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Illustration of the Quantum Central Limit Theorem by Independent Addition of Spins

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Coin tossing is one of the basic examples of classical probability. The distribution of the number of heads in N successive tosses can be calculated explicitely. It is given by the binomial distribution which converges to the normal distribution for $N \to \infty$. This is the content of the theorem of de Moivre-Laplace, which can be proved by using Stirling's formula. There are more powerful central limit theorems and more elegant proofs, but nevertheless the theorem of de Moivre-Laplace provides an easy access to the central limit theorem where the convergence can be seen nearly by looking with the naked eye.

One of the easiest non-trivial examples of quantum probability is provided by independent addition of spins. The limit distribution is a non-commutative gaussian state. This has been proven by many previous papers e.g. [1], [2], [3] . The object of this paper is to calculate the distribution explicitely for finite N and to indicate how for large N the limit distribution is obtained. The central limit theorem will not be proven but only the asymptotic behaviour will be discussed.

Let us at first state the quantum central theorem in this context. We consider the spin matrices

(1) $$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and their linear combinations

(2) $$\sigma_+ = \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_- = \sigma_1 - i\sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

The table of multiplication is given by

(3)

	σ_1	σ_2	σ_3
σ_1	$\frac{1}{4}$	$\frac{i}{2}\sigma_3$	$-\frac{i}{2}\sigma_2$
σ_2	$\frac{i}{2}\sigma_3$	$\frac{1}{4}$	$\frac{i}{2}\sigma_1$
σ_3	$-\frac{i}{2}\sigma_2$	$\frac{i}{2}\sigma_1$	$\frac{1}{4}$

A state ω on the algebra M_2 of complex 2×2-matrices is given by a density matrix ρ which we assume to be given in the form

(4) $$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} = \begin{pmatrix} 1/2+z & 0 \\ 0 & 1/2-z \end{pmatrix}$$

$$0 \le \rho_i \le 1 \ , \ \rho_1+\rho_2 = 1 \ , \ \rho_1 \ge \rho_2 \ , \ 0 \le z \le 1/2 \ .$$

This is the most general case as any density matrix can be brought into that form by a unitary change of base and as the σ_i by a unitary change of base are transformed into linear combinations of the σ_i. If $A \in M_2$ then

(5) $$\omega(A) = \mathrm{Tr}\, \rho A$$

so

(6) $$\omega(\sigma_1) = \omega(\sigma_2) = 0 , \qquad \omega(\sigma_3) = \frac{1}{2}(\rho_2 - \rho_1) = -z \ .$$

Consider $(C^2)^{\otimes N}$ and $(M_2)^{\otimes N}$ and on this algebra the state $\omega^{\otimes N}$ given by the density matrix $\rho^{\otimes N}$. Define

(7) $$\sigma_i^{(N)} = \sigma_i \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes \sigma_i \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes \sigma_i \ .$$

The quantum weak law of large numbers states in its simplest form, cf. [2] : let f be a polynomial in three non-commutative indeterminates, then for $N \to \infty$

(8) $$\omega^{\otimes N}\left(f\left(\frac{\sigma_1^{(N)}}{N}, \frac{\sigma_2^{(N)}}{N}, \frac{\sigma_3^{(N)}}{N}\right)\right) \to f(\omega(\sigma_1), \omega(\sigma_2), \omega(\sigma_3)) = f(0, 0, -z) \ .$$

Roughly speaking the quantities $\sigma_i^{(N)} / N$ behave for large N like the constants $\omega(\sigma_i)$.The quantum central limit theorem states for any such polynomial f

(9) $$\omega^{\otimes N}\left(f\left(\frac{\sigma_i^{(N)} - \omega(\sigma_i)}{\sqrt{N}} \ , \ i = 1, 2, 3\right)\right) = \omega\left(f\left(\frac{\sigma_1^{(N)}}{\sqrt{N}}, \frac{\sigma_2^{(N)}}{\sqrt{N}}, \frac{\sigma_3^{(N)} + Nz}{\sqrt{N}}\right)\right)$$

$$\to \gamma_Q(f(\xi, \eta, \zeta))$$

there Q is the covariance matrix

(10) $$Q_{ik} = \omega(\sigma_i \sigma_k) - \omega(\sigma_i)\, \omega(\sigma_k)$$

which can be easily calculated with the help of (3).

(11) $$Q = \begin{pmatrix} \frac{1}{4} & -\frac{iz}{2} & 0 \\ +\frac{iz}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} - z^2 \end{pmatrix} = Q_1 \otimes Q_2$$

with

(12) $$Q_1 = \begin{pmatrix} \frac{1}{4} & -\frac{iz}{2} \\ +\frac{iz}{2} & \frac{1}{4} \end{pmatrix} , \qquad Q_2 = \frac{1}{4} - z^2 \ .$$

For $\rho_2 < \rho_1$, $z > 0$ the gaussian functional γ_Q may be considered as a state on the tensor product of $\mathcal{B}(\ell^2(\mathbf{N}))$, (i.e. the bounded operators on $\ell^2(\mathbf{N})$, $\mathbf{N} = \{0, 1, 2, \ldots\}$) and $L^\infty(\mathbf{R})$

(13) $$\gamma_Q = \gamma_{Q_1} \otimes \gamma_{Q_2} : \quad \mathcal{B}(\ell(\mathbf{N})) \otimes L^\infty(\mathbf{R}) \to \mathbf{C}$$

with

(14) $$\gamma_{Q_1}(A) = \sum_{k=0}^{\infty} \left(1 - \frac{\rho_2}{\rho_1}\right)\left(\frac{\rho_2}{\rho_1}\right)^k \cdot \langle e_k \,|\, A e_k \rangle$$

where e_k is the k-the vector of the standard basis,

(15) $$\gamma_{Q_2}(f) = \frac{1}{\sqrt{2\pi Q_2}} \int \exp(-\xi^2/2Q_2) f(\xi)\, d\xi = \int g_{Q_2}(\xi) f(\xi) d\xi$$

and

(16) $$g_q(\xi) = \frac{1}{\sqrt{2\pi q}} \exp - \xi^2/2q \quad .$$

So γ_{Q_2} is a classical gaussian probability distribution. We shall not consider the degenerate case $z = 0$, $\rho_1 = \rho_2$, where γ_Q is the tensor produced of threee gaussian probability distribution. In (9) ξ and η are unbounded operators on $\ell^2(\mathbf{N})$ given by the equations

(17) $$a = \frac{\xi - i\eta}{\sqrt{2z}}, \qquad a^* = \frac{\xi + i\eta}{\sqrt{2z}}$$

where

(18) $$a = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & 0 & 0 & \vdots \\ 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \\ & & \cdots & & \end{pmatrix}, \qquad a^* = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & \vdots \\ 0 & 0 & 0 & \sqrt{3} & \\ 0 & 0 & 0 & 0 & \\ & & \cdots & & \end{pmatrix}$$

are the wellknown annihilation and creation operators. It is clear that γ_{Q_1} can be extended to any polynomial in a and a* and hence to any polynomial in ξ and η . The variable ζ in (9) may be just a real integration variable as in (15).

We want to make these results a bit more transparent by discussing them more explicitly for large N.

We observe the $\sigma_i^{(N)}$ have the same commutation rules as the σ_i

(19) $$\left[\sigma_1^{(N)}, \sigma_2^{(N)}\right] = i\sigma_3^{(N)}$$

(and cyclic permutations) so they form a representation of the spin operators or, what amounts to the same, of the Lie algebra of the group SU (2). We use that fact in order to split $(\mathbf{C}^2)^{\otimes N}$ into invariant subspaces.

Let V be a finite dimensional unitary vector space and let S_1, S_2, S_3 be hermitian operators on V with the commutation rules

(20) $$[S_1, S_2] = iS_3, \ldots \quad .$$

Then

(21) $$S^2 = S_1^2 + S_2^2 + S_3^2 \quad .$$

Define

(22) $$S_\pm = S_1 \pm iS_2 \quad .$$

Assume at first that V is irreducible. Then it induces an irreducible representation $\mathcal{D}_\ell$, where ℓ may take one of the values $\ell = 0, 1/2, 1, 3/2, 2, \ldots$. The dimension of V is $2\ell+1$. It is possible to introduce an orthogonal basis ψ_m, $m = -\ell, -\ell+1, \ldots, +\ell$ in V, such that

(23) $$\begin{aligned} S_3\psi_m &= m\psi_m \\ S_+\psi_m &= \sqrt{\ell(\ell+1) - m(m+1)}\,\psi_{m+1} \\ S_-\psi_m &= \sqrt{\ell(\ell+1) - m(m-1)}\,\psi_{m-1} \\ S^2\psi_m &= \ell(\ell+1)\psi_m \quad . \end{aligned}$$

If V is not irreducible, it can be split into irreducible parts. This means e.g. it is possible to introduce a basis $\psi_{\ell,m,j}$ with

(24) $$\begin{aligned} &\ell \in \Lambda \subset \{0, 1/2, 1, 3/2, \ldots\}, \\ &m = -\ell, -\ell+1, \ldots, +\ell, \\ &j = 1, \ldots, d_\ell \; . \end{aligned}$$

So all $\psi_{\ell,m,j}$ for fixed ℓ,j span an irreducible representation of type $\mathcal{D}_\ell$ and d_ℓ is the multiplicity of $\mathcal{D}_\ell$. One has

(25) $$\begin{aligned} S_3\psi_{\ell,m,j} &= m\psi_{\ell,m,j} \\ S_\pm\psi_{\ell,m,j} &= \sqrt{\ell(\ell+1) - m(m\pm1)}\,\psi_{\ell,m\pm1,j} \\ S^2\psi_{\ell,m,j} &= \ell(\ell+1)\psi_{\ell,m,j} \quad . \end{aligned}$$

Let

(26) $$E_{\ell,m} = \{x \in V:\ S^2x = \ell(\ell+1)x\ ,\ \ S_3x = mx\} \quad .$$

Then

(27) $$d_\ell = \dim\ E_{\ell,m}$$

and $S_\pm$ maps $E_{\ell,m}$ into $E_{\ell,m\pm1}$. The algebra generated by the S_i in $\mathcal{L}(V)$ is in the basis $\psi_{\ell,m,j}$ the algebra $\mathcal{A}$ of all matrices A with

(28) $$\langle \psi_{\ell,m,j} | A | \psi_{\ell,m,j'} \rangle = \delta_{\ell\ell'}\,\delta_{jj'}\,(A_\ell)_{m,m'}$$

where A_ℓ is a $(2\ell+1)$-dimensional matrix. We may write

(29) $$A = \bigoplus_{\ell \in \Lambda} A_\ell \otimes 1_{d_\ell} \quad .$$

We take now $V = (C^2)^{\otimes N}$ and $S_i = \sigma_i^{(N)}$. We choose in C^2 the basis

(30) $$\varphi\left(-\tfrac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \varphi\left(\tfrac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and in $(C^2)^{\otimes N}$ the basis

(31) $$\varphi(\varepsilon_1, .., \varepsilon_N) = \varphi(\varepsilon_1) \otimes \ldots \otimes \varphi(\varepsilon_N)$$

with $\varepsilon_i = \pm 1/2$. Then

(32) $$S_3 \varphi(\varepsilon_1, .., \varepsilon_N) = (\varepsilon_1 + \ldots + \varepsilon_N) \ \varphi(\varepsilon_1, \ldots, \varepsilon_N) \ .$$

So m can only take the values

(33) $$m = 0, \pm 1, \pm 2, \ldots, \pm N/2 \ \text{(N even)}$$
$$m = \pm 1/2, \pm 3/2, \ldots, \pm N/2 \ \text{(N odd)}$$

and hence ℓ can only take the values

(34) $$\ell = 0, 1, \ldots, N/2 \ \text{(N even)}$$
$$\ell = 1/2, 3/2, \ldots, N/2 \ \text{(N odd)} \ .$$

Let

(35) $$F_m = \left\{ x \in (C^2)^{\otimes N} \ , \ S_3 x = m \right\} .$$

Then

(36) $$\dim F_m = \begin{pmatrix} N \\ \frac{N}{2} - m \end{pmatrix} .$$

As

(37) $$F_m = E_{m,m} \oplus E_{m+1,m} \oplus \ldots \oplus E_{N/2,m}$$

and as $d_\ell = \dim E_{\ell,m} = d_\ell$ is independent of m one obtains

$$\begin{pmatrix} N \\ \frac{N}{2} - m \end{pmatrix} = d_m + d_{m+1} + \ldots + d_{N/2}$$

and finally

(38) $$d_\ell = \begin{pmatrix} N \\ \frac{N}{2} - \ell \end{pmatrix} - \begin{pmatrix} N \\ \frac{N}{2} - \ell - 1 \end{pmatrix} = \frac{2\ell + 1}{\frac{N}{2} + \ell + 1} \begin{pmatrix} N \\ \frac{N}{2} - \ell \end{pmatrix} .$$

By (4) and (31) we obtain

(39) $$\rho^{\otimes N} \varphi(\varepsilon_1, .., \varepsilon_N) = \rho_1^{\frac{N}{2} - m} \ \rho_2^{\frac{N}{2} + m} \ \varphi(\varepsilon_1, \ldots, \varepsilon_N)$$

with $m = \varepsilon_1 + \ldots + \varepsilon_N$. So $\rho^{\otimes N}$ is diagonal in the basis $\psi_{\ell,m,j}$ and we obtain for $A \in \mathcal{A}$ given in the form (29)

(40) $$\omega^{\otimes N}(A) = \sum_{\ell,m} p_{\ell,m} \ (A_\ell)_{m,m}$$

with

(41) $$p_{\ell,m} = \rho_1^{\frac{N}{2}-m} \quad \rho_2^{\frac{N}{2}+m} \quad d_\ell \quad .$$

Hence by (38)

(42) $$p_{\ell,-l+k} = \frac{2\ell+1}{\frac{N}{2}+\ell+1} \left(\frac{\rho_2}{\rho_1}\right)^k \begin{pmatrix} N \\ \frac{N}{2}-\ell \end{pmatrix} \rho_1^{\frac{N}{2}+\ell} \rho_2^{\frac{N}{2}-\ell} \quad .$$

The approximation of the binomial distribution via Stirling's formula gives

(43) $$p_{\ell,-\ell+k} \sim \frac{2\ell+1}{\frac{N}{2}+\ell+1} \left(\frac{\rho_2}{\rho_1}\right)^k \frac{1}{\sqrt{2\pi N\left(\frac{1}{4}-\frac{\ell^2}{N^2}\right)}} \exp(-N\eta_N)$$

where η_N is

(44) $$\eta_N = \left(\frac{1}{2}-\frac{\ell}{N}\right) \left(\log\left(\frac{1}{2}-\frac{\ell}{N}\right)-\log\left(\frac{1}{2}-z\right)\right) + \left(\frac{1}{2}+\frac{\ell}{N}\right) \left(\log\left(\frac{1}{2}+\frac{\ell}{N}\right)-\log\left(\frac{1}{2}+z\right)\right) \quad .$$

This shows at first that for large N all ℓ which are not near Nz can be neglected and that for those ℓ, which are near Nz

(45) $$p_{\ell,-\ell+k} \approx \left(1-\frac{\rho_2}{\rho_1}\right) \left(\frac{\rho_2}{\rho_1}\right)^k \frac{1}{\sqrt{2\pi\, NQ_2}} \exp \; - \frac{(\ell - Nz)^2}{eNQ_2}$$

with Q_2 given by (12).

We imbed $\mathcal{A}$ into the algebra $\mathcal{M}_N \otimes \mathbf{C}^\Lambda$, where $\mathbf{C}^\Lambda$ is the algebra of complex functions on Λ with pointwise multiplication (recall that Λ was the set of possible ℓ) and where $\mathcal{M}_N$ is the algebra all N×N-matrices, where all entries except finitely many ones vanish. If $A \in \mathcal{A}$ is given by the form (27) then

(46) $$j: \; A \to \sum_\ell \tilde{A}_\ell \otimes e_\ell$$

where

(47) $$(\tilde{A}_\ell)_{k,k'} = \begin{cases} (A_\ell)_{-\ell+k,\,-\ell+k'} = \langle \psi_{\ell,-\ell+k,j} | A | \psi_{\ell,-\ell k',j} \rangle \text{ for } 0 \le k, k' \le 2\ell \text{ for } 0 \le k, k' \le 2\ell \\ 0 \text{ else} . \end{cases}$$

and where e_ℓ is the ℓ- the vector in the standard basis. Then by (40) and (42)

(48) $$\omega^{\otimes N}(A) = \pi^{(N)}(j(A)) = \sum q_\ell^{(N)} \gamma_{Q_2}(\tilde{A}_\ell)$$

and by (45)

(49) $$q_\ell^{(N)} = \frac{2\ell+1}{\frac{N}{2}+\ell+1}\,\frac{1}{1-\frac{\rho_2}{\rho_1}}\begin{pmatrix} N \\ \frac{N}{2}-\ell \end{pmatrix}\rho_1^{\frac{N}{2}+\ell}\,\rho_2^{\frac{N}{2}-\ell} \approx g_{NQ_2}(\ell-Nz)$$

for $\ell \approx Nz$. So

(50) $$\pi^{(N)} = \gamma^{(N)} \otimes \gamma_{Q_2}$$

with

(51) $$\gamma^{(N)}(e_\ell) = q_\ell^{(N)} \approx g_{NQ_2}(\ell-Nz) \ .$$

Put

$$j\left(\frac{\sigma_i^{(N)} - N\omega(\sigma_i)}{\sqrt{N}}\right) = \sum_{\ell\in\Lambda} T_i^{(\ell)} \otimes e_\ell \ .$$

Then

$$\left(T_3^{(\ell)}\right)_{kk'} = \delta_{kk'}\ \frac{-\ell - k + Nz}{\sqrt{N}} \approx \delta_{kk'}\ \frac{Nz-\ell}{\sqrt{N}}$$

as $k << \sqrt{N}$. Hence for $\ell \approx Nz$:

(52) $$j\left(\frac{\sigma_3 + Nz}{\sqrt{N}}\right) \approx 1 \otimes X_3^{(N)}$$

with

$$X_3(\ell) = \frac{Nz - \ell}{\sqrt{N}} \ .$$

One has

$$\left(T_+^{(\ell)}\right)_{k',k} = \delta_{k',k+1}\,\frac{\sqrt{2\ell(k+1)-k-k^2}}{\sqrt{N}} \approx \delta_{k',k+1}\sqrt{2z(k+1)}$$

$$\left(T_-^{(\ell)}\right)_{k',k} = \delta_{k',k-1}\,\frac{\sqrt{2\ell k + k - k^2}}{\sqrt{N}} \approx \delta_{k',k-1}\sqrt{2zk} \ .$$

So finally

(53) $$j\left(\frac{\sigma_+^{(N)}}{\sqrt{N}}\right) \approx \sqrt{2z}\left(a^* \otimes 1\right)$$

(54) $$j\left(\frac{\sigma_-^{(N)}}{\sqrt{N}}\right) \approx \sqrt{2z}\left(a \otimes 1\right) \ .$$

Equations (50) to (54) show, how the postulated limit behaviour may arise.

<u>Literature</u>

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