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MARKOV CHAINS AS EVANS-HUDSON DIFFUSIONS IN FOCK SPACE

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1. Introduction.

M.P. Evans and R.L. Hudson have recently formulated and developed an algebraic theory of quantum diffusion processes in a series of papers [1–4]. In his two recent notes [6,7] P.A. Meyer has pointed out how a classical finite Markov chain in continuous time can be viewed upon an an Evans–Hudson diffusion, and also exploited to develop chaos expansions or, more specifically, Isobe–Sato expansions in terms of multiple stochastic integrals with respect to a fixed finite family of martingales determined by the Markov chain. The present note is motivated by some of Meyer's observations on Markov chains. It is shown that whenever the structure maps of Evans–Hudson are defined on a commutative *–algebra of operators the whole diffusion is commutative or, equivalently, is a classical stochastic process eventhough the driving quantum noise is noncommutative. This striking fact enables us to construct a whole class of Markov processes as Evans–Hudson diffusions by using general group actions. Such processes are realized by conjugations with respect to unitary operator valued adapted processes satisfying a quantum stochastic differential equation in the sense of Hudson–Parthasarathy [5]. In the special case of a cyclic group acting on itself by translation our construction reduces to that of Meyer. An ergodic theorem is proved for the homomorphisms that describe the Evans–Hudson diffusion in some special cases.

2. Abelian diffusions in the sense of Evans–Hudson.

All the Hilbert spaces that we deal with are assumed to be complex and separable with scalar product $<,>$ linear in the second variable. For any Hilbert space $\mathcal{H}$ we denote by $\Gamma(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ respectively the boson Fock space over $\mathcal{H}$ and the *–algebra of all bounded operators in $\mathcal{H}$. For any $u \in \mathcal{H}$ we denote by $e(u)$ the exponential or coherent vector in $\Gamma(\mathcal{H})$ associated with u. Let

$$\mathcal{H} = L_2(\mathbb{R}_+) \otimes \mathbb{C}^n \quad ; \quad \widetilde{\mathcal{H}} = \mathcal{H}_0 \otimes \Gamma(\mathcal{H}) \tag{2.1}$$

where $\mathcal{H}_0$ is a fixed Hilbert space. Denote by $\mathcal{E}$ the set of all vectors of the form $f \otimes e(u)$, $f \in \mathcal{H}_0$, $u \in \mathcal{H}$. We adopt the convention of writing $f \otimes e(u)$ as $fe(u)$. It is important to note that $\mathcal{E}$ is total in $\widetilde{\mathcal{H}}$.

Suppose that $A \subset \mathcal{B}(H_0)$ is a unital *–subalgebra of $\mathcal{B}(\mathcal{H}_0)$. Consider a family of bounded linear "structure maps" $\{\tau^i_k, 0 \le i, k \le n\}$ mapping $\mathcal{A}$ into $\mathcal{A}$ and satisfying the Evans–Hudson structure equations

$$\tau^i_j(I) = 0\ ,\ \left(\tau^i_j(X)\right)^* = \tau^j_i(X^*)$$
$$\tau^i_j(XY) = \tau^i_j(X)Y + X\tau^i_j(Y) + \sum_{k=1}^{n} \tau^i_k(X)\tau^k_j(Y) \tag{2.2}$$

and the inequality

$$\| \tau^i_j(X) \| \leq M \|X\| \quad \text{for all} \quad X \in \mathcal{A},\ 0 \leq i,j \leq n \tag{2.3}$$

M being a positive constant. Let $\Lambda^i_j(t)$, $1 \leq i,j \leq n$ be the conservation or gauge processes in $\widetilde{\mathcal{H}}$ with reference to the standard orthonormal basis e_i, $1 \leq i \leq n$ in $\mathbb{C}^n$. Denote by $\Lambda^i_0(t) = A_i(t)$, $\Lambda^0_i(t) = A^+_i(t)$, $1 \leq i \leq n$ the annihilation and creation processes with respect to the same basis. Write $\Lambda^0_0(t) = t$ where t denotes tI, the identity in $\widetilde{\mathcal{H}}$. By the quantum Ito formula we have

$$d\Lambda^i_j d\Lambda^k_\ell = \widehat{\delta}^i_\ell d\Lambda^k_j\ ,\ 0 \leq i,j,k,\ell \leq n \tag{2.4}$$

$$\widehat{\delta}^i_\ell = 0 \quad \text{if } i=0 \text{ or } \ell=0,\ \delta^i_\ell \text{ otherwise} \tag{2.5}$$

Define

$$T_t(X) = e^{t\tau^0_0}(X) \quad \text{for all } t \geq 0,\ X \in \mathcal{A}. \tag{2.6}$$

THEOREM 2.1. *There exists a unique adapted family* $\{ j_t,\ t \geq 0 \}$ *of identity preserving contractive* **-homomorphisms from* $\mathcal{A}$ *into* $\mathcal{B}(\widetilde{\mathcal{H}})$ *satisfying the quantum stochastic differential equations*

$$j_0(X) = X\ ,\ dj_t(X) = \sum_{0 \leq i,k \leq n} j_t(\tau^i_k(X))\, d\Lambda^k_i(t) \tag{2.7}$$

for all $X \in \mathcal{A}$. *Furthermore, the map* $t \longmapsto j_t(X)$ *is strongly continuous for each* X *and*

$$< f e(0), j_{t_1}(X_1) j_{t_2}(X_2) \ldots j_{t_k}(X_k) g e(0) > = \\ < f\ ,\ T_{t_1}(X_1 T_{t_2 - t_1}(X_2 \ldots T_{t_k - t_{k-1}}(X_k)) \ldots) g > \tag{2.8}$$

for $0 \leq t_1 < t_2 \ldots < t_k < \infty$, $X_j \in \mathcal{A}$ *and* $f,g \in \mathcal{H}_0$.

PROOF. The first part is proved in [1,2]. For fixed $f,g \in \mathcal{H}_0$, $u,v \in \mathcal{H}$ write

$$\lambda_t(X) = < f e(u), j_t(X) g e(v) > .$$

Then (2.7) implies

$$\frac{d\lambda_t(X)}{dt} = \sum_{0 \leq i,k \leq n} u_i(t) v^k(t) \lambda_t(\tau^i_k(X)) \tag{2.9}$$

where we have expressed $\mathcal{H}$ in (2.1) as the n-fold direct sum of $L_2(\mathbb{R}_+)$ and denoted by u^i the i-th component of u in $L_2(\mathbb{R}_+)$, $u_i = \bar{u}^i$ for $1 \leq i \leq n$ and $u_0 = u^0 \equiv 1$ for every u in $\mathcal{H}$. In particular, the map $t \longmapsto \lambda_t(X)$ is continuous. Since $\|j_t(X)\| \leq \|X\|$

the totality of $\mathcal{E}$ in $\widetilde{\mathcal{H}}$ implies the weak continuity of $j_t(X)$ in t. The required strong continuity follows from the relation

$$\|(j_t(X)-j_s(X)\|^2 = \langle fe(u), (j_t(X^*X)+j_s(X^*X))fe(u)\rangle$$
$$-2\Re e\langle j_t(X)fe(u), j_s(X)fe(u)\rangle .$$

Equation (2.8) is a straightforward consequence of (2.7). □

THEOREM 2.2. *In theorem 2.1. let $\mathcal{A}$ be abelian. Then*

$$j_s(X)j_t(Y)=j_t(Y)j_s(X) \quad \text{for all} \quad X,Y\in\mathcal{A}, \quad 0\le s,t<\infty .$$

PROOF. Without loss of generality we assume $s<t$. Since j_s is a homomorphism and $\mathcal{A}$ is abelian we have

$$j_s(X)j_s(Y)=j_s(XY)=j_s(YX)=j_s(Y)j_s(X) .$$

By (2.7) we have

$$j_t(Y)=j_s(Y)+\sum_{i,k}\int_s^t j_r(\tau_k^i(Y))\,d\Lambda_i^k(r) .$$

Thus

$$[j_s(X), j_t(Y)] = \int_s^t \sum_{i,k}[j_s(X), j_r(\tau_k^i(Y))]\,d\Lambda_i^k(r) , \tag{2.10}$$

thanks to the fact that $j_t(X)$ is adapted and $j_s(X)$ commutes with the increments of Λ_k^i in $[s,\infty)$. Fix $f,g\in\mathcal{H}_0$, $u,v\in\mathcal{H}$ and put

$$K(s,t;X,Y)=\langle fe(u), [j_s(X), j_t(Y)]\,ge(v)\rangle \tag{2.11}$$

and write (2.10) as

$$K(s,t;X,Y)=\int_s^t \sum_{i,k} u_i(\alpha)v^k(\alpha)K(s,\alpha;X,\tau_k^i(Y))\,d\alpha \tag{2.12}$$

where we have adopted the notations in (2.9). Iterating (2.12) N times we get

$$K(s,t;X,Y)=\sum_{(i_1,k_1)\dots(i_N,k_N)}\int_s^t dt_N\, u_{i_N}(t_N)v^{k_N}(t_N)\int_s^{t_N} dt_{N-1}\, u_{i_{N-1}}(t_{N-1})v^{k_{N-1}}(t_{N-1})$$
$$\dots \int_s^{t_2} dt_1\, u_{i_1}(t_1)v^{k_1}(t_1)K(s,t;X,\tau_{k_1}^{i_1}\tau_{k_2}^{i_2}\dots\tau_{k_N}^{i_N}(Y)) \tag{2.13}$$

Restrict u,v to be $\mathbb{C}^n$–valued bounded functions, set

$$\beta(T)=\sup_{0\le s\le T}\max\{\|u(s)\|, \|v(s)\|, \|u(s)\|\|v(s)\|, 1\} \tag{2.14}$$

and note that

$$|K(s,t;X,Y)|\le 2\|f\|\ \|g\|\ \|X\|\ \|Y\|e^{(\|u\|^2+\|v\|^2)/2} \tag{2.15}$$

Inserting (2.14) and (2.15) in (2.13) and using (2.3) we conclude

$$|K(s,t;X,Y)| \leq C(n+1)^2 N \frac{(\beta(T)M(t-s))^N}{N!} \tag{2.16}$$

for all $0 \leq s \leq t \leq T$ where C is the constant on the right side of (2.15). Since the right hand side of (2.16) tends to 0 as $N \to \infty$ and the set of all $fe(u)$ with $f \in \mathcal{H}_0$ and u a $\mathbb{C}^n$-valued bounded Borel function is total in $\widetilde{\mathcal{H}}$ it follows from (2.11) that $[j_s(X), j_t(Y)] = 0$. □

THEOREM 2.3. *In Theorem 2.1 suppose there exists a bounded linear map $T_\infty : \mathcal{A} \to \mathcal{A}$ such that*

$$\lim_{t\to\infty} \|T_t(X) - T_\infty(X)\| = 0 \quad \text{for all } X \in \mathcal{A} \tag{2.17}$$

Then there exists a unique contractive linear map $j_\infty : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{H}})$ such that (w.lim denoting a weak operator limit)

$$\text{w.}\lim_{t\to\infty} j_t(X) = j_\infty(X) \quad \text{for all } X \in \mathcal{A} \tag{2.18}$$

and $j_\infty = j_\infty \circ T_\infty$.

PROOF. Fix any $0 < t_0 < \infty$. Considering any element in $\mathcal{H}$ as a $\mathbb{C}^n$-valued function on $\mathbb{R}_+$ choose $u, v \in \mathcal{H}$ such that $u(t) = v(t) = 0$ for all $t > t_0$ and $f, g \in \mathcal{H}_0$. Then consider $\lambda_t(X) = \langle fe(u), j_t(X) ge(v) \rangle$. The differential equation (2.9) now assumes the form

$$\frac{d\lambda_t(X)}{dt} = \lambda_t(\tau_0^0(X))) \quad ; \quad X \in \mathcal{A},\ t \geq t_0$$

Thus

$$\lambda_t(X) = \lambda_{t_0}(T_{t-t_0}(X)) \qquad \text{for } t \geq t_0$$

By (2.17) we have

$$\lim_{t\to\infty} \lambda_t(X) = \lambda_{t_0}(T_\infty(X)).$$

The totality in $\widetilde{\mathcal{H}}$ of the set $\{fe(u)\}$ where $f \in \mathcal{H}_0$ and u has compact support, and the inequality $\|j_t(X)\| \leq \|X\|$ imply (2.18). Since $T_\infty = T_t T_\infty$ for each $t \geq 0$ it follows that $j_\infty = j_\infty \circ T_\infty$.

COROLLARY 1. *If $j_\infty(X^*X) = j_\infty(X^*) j_\infty(X)$, the weak limit in (2.18) can be replaced by a strong limit.*

PROOF. For $f \in \mathcal{H}_0$, $u \in \mathcal{H}$ we have from Theorem 2.3

$$\lim_{t\to\infty} \|(j_t(X) - j_\infty(X)) fe(u)\|^2 =$$

$$\lim_{t\to\infty} \{\langle fe(u), \{j_t(X^*X) + j_\infty(X^*) j_\infty(X)\} fe(u)\rangle - 2\,\Re e \langle j_t(X) fe(u), j_\infty(X) fe(u)\rangle\}$$

$$= \langle fe(u), \{j_\infty(X^*X) - j_\infty(X^*) j_\infty(X)\} fe(u)\rangle . \quad \square$$

COROLLARY 2. *If $j_\infty(X^*T_\infty(X)) = j_\infty(X^*)j_\infty(X)$, we have in theorem 2.3*

$$\operatorname*{s.lim}_{t\to\infty} t^{-1}\int_0^t j_s(X)\,ds = j_\infty(X)\,.$$

PROOF. Let $f \in \mathcal{H}_0$, $u \in \mathcal{H}$ with compact support. We have

$$\begin{aligned}\|(t^{-1}\int_0^t j_s(X)ds - j_\infty(X))fe(u)\|^2 = \\ t^{-2}\int_{0<s_1<s_2<t} 2\,\Re e< j_{s_1}(X)fe(u),\, j_{s_2}(X)fe(u)> ds_1 ds_2 \qquad (2.19)\\ +< fe(u),\, j_\infty(X^*)j_\infty(X)fe(u)> - 2t^{-1}\int_0^t \Re e< j_s(X)fe(u),\, j_\infty(X)fe(u)> ds\,.\end{aligned}$$

If $u(t) = 0$ for all $t \geq t_0$ and $t_0 \leq s_1 \leq s_2 < \infty$ then the adaptedness of $\{j_t(X)\}$ implies that

$$< j_{s_1}(X)fe(u),\, j_{s_2}(X)fe(u)> = < j_{s_1}(X)fe(u),\, j_{s_1}(T_{s_2-s_1}(X))fe(u)>\,.$$

This together with (2.17) and elementary analysis yields

$$\begin{aligned}\lim_{t\to\infty} t^{-2}\int_{0<s_1<s_2<t} 2\,\Re e< j_{s_1}(X)fe(u),\, j_{s_2}(X)fe(u)> ds_1 ds_2 \\ = < fe(u),\, j_\infty(X^* T_\infty(X))fe(u)>\,.\end{aligned}$$

Now (2.19) becomes

$$\begin{aligned}\lim_{t\to\infty} \|(t^{-1}\int_0^t j_s(X)ds - j_\infty(X))fe(u)\|^2 = \\ < fe(u),\, (j_\infty(X^* T_\infty(X)) - j_\infty(X^*)j_\infty(X))fe(u)>\,. \qquad \square\end{aligned}$$

REMARK. We may say that the semigroup $\{T_t\}$ is *ergodic* if (2.17) holds and $T_\infty(X)$ is a scalar multiple of the identity for each X. In such a case, since $j_\infty = j_\infty \circ T_\infty$ and $j_\infty(I) = I$ the condition of Corollary 2 holds for all $X \in \mathcal{A}$ and $t^{-1}\int_0^t j_s(X)ds$ converges strongly to $T_\infty(X)I$ as $t \to \infty$.

3. Markov chains as Evans-Hudson diffusions.

Let G be a measurable group acting on a separable σ-finite measure space $(\mathcal{X}, \mathcal{F}, \mu)$ so that μ is quasi invariant under G action. For any $g \in G$ define the unitary operator S_g in $L_2(\mu)$ by

$$(S_g f)(x) = \sqrt{\frac{d\mu}{d\mu g}}(g^{-1}x)\, f(g^{-1}x)\,,\ f \in L_2(\mu) \qquad (3.1)$$

where $(\mu g)(E) = \mu(gE)$, $E \in \mathcal{F}$. Then the map $g \mapsto S_g$ is a unitary representation of G in $L_2(\mu)$. Let m be any complex valued bounded measurable function on the product Borel space $G \times \mathcal{X}$. For any $\varphi \in L_\infty(\mu)$ denote by the same letter φ the bounded operator of multiplication by φ in $L_2(\mu)$ with norm $\|\varphi\|_\infty$. Then $L_\infty(\mu) = \mathcal{A}$ is an abelian *-subalgebra of $\mathcal{B}(L_2(\mu))$. Define bounded operators L_g, $g \in G$ in $L_2(\mu)$ by

$$(L_g f)(x) = m(g, g^{-1}x)(S_g f)(x) . \tag{3.2}$$

For any finite set $F \subset G$ consider the Hilbert space

$$\mathcal{H}_F = L_2(\mu) \otimes \Gamma(L_2(\mathbb{R}_+) \otimes L_2(F)) , \tag{3.3}$$

where $L_2(F)$ is the Hilbert space when F is equipped with the counting measure. If the cardinality of F is n then $L_2(F)$ can be identified with $\mathbb{C}^n$ and we may write

$$d\Lambda_0^0 = dt \ , \ d\Lambda_g^0 = dA_g^\dagger \ , \ d\Lambda_0^g = dA_g \ , \ d\Lambda_g^g = d\Lambda_g \ , \ g \in F$$

with respect to the orthonormal basis of indicators of singletons in F. Now consider the following quantum stochastic differential equations

$$dW_F(t) = \{ \sum_{g\in F} (L_g dA_g^\dagger + (S_g - 1) d\Lambda_g - L_g^* S_g dA_g) - \frac{1}{2} \sum_{g \in F} L_g^* L_g \, dt \} W_F(t) \tag{3.4}$$

with initial value $W_F(0) = 1$. By the basic results of quantum stochastic calculus [5] there exists a unique unitary operator valued adapted process $\{W_F(t),\ t \geq 0\}$ satisfying (3.4). Define

$$j_t(X) = W_F(t)^* X \otimes 1 \, W_F(t) \ , \quad X \in \mathcal{B}(L_2(\mu)) . \tag{3.5}$$

Then

$$\begin{aligned} dj_t(X) = & \sum_{g\in F} \{ j_t(S_g^{-1}[X, L_g]) dA_g^\dagger + j_t(S_g^{-1} X S_g - X) d\Lambda_g + j_t([L_g^*, X] S_g) dA_g \} \\ & + j_t(\mathcal{L}_F(X)) dt \end{aligned} \tag{3.6}$$

where

$$\mathcal{L}_F(X) = -\tfrac{1}{2} \sum_{g\in F} (L_g^* L_g X + X L_g^* L_g - 2 L_g^* X L_g) . \tag{3.7}$$

We now specialize to the case when $X = \varphi \in L_\infty(\mu)$. We then have

$$(S_g^{-1}[\varphi, L_g] f)(x) = m(g,x) \{ \varphi(gx) - \varphi(x) \} f(x) \tag{3.8}$$

$$(S_g^{-1} \varphi S_g f)(x) = \varphi(gx) f(x) \tag{3.9}$$

$$([L_g^*, \varphi] S_g f)(x) = \overline{m(g,x)} \{ \varphi(gx) - \varphi(x) \} f(x) \tag{3.10}$$

$$(\mathcal{L}_F(\varphi) f)(x) = \sum_{g\in F} |m(g,x)|^2 \{ \varphi(gx) - \varphi(x) \} f(x) . \tag{3.11}$$

Equation (2.8) and Theorem (2.2) imply that the *-homomorphisms $\{j_t,\ t \geq 0\}$ of the abelian algebra $L_\infty(\mu)$ constitute an Evans-Hudson diffusion which describes the classical Markov process with infinitesimal generator L_F given by

$$(L_F(\varphi))(x) = \sum_{g\in F} |m(g,x)|^2 (\varphi(gx) - \varphi(x))) . \tag{3.12}$$

EXAMPLE 3.1. Consider a continuous time Markov chain with finite state space $\mathcal{X}$ and stationary transition probabilities $p_t(x,y)$, $x,y \in \mathcal{X}$ such that

$$\frac{d}{dt} p_t(x,y) \big|_{t=0} = \ell(x,y)\,, \quad x,y \in \mathcal{X}\,. \tag{3.13}$$

Then $\ell(x,y) \geq 0$ if $x \neq y$ and $\sum_y \ell(x,y) = 0$. We can realize such a Markov chain as an Evans-Hudson abelian diffusion in several ways. For example impose any group structure on $\mathcal{X}$ so that $G = \mathcal{X}$, μ is the counting measure and G acts on itself by left translation, $F = \mathcal{X} \setminus \{e\}$ where e is the identity element and put

$$\begin{aligned} m(x,y) &= \sqrt{\ell(y,xy)}\, e^{i\theta(x,y)} \quad \text{if } x \neq e \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\theta(x,y)$ is an arbitrary real valued function. Then j_t defined by (3.5) and restricted to the algebra $\mathcal{A}$ of all complex valued functions on $\mathcal{X}$ yields an Evans-Hudson diffusion with

$$(L_F\varphi)(x) = \sum_{y \in \mathcal{X}} \ell(x,y)\varphi(y)\,.$$

We may interpret $m(x,y)\sqrt{dt}$ as the probability amplitude for a transition from the state y to the state xy in time dt. When $\theta \equiv 0$ and $\mathcal{X}$ is the cyclic group with n elements we obtain Meyer's construction in [6,7].

EXAMPLE 3.2. Let $\ell(x,y)$ be as in (3.13). Choose G to be the group of all permutations of $\mathcal{X}$. Define

$$\begin{aligned} m(g,x) &= \sqrt{\ell(y,xy)}\, e^{i\theta(x,y)} \quad \text{if } gx = y,\ x \neq y \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

By a transposition we mean an element g satisfying the following : there exists a pair x, y in $\mathcal{X}$ such that $gx = y$, $gy = x$, $gz = z$ whenever z is different from both x and y. Let F be the set of all transpositions. Then (3.12) becomes

$$(L_F\varphi)(x) = \sum_{y \in \mathcal{X}} \ell(x,y)\varphi(y)\,.$$

In this description, for any $g \in F$, $m(g,x)\sqrt{dt}$ is the probability amplitude for a transition from x to gx in time dt. Thus we obtain another realization of the finite Markov chain described in example 3.1 as an abelian Evans-Hudson diffusion.

EXAMPLE 3.3. Choose $G = \mathcal{X} = \mathbb{Z}$, the additive group of all integers, $F = \{1, -1\}$,

$$\begin{aligned} m(g,x) &= \lambda(x)^{1/2} \quad \text{if } g = 1 \\ &= \mu(x)^{1/2} \quad \text{if } g = -1 \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where λ, μ are nonnegative bounded functions on $\mathbb{Z}$ satisfying $\lambda(x) = 0$ if $x < 0$, $\mu(x) = 0$ if $x \leq 0$. When $\mathbb{Z}$ acts on itself by translation the generator L_F in (3.12) assumes the form

$$(L_F\varphi)(x) = \lambda(x)(\varphi(x+1) - \varphi(x)) + \mu(x)(\varphi(x-1) - \varphi(x)) .$$

In this case the Evans–Hudson diffusion restricted to $L_\infty(\mathbb{Z})$ becomes the classical birth and death process with bounded birth and death rates λ and μ respectively.

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