MARTIN T. BARLOW
PHILIP PROTTER

On convergence of semimartingales

Séminaire de probabilités (Strasbourg), tome 24 (1990), p. 188-193
<http://www.numdam.org/item?id=SPS_1990__24__188_0>

ON CONVERGENCE OF SEMIMARTINGALES

Martin T. Barlow[1] and Philip Protter[2]

Statistical Laboratory
16 Mill Lane
Cambridge CB2 15B
England

Mathematics and Statistics Departments
Purdue University
W. Lafayette, IN 47907
U.S.A.

Let X be a semimartingale. A norm commonly used on the space of semimartingales is the $\mathcal{H}^p$ norm: One defines

$$j_p(M,A) = ||[M,M]_\infty^{1/2} + \int_0^\infty |dA_s|||_{L^p}$$

for any decomposition $X = M + A$ with M a local martingale and A an adapted, right continuous process with paths of finite variation on compacts. Then

$$||X||_{\mathcal{H}^p} = \inf_{X=M+A} j_p(M,A)$$

where the infimum is taken over all such decompositions of X. Then as is well known (see, for example, Emery [2], Meyer [7], or Protter [8], Theorem 2 of Chapter V):

$$||X^*||_{L^p} \le c_p ||X||_{\mathcal{H}^p} \qquad (1 \le p < \infty)$$

where $X^* = \sup_t |X_t|$, and c_p is a universal constant. An immediate consequence is that if a sequence of semimartingales X^n converges to X in $\mathcal{H}^1$, then

$$\lim_{n\to\infty} E\{(X^n - X)^*\} = 0$$

as well.

In this paper we examine the converse question: if $X^n = M^n + A^n$ is a sequence of semimartingales converging uniformly in L^1 to a process X, what can be said about the convergence of the M^n and A^n processes of the decompositions? Such a question is closely related to recent work on weak convergence of semimartingales: In particular Jacod-Shiryaev [3], Jakubowski-Mémin-Pages [4], and Kurtz-Protter [5].

The examination of two simple examples illustrates the problems that arise and shows that one cannot expect a full converse.

[1] Supported by a NSF grant while visiting Cornell University

[2] Supported in part by NSF grant #DMS-8805595

Let Y be any continuous, adapted process with $Y_0 = 0$ and Y constant on $[1, \infty)$; set

$$X_t^n = n \int_{t-1/n}^t Y_s ds 1_{\{t>1/n\}}.$$

Then X^n is a differentiable function of t in $[\frac{1}{n}, \infty)$ for each n and in particular each X^n is of finite variation (and hence it is a semimartingale). However the limit Y need not be a semimartingale.

The preceding example indicates that we have to impose some type of uniform bound on the total variation of the A^n processes. But even if we do this we cannot hope always to obtain convergence of the A^n processes in total variation norm. Indeed, let $0 \leq t \leq \frac{\pi}{2}$, and define $A_t^n = \frac{1}{n} \sin nt$. Then $\int_0^{\pi/2} |dA_s^n| = 1$, but $(A^n)^*$ converges to zero.

The following theorem avoids the pathologies of the two preceding examples. Recall that a semimartingale X in $\mathcal{H}^1$ is special: that is, it always has a unique decomposition $X = X_0 + M + A$, where $M_0 = A_0 = 0$, and the finite variation process A is predictable. Such a decomposition is said to be the canonical decomposition.

Theorem 1. Let X^n be a sequence of semimartingales in $\mathcal{H}^1$ with canonical decomposition $X^n = X_0^n + M^n + A^n$, satisfying for some constant K,

(1a) $$E\{\int_0^\infty |dA_s^n|\} \leq K$$

(1b) $$E\{(M^n)^*\} \leq K.$$

Let X be a process, and suppose that

(2) $$E\{(X^n - X)^*\} \to 0 \qquad \text{as } n \to \infty.$$

Then X is a semimartingale in $\mathcal{H}^1$, and if $X = X_0 + M + A$ is its canonical decomposition we have

(3) $$E\{M^*\} \leq K, \qquad E\{\int_0^\infty |dA_s|\} \leq K$$

and

(4a) $$\lim_{n\to\infty} ||M^n - M||_{\mathcal{H}^1} = 0,$$

(4b) $$\lim_{n\to\infty} E\{(A^n - A)^*\} = 0.$$

Corollary 2. Let (X^n) be a sequence of special semimartingales with canonical decomposition $X^n = X_0^n + M^n + A^n$, where the A^n satisfy (1a). Then if X is a process such that $\lim_{n\to\infty} ||(X^n - X)^*||_{L^1} = 0$, X is a special semimartingale. Further if $X = X_0 + M + A$ is its canonical decomposition, then

$$\lim_{n\to\infty} ||M^n - M||_{\mathcal{H}^1} = 0, \qquad \lim_{n\to\infty} E\{(A^n - A)^*\} = 0, \qquad E\{\int_0^\infty |dA_s|\} \le K.$$

Proof. By deleting a finite number of terms in the sequence (X^n), we may suppose that $E\{(X^n - X)^*\} \le K$ for $n \ge 1$. But then

$$\begin{aligned} E\{(M^n - M^1)^*\} &\le E\{|X_0^n - X_0^1|\} + E\{(X^n - X)^*\} + E\{(A^n - A^1)^*\} \\ &\le 4K. \end{aligned}$$

So write $\tilde{X}^n = X^n - M^1 = X_0^n + (M^n - M^1) + A^n$, $\tilde{X} = X - M^1$. Then the hypotheses of Theorem 1 hold for $\tilde{X}^n$, $\tilde{X}$ and the conclusion follows easily. □

The proof of Theorem 1 uses some ideas from Kurtz and Protter [5], and it also needs the following martingale inequality.

Proposition 3. Let $p \ge 1/2$, M be a martingale in $\mathcal{H}^{2p}$ and K be a predictable process with $K^* \in L^{2p}$. Then

$$||(K \cdot M)^*||_{L^p} \le c_p ||K^*||_{L^{2p}} ||M^*||_{L^{2p}}.$$

Proof. Recall the Davis decomposition of M — see Meyer [6, p. 80–81]. Let $\Delta M_s = M_s - M_{s-}$. Let $A_t = \sup_{s\le t} |\Delta M_s|$: then $M = N + U$, where N is a martingale with $|\Delta N_t| \le A_{t-}$, and U is a martingale with paths of integrable variation satisfying

$$|| \int |dU_s| ||_{L^q} \le c_q ||A_\infty||_{L^q}, \quad q \ge 1.$$

Further, we have the pointwise inequalities

$$\begin{aligned} A_\infty &\le 2M^*, \\ [N]_\infty^{1/2} &\le [M]_\infty^{1/2} + [U]_\infty^{1/2}, \\ [U]_\infty^{1/2} &\le 4A_\infty. \end{aligned}$$

Now $(K \cdot M)^* \le (K \cdot N)^* + (K \cdot U)^*$, and $|\Delta(K \cdot N)_t| \le K_t^* A_t$. Hence, by Meyer [6], Theorem 2 on p. 76,

$$\begin{aligned} ||(K \cdot M)^*||_{L^p} &\le c_p(||([K \cdot N]_\infty + (K^* A_\infty)^2)^{1/2}||_{L^p} + ||(K \cdot U)^*||_{L^p}) \\ &\le c_p(||[K \cdot N]_\infty^{1/2} + K^* A_\infty||_{L^p} + ||(K \cdot U)^*||_{L^p}) \\ &\le c_p(||K^*[N]_\infty^{1/2}||_{L^p} + ||K^* M^*||_{L^p} + || \int |K_s||dU_s| ||_{L^p}) \\ &\le c_p(||K^*[M]_\infty^{1/2}||_{L^p} + ||K^* M^*||_{L^p} + ||K^* \int |dU_s| ||_{L^p}). \end{aligned}$$

The proof is concluded by applying Holder's inequality, and noting that $\|\int |dU_s|\|_{L^{2p}} \leq c_p\|M^*\|_{L^{2p}}$. (The constant c_p changes from place to place in the preceding.) $\square$

Remarks. 1. Of course, for $p \geq 1$ this inequality is an immediate consequence of the Burkholder-Davis-Gundy inequalities.
2. This inequality is not true in general for $0 < p < 1/2$.

Proof of Theorem 1. First note that as X is the a.s. uniform limit of a subsequence of the X^n, X is cadlag. Also, as $\|X_0^n - X_0\|_{L^1} \to 0$, we may take $X_0^n = X_0 = 0$.

Let H be an elementary predictable process, that is a process of the form

$$H_t = \sum_{i=1}^{k} h_i 1_{(t_i,t_{i+1}]}(t),$$

where $h_i \in \mathcal{F}_{t_i}$, $|h_i| \leq 1$, and $t_1 < t_2 < \ldots < t_k$. Then writing $H \cdot X$ for the elementary stochastic integral of H with respect to X, $t_{k+1} = \infty$, we have

$$\begin{aligned} E\{(H \cdot X)_\infty\} &= E\{\sum_{i=1}^{k+1} h_i(X_{t_{i+1}} - X_{t_i})\} \\ &= \lim_{n\to\infty} E\{\sum_{i=1}^{k+1} h_i(X^n_{t_{i+1}} - X^n_{t_i})\} \\ &= \lim_{n\to\infty} E\{\int_0^\infty H_t dA_t^n\} \leq K. \end{aligned}$$

So by the Bichteler-Dellacherie theorem (e.g., Dellacherie-Meyer [1]) X is a quasimartingale, and therefore a special semimartingale. Hence X has a canonical decomposition $X = M + A$, with M a local martingale and A a predictable finite variation process. Choose a sequence (T_k) reducing M. Then, if H is an elementary predictable process, $E\{(H \cdot A)_{T_k}\} = E\{(H \cdot X)_{T_k}\} = \lim_n E\{(H \cdot X^n)_{T_k}\} \leq K$. Thus

$$E\{\int_0^{T_k} |dA_s|\} \leq K, \quad \text{for each } k \geq 1,$$

and hence $E\{\int_0^\infty |dA_s|\} \leq K$.

Now $M = X - A = (X - X^n) + (M^n + A^n) - A$, and so

$$M^* \leq (X - X^n)^* + (M^n)^* + \int_0^\infty |dA_s^n| + \int_0^\infty |dA_s|.$$

Thus $E\{M^*\} \leq 3K < \infty$, and M is a martingale in $\mathcal{H}^1$. Set $Y^n = X^n - X$, $N^n = M^n - M$, $B^n = A^n - A$: We have

$$E\{\int_0^\infty |dB_s^n|\} \leq 2K, \qquad E\{(N^n)^*\} \leq 2K, \quad \lim_n E\{(Y^n)^*\} = 0.$$

To complete the proof it is enough to prove that

$$\lim_{n\to\infty} E\{[Y^n]_\infty^{1/2}\} = 0. \tag{5}$$

For then, by Dellacherie and Meyer [1], section VII.95, we have $E\{[B^n]^{1/2}\} \leq 2E\{[Y^n]^{1/2}\}$. Hence, as $[N^n]^{1/2} \leq [B^n]^{1/2} + [Y^n]^{1/2}$, $E\{[N^n]_\infty^{1/2}\} \leq 3E\{[Y^n]_\infty^{1/2}\}$, so that $\lim_{n\to\infty} ||N^n||_{\mathcal{H}^1} = 0$. This implies that $E\{(M^n - M)^*\} \to 0$, and hence that $\infty E\{(A^n - A)^*\} \to 0$. Finally, $E\{M^*\} \leq K$ follows from (4a) and (1b).

To show that $\lim_{n\to\infty} E\{[Y^n]_\infty^{1/2}\} = 0$, use integration by parts to conclude

$$[Y^n]_\infty = (Y_\infty^n)^2 - 2\int_0^\infty Y_{s-}^n dN_s^n - 2\int_0^\infty Y_{s-}^n dA_s^n,$$

and so, writing $U^n = Y_-^n \cdot N^n$,

$$E\{[Y^n]_\infty^{1/2}\} \leq E\{(Y^n)^*\} + 2^{1/2}E\{((U^n)^*)^{1/2}\} + 2^{1/2}E\{(\int_0^\infty |Y_{s-}^n||dA_s^n|)^{1/2}\}. \tag{6}$$

By Proposition 2

$$\begin{aligned} E\{((U^n)^*)^{1/2}\} &\leq c(E\{(Y^n)^*\})^{1/2}(E\{(N^n)^*\})^{1/2} \\ &\leq cK^{1/2}(E\{(Y^n)^*\})^{1/2}. \end{aligned}$$

Similarly, the third term in (6) is dominated by

$$\begin{aligned} E\{((Y^n)^* \int_0^\infty |dA_s^n|)^{1/2}\} &\leq (E\{(Y^n)^*\})^{1/2}(E\{\int_0^\infty |dA_s^n|\})^{1/2} \\ &\leq K^{1/2}(E\{(Y^n)^*\})^{1/2}. \end{aligned}$$

Thus $\lim_{n\to\infty} E\{[Y^n]_\infty^{1/2}\} = 0$. □

REFERENCES

1. C. Dellacherie, P. A. Meyer: ***Probabilities and Potential B***, North-Holland, Amsterdam New York 1982.
2. M. Emery: Stabilité des solutions des équations différentielles stochastiques; applications aux intégrales multiplicatives stochastiques; Z. Wahrscheinlichkeitstheorie **41**, 241–262, 1978.
3. J. Jacod, A. N. Shiryaev: "Limit Theorems for Stochastic Processes," Springer, Berlin Heidelberg New York 1987.
4. A. Jakubowski, J. Mémin, G. Pages: Convergence en loi des suites d'intégrales stochastiques sur l'espace D^1 de Skorohod; Probability Theory and Related Fields **81**, 111–137, 1989.

5. T. Kurtz, P. Protter: Weak limit theorems for stochastic integrals and stochastic differential equations; preprint.
6. P. A. Meyer: "Martingales and Stochastic Integrals I," Springer Lecture Notes in Mathematics **284**, 1972.
7. P. A. Meyer: 'Inégalités de normes pour les integrales stochastiques," Séminaire de Probabilités XII, Springer Lecture Notes in Math. **649**, 757–762, 1978.
8. P. Protter: *Stochastic Integration and Differential Equations: A New Approach*, Springer-Verlag, forthcoming.