (k+1)/n]$ , we deduce that each of the sets

$$H_n = \bigcup_{k \geq 0} (B_{k,n} \times \Omega_{B_{k,n}}) \cup (\{\infty\} \times \Omega_{\{\infty\}}) \quad (n \geq 1)$$

belongs to  $\mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ . Also, since almost every  $\omega$ -section of  $J$  is compact, the same is true of  $H$ , and this easily implies that  $\bigcap_{n \geq 1} H_n = H$ . Hence  $H \in \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ , justifying the claim.  $\square$

We now give an application to the predictability of an uncountable supremum of processes. Note that by this is meant the *actual* supremum, not just an essential supremum in the sense of [1] for example.

**Corollary.** *Let  $\Psi : C \times [0, \infty] \times \Omega \rightarrow [-\infty, \infty]$  be a map such that*

- (i)  $\Psi$  is  $\mathcal{B}(C) \otimes \mathcal{P}$ -measurable, and
- (ii) the map  $(x, t) \mapsto \Psi(x, t, \omega)$  is upper semicontinuous almost surely.

*Then the process  $\Phi : [0, \infty] \times \Omega \rightarrow [-\infty, \infty]$  is predictable, where*

$$\Phi(t, \omega) = \sup_{x \in C} \Psi(x, t, \omega) \quad ((t, \omega) \in [0, \infty] \times \Omega).$$

*Proof.* By upper semicontinuity, the supremum in the definition of  $\Phi$  is always attained. Hence, given  $\alpha \in \mathbb{R}$ , we have

$$\{\Phi \geq \alpha\} = \pi(\{\Psi \geq \alpha\}),$$

where  $\pi$  is as in Theorem 2. The hypotheses on  $\Psi$  guarantee that  $\{\Psi \geq \alpha\} \in \mathcal{P}(C)$ , so by Theorem 2 it follows that  $\{\Phi \geq \alpha\} \in \mathcal{P}$ . Thus  $\Phi$  is predictable.  $\square$

### 3. Set-Valued Processes.

One way to extend the last corollary is to allow the supremum to be taken over a set which itself varies, namely, a set-valued stochastic process. Set-valued processes arise in a number of contexts, and in [4] at least, such suprema play a fundamental rôle.

As before, let  $C$  be a compact metrizable space, and now denote by  $\mathcal{K}(C)$  the collection of all compact subsets of  $C$ . A set-valued map  $K : [0, \infty] \times \Omega \rightarrow \mathcal{K}(C)$  is:

(i) *predictable* if for every  $F \in \mathcal{K}(C)$

$$\{(t, \omega) \in [0, \infty] \times \Omega : K(t, \omega) \cap F \neq \emptyset\} \in \mathcal{P};$$

(ii) *upper semicontinuous* if, for almost all  $\omega$ , for every  $F \in \mathcal{K}(C)$

$$\{t \in [0, \infty] : K(t, \omega) \cap F \neq \emptyset\} \in \mathcal{K}[0, \infty].$$

These two properties can be characterized very simply in terms of the graph of  $K$ .

**Theorem 3.** *A map  $K : [0, \infty] \times \Omega \rightarrow \mathcal{K}(C)$  is predictable and upper semicontinuous if and only if  $\Gamma(K) \in \mathcal{P}(C)$ , where*

$$\Gamma(K) = \{(x, t, \omega) \in C \times [0, \infty] \times \Omega : x \in K(t, \omega)\}.$$

*Proof.* First suppose that  $\Gamma(K) \in \mathcal{P}(C)$ . Then given  $F \in \mathcal{K}(C)$ , we have

$$\{(t, \omega) : K(t, \omega) \cap F \neq \emptyset\} = \pi(\Gamma(K) \cap (F \times [0, \infty] \times \Omega)),$$

so by Theorem 2 it follows that  $K$  is predictable. As almost every  $\omega$ -section of  $\Gamma(K)$  is compact, it is plain that  $K$  is upper semicontinuous.

Conversely, suppose that  $K$  is predictable and upper semicontinuous. In particular it then follows that for each  $F \in \mathcal{K}(C)$  we have  $J(F) \in \mathcal{P}(C)$ , where

$$J(F) = ((C \setminus \text{int}(F)) \times [0, \infty] \times \Omega) \cup (C \times \{(t, \omega) : K(t, \omega) \cap F \neq \emptyset\}).$$

Now as  $C$  is compact metrizable, we may choose a sequence  $(F_n)$  in  $\mathcal{K}(C)$  with the following property: given  $C' \in \mathcal{K}(C)$  and  $x \in C \setminus C'$ , there exists  $n$  such that  $x \in \text{int}(F_n)$  and  $C' \cap F_n = \emptyset$ . With this sequence it is then elementary to check that  $\Gamma(K) = \bigcap_{n \geq 1} J(F_n)$ . Hence  $\Gamma(K) \in \mathcal{P}(C)$ .  $\square$

Finally we can read off the result that was hinted at earlier.

**Corollary.** *Let  $\Psi : C \times [0, \infty] \times \Omega \rightarrow [-\infty, \infty]$  be a map satisfying the same conditions as in the Corollary to Theorem 2. Let  $K : [0, \infty] \times \Omega \rightarrow \mathcal{K}(C)$  be a predictable, upper semicontinuous process. Then  $\Phi : [0, \infty] \times \Omega \rightarrow [-\infty, \infty]$  is a predictable process, where*

$$\Phi(t, \omega) = \sup_{x \in K(t, \omega)} \Psi(x, t, \omega) \quad ((t, \omega) \in [0, \infty] \times \Omega).$$

*Proof.* This time, given  $\alpha \in \mathbf{R}$ , we have

$$\{\Phi \geq \alpha\} = \pi(\{\Psi \geq \alpha\} \cap \Gamma(K)).$$

Using Theorem 3,  $(\{\Psi \geq \alpha\} \cap \Gamma(K)) \in \mathcal{P}(C)$ , so as before  $\{\Phi \geq \alpha\} \in \mathcal{P}$  and  $\Phi$  is predictable.  $\square$

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