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On Semi-Martingales Associated with Crossings

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Introduction. Let $(X_t)_{t \geq 0}$ be a Brownian motion, $X_0 = x$ almost surely, $x < a < b$. Let σ_t be the last exit time of X before t from $(a,b)^c$, defined in sec. 1.1. We note that when $b = \infty$, $X_t - X_{\sigma_t} = (X_t - a)^+$ and by Tanaka's formula it follows that $X_t - X_{\sigma_t}$ and hence X_{σ_t} , are semi-martingales. It is easy to see from Theorem 1 of [6] that when $b < \infty$, $|X_t - X_{\sigma_t}|$ is a semi-martingale given by

$$(b-a)c(t) + |X_t - X_{\sigma_t}| = \int_0^t I_{(a,b)}(X_s) \Theta(s) dX_s + \frac{1}{2}(L(t,a) + L(t,b))$$

where $c(t)$ is the numbers of crossings of (a,b) in time t , $\Theta(s,w)$ is 1 during an upcrossing and -1 during a downcrossing and $L(t,.)$ is the local time of X .

In the case of a continuous semi-martingale (X_t, \mathcal{F}_t) , where \mathcal{F}_t is the underlying filtration and σ_t as above, it is an immediate consequence of Tanaka's formula that $(X_{\sigma_t}, \mathcal{F}_t)$, $(X_t - X_{\sigma_t}, \mathcal{F}_t)$ are semi-martingales (Theorem 2.1). In this case, time changing by σ_t does not change the underlying filtration. In this paper, as our main result we determine the martingale and bounded variation parts of $|X_t - X_{\sigma_t}|$ (Theorem 4.1). In sec. 5, we state a few applications of this result. These include Levy's crossing theorem, an asymptotic relationship between local times and crossings of Brownian motion and a probabilistic approximation of the remainder term in the 2nd order Taylor expansion of a function.

1. Preliminaries

Let (Ω, \mathcal{F}, P) be a probability space and $(\mathcal{F}_t)_{t \geq 0}$ a filtration on it satisfying usual conditions. For a continuous adapted process $(X_t)_{t \geq 0}$ and $a < b$, the upcrossing intervals $(\sigma_{2k}, \sigma_{2k+1}]$, $k = 0, 1, 2, \dots$, are defined by $\sigma_0 = \inf \{s \geq 0, X_s \leq a\}$, $\sigma_{2k} = \inf \{s > \sigma_{2k-1}, X_s \leq a\}$ and $\sigma_{2k+1} = \inf \{s > \sigma_{2k} : X_s \geq b\}$. As usual the infimum over the empty set is infinity. The downcrossing intervals $(\tau_{2k}, \tau_{2k+1}]$, $k = 0, 1, \dots$ are similarly defined. Let $\theta^u(s) = \sum_{k=0}^{\infty} I_{(\sigma_{2k}, \sigma_{2k+1}]}(s)$; $\theta^d(s) = \sum_{k=0}^{\infty} I_{(\tau_{2k}, \tau_{2k+1}]}(s)$ and $\theta(s) = \theta^u(s) - \theta^d(s)$. The number of upcrossings in time t , denoted by $U(t)$ is defined as $U(t) = \max \{k : \sigma_{2k+1} \leq t\}$. The number of downcrossings is similarly defined. $C(t) = U(t) + D(t)$ is the total number of crossings. Let $\tau = \inf \{s > 0 : X_s \notin (a, b)\}$.

$$\text{Let } \sigma_t = \begin{cases} t & t \leq \tau \\ \max_{s < t : X_s \notin (a, b)^c} & , t > \tau \end{cases}$$

σ_t is in general not a stop time, but is however \mathcal{F}_t measurable. Consequently X_{σ_t} is \mathcal{F}_t measurable.

2. The Semi-Martingale $X_t - X_{\sigma_t}$

From now on we fix a continuous \mathcal{F}_t -semi-martingale $X_t = X_0 + M_t + V_t$ and $a < b$. Let $L(t, x, w)$ be a jointly (t, x, w) measurable version of the local time of X which is continuous in t and right continuous in x . For the existence of such versions see [10]. Let $Y_t = X_t - X_{\sigma_t}$ and $Z_t = X_{\sigma_t}$.

Theorem 2.1. The process Y_t is an \mathcal{F}_t semi-martingale and we have

$$Y_t = \int_{\tau_t}^t I_{(a, b]}(X_s) dX_s + \frac{1}{2} (L(t, a) - L(t, b)) - (b-a)(U(t) - D(t)) \quad (1)$$

Proof. The proof is immediate from Tanaka's formula and the following pathwise identity :

$$\begin{aligned} (X_{\tau_t} - X_0) + (b-a)[U(t)-D(t)] + (X_t - X_{\sigma_t}) \\ = (X_t - a)^+ - (X_0 - a)^+ - (X_t - b)^+ + (X_0 - b)^+ \end{aligned} \quad (2)$$

Remarks.

2.2 It is immediate from Theorem 2.1 that X_{σ_t} is also a semi-martingale whose components can be got by subtracting (X_t) from both sides of eqn. (1).

2.3 The sum of the jumps of Y in time $t = \sum_{s \leq t} \Delta Y_s$ is precisely $(b-a)[D(t)-U(t)]$. Since $|U(t) - D(t)| \leq 1$ this implies that Y (and hence Z) is a special semimartingale. Further the representation (1) of Y_t is unique (see [9]). The jump times of these processes are precisely the times of crossings of (a,b) by X and $|\Delta Y_s| = b-a$ or 0 .

2.4 Equation (2) and hence Theorem 2.1 are still valid for a semi-martingale (X_t) with $\sum_{s \leq t} |\Delta X_s| < \infty \quad \forall t$, almost surely.

Now $(b-a)[U(t)-D(t)]$ is replaced by $-\sum_{s \leq t} \Delta Y_s$ and $(X_t - a)^+$, $(X_t - b)^+$ are replaced by $(X_t - a)^+ - \sum_{s \leq t} \Delta (X_s - a)^+$, $(X_t - b)^+ - \sum_{s \leq t} \Delta (X_s - b)^+$ respectively.

3. Local times of $X_t - X_{\sigma_t}$

We now determine the local times of Y in terms of that of X . We note that the process lives in $[0, b-a)$ during an upcrossing of (a,b) and in $-(b-a), 0]$ during a downcrossing. Also $Y_t = 0$ whenever $X_t = a$ or b . Let $I(t,x)$ denote the local time of the Y process.

Lemma 3.1(i) For $x \in [0, b-a)$,

$$(Y_{t-x})^+ = \int_{\tau_t}^t I_{(a,b]}(X_s) I_{(x,\infty)}(Y_{s-}) dX_s - (b-a-x)U(t) + \frac{1}{2}I(t,x) \quad (3)$$

(ii) For $x \in (-(b-a), 0]$

$$\begin{aligned} (Y_{t-x})^- &= - \int_{\tau_t}^t I_{(a,b]}(X_s) I_{(-\infty,x]}(Y_{s-}) dX_s - (b-a+x)D(t) + \frac{1}{2}I(t,x) \\ &\quad + \frac{1}{2} \left(\int_0^t I_{(-\infty,x]}(Y_{s-}) L(ds,b) - \int_0^t I_{(-\infty,x]}(Y_{s-}) L(ds,a) \right) \end{aligned} \quad (4)$$

Remark 3.2 Observe that in case $x < 0$, the 2nd term on the RHS of (4) is zero whereas when $x = 0$ it is $\frac{1}{2}(L(t,b) - L(t,a))$.

Proof. Tanaka's formula (see [4]) applied to Y at the point $x \in [0, b-a)$ gives

$$\begin{aligned} (Y_{t-x})^+ &= (Y_0-x)^+ + \int_0^t I_{(x,\infty)}(Y_{s-}) dY_s \\ &\quad + \sum_{0 < s \leq t} I_{(x,\infty)}(Y_{s-})(Y_s - x)^- \\ &\quad + \sum_{0 < s \leq t} I_{(-\infty,x]}(Y_{s-})(Y_s - x)^+ + \frac{1}{2}I(t,x) \\ &= I_0 + I_1 + I_2 + I_3 + \frac{1}{2}I(t,x). \end{aligned}$$

Since $Y_0 \equiv 0$, $I_0 \equiv 0$. Using eqn. (2) for Y_t and noting that the measures $L(ds,a)$, $L(ds,b)$, $D(ds)$ have no support on the set $s : Y_{s-} > x$ we get

$$I_1(t) = \int_{\tau_t}^t I_{(a,b]}(X_s) I_{(x,\infty)}(Y_{s-}) dX_s - (b-a)U(t).$$

Since the jumps of Y occur at the crossing times σ_{2k+1} , τ_{2k+1} it is easy to see that almost surely for $x \in [0, b-a)$, $I_2(t) = x U(t)$, $I_3(t) \equiv 0$. This proves the first part of the lemma. The proof of (4) is similar using the Tanaka formula for $(Y_{t-x})^-$.

The following theorem gives I in terms of L .

Theorem 3.3

(i) For $x \in (0, b-a)$, almost surely,

$$I(t, x) = \int_0^t \Theta^u(s) L(ds, a+x) \quad (5)$$

(ii) For $x \in (-(b-a), 0)$, almost surely,

$$I(t, x) = \int_0^t \Theta^d(s) L(ds, b+x) \quad (6)$$

(iii) For $x = 0$, almost surely,

$$I(t, 0) = L(t, a) \quad (7)$$

Proof.

(i) Let $x \in (0, b-a)$. Fix $k \geq 0$. Let $Y_1(t) = (Y_t - x)^+$
 $Y_2(t) = (X_t - (a+x))^+$. We note that, $\forall t \in (\sigma_{2k}, \sigma_{2k+1})$

$$\int_0^t I_{(\sigma_{2k}, \sigma_{2k+1}]}(s) dY_1(s) = \int_0^t I_{(\sigma_{2k}, \sigma_{2k+1}]}(s) dY_2(s) \quad (8)$$

By Tanaka's formula,

$$\begin{aligned} \int_0^t I_{(\sigma_{2k}, \sigma_{2k+1}]} dY_2(s) &= \int_0^t I_{(\sigma_{2k}, \sigma_{2k+1}]}(s) I_{(a, b]}(X_s) I_{(x, \infty)}(Y_{s-}) dX_s \\ &\quad + \frac{1}{2} (L(t \wedge \sigma_{2k+1}, a+x) - L(t \wedge \sigma_{2k}, a+x)). \end{aligned}$$

By eqn. (3), $\forall t \in (\sigma_{2k}, \sigma_{2k+1})$

$$\begin{aligned} \int_0^t I_{(\sigma_{2k}, \sigma_{2k+1}]} dY_1(s) &= \int_0^t I_{(\sigma_{2k}, \sigma_{2k+1}]}(s) I_{(a, b]}(X_s) I_{(x, \infty)}(Y_{s-}) dX_s \\ &\quad + \frac{1}{2} (I(t \wedge \sigma_{2k+1}, x) - I(t \wedge \sigma_{2k}, x)) \end{aligned}$$

eqn. (8) now implies that $\forall t \geq 0$,

$$I(t \wedge \sigma_{2k+1}, x) - I(t \wedge \sigma_{2k}, x) = L(t \wedge \sigma_{2k+1}, a+x) - L(t \wedge \sigma_{2k}, a+x)$$

since $I(ds, x)$ is supported on the upcrossing intervals,
the proof of (i) is complete.

(ii) Let $x \in (-(b-a), 0)$. Then $Y_1(t) = (Y_t - x)^-$ and

$Y_2(t) = (X_t - (b+x))^-$ agree on the downcrossing intervals.

Applying Tanaka's formula for Y_1 and eqn. (4) to Y_2 the proof is completed as in (i) above.

(iii) Let $x = 0$. Proceeding as in case (i) we show that

$$\int_0^t \Theta^u(s) I(ds, 0) = \int_0^t \Theta^u(s) L(ds, a) = L(t, a).$$

To complete the proof we show that $\int_0^t \Theta^d(s) I(ds, 0) = 0$.

To see this we fix k and as in case (ii), compare the expressions for $(X_t - b)^-$ and $(Y_t)^-$ for $t \in (\tau_{2k}, \tau_{2k+1})$ given by Tanaka's formula and eqn. (4) respectively. Using Remark 3.2 we see that

$$\begin{aligned} L(t \wedge \tau_{2k+1}, b) - L(t \wedge \tau_{2k}, b) + (I(t \wedge \tau_{2k+1}, 0) - I(t \wedge \tau_{2k}, 0)) \\ = L(t \wedge \tau_{2k+1}, b) - L(t \wedge \tau_{2k}, b) \end{aligned}$$

whence $I(t \wedge \tau_{2k+1}, 0) - I(t \wedge \tau_{2k}, 0) = 0$.

Remarks.

3.4 We recall from [10] that for the semi-martingale (X_t) with $-\sum_{s \leq t} \Delta X_s + X_t = X_0 + M_t + V_t$, where M and V are the continuous martingale and bounded variation parts respectively, the jumps of the local time $L(t, x)$ is given by the formula : almost surely,

$$L(t, x) - L(t, x-) = \int_0^t I_{\{X_s = x\}} dV_s \quad (9)$$

Using (9) it is easy to see that for $x \in (0, b-a)$, $I(t, x)$ is continuous at x if $L(t, \cdot)$ is continuous at $a+x$. The case $x \in (-(b-a), 0)$ is similar. When $x = 0$, it is easy to see that $I(t, 0-) = L(t, b-) \neq L(t, a)$.

3.5 Let $\bar{I}(t, x)$ denote the local time process of $Z_t = X_{\sigma_t}$. The martingale, bounded variation part and the jumps of Z_t are easily calculated from eqn. (1). By using Tanaka's formula it is easily verified that $\bar{I}(t, x) = L(t, x)$,

$\forall x \in (-\infty, a) \cup [b, \infty)$, $\bar{I}(t, a) = 0$ and $\bar{I}(t, a) = L(\tau \wedge t, x)$,
 $\forall x \in (a, b)$.

4. The Semi-Martingale $|X_t - X_{\sigma_t}|$

We now determine the continuous martingale and the continuous bounded variation parts of $|X_t - X_{\sigma_t}|$. We note that the sum of the jumps upto time t is $-(b-a)c(t)$.

Theorem 4.1 For $a < b$, we have almost surely,

$$(b-a)c(t) + |X_t - X_{\sigma_t}| = \int_0^t \Theta(s, w) I_{(a,b)}(X_s) dX_s + \frac{1}{2}(L(t, a) + L(t, b-)) \quad (10)$$

Proof. Lemma 3.1 and Theorem 3.3 together give

$$\begin{aligned} |X_t - X_{\sigma_t}| &= (X_t - X_{\sigma_t})^+ + (X_t - X_{\sigma_t})^- \\ &= \int_{\tau_t}^t (I_{(0,\infty)}(Y_{s-}) - I_{(-\infty,0]}(Y_{s-})) I_{(a,b)}(X_s) dX_s \\ &\quad - (b-a)c(t) + \frac{1}{2} (L(t, a) + L(t, b)) \\ &= \int_0^t \Theta(s) I_{(a,b)}(X_s) dX_s - (b-a)c(t) \\ &\quad + \frac{1}{2} (L(t, a) + L(t, b-)) \end{aligned}$$

where in the last equality we have used eqn. (9).

Remark 4.2 We refer to [6] for an analogous result on crossings of closed intervals by a continuous martingale.

5. Applications

We now give some applications of the previous results. We mention only the results and refer the proofs to [5], [6] and [7].

Firstly we note that letting $a \uparrow b$ in Theorem 4.1 eqn. (10) yields Levy's crossing theorem. We note that if $\varepsilon_1 \leq \varepsilon \leq \varepsilon_2$ then $\varepsilon_1 C_{\varepsilon_2}(t) \leq \varepsilon C_{\varepsilon}(t) \leq \varepsilon_2 C_{\varepsilon_1}(t)$ where

$C_\varepsilon(t)$ = number of crossings of $(b-\varepsilon, b)$ in time $t = C((b-\varepsilon, b), t)$.
Hence sufficient to let $a \uparrow b$ along a sequence. This is done via the Borel-Cantelli lemma and an estimate due to Yor (Theorem 1, [10]). The following theorem (Levy's (down) crossing theorem) was first proved in the case of a continuous semi-martingale in El Karoui [3] where the discontinuous case is also discussed.

Theorem 5.1 Let (X_t) be a continuous semi-martingale. Then

(a) almost surely, $\lim_{a \uparrow b} (b-a)C((a,b), t) = L(t, b-)$

$$\lim_{b \downarrow a} (b-a)C((a,b), t) = L(t, a)$$

(b) If further $(X_t) \in H^p$, $p \geq 1$ then the above limits hold in H^p .

Next let (X_t) be a Brownian motion. We now state a result somewhat related to Theorem 5.1 above and whose proof can be found in [6], [7]. The crossing theorem says, that $(b-a)C(t) \sim L(t, a)$ as $b \downarrow a$, the parameter t being fixed. It is an interesting fact that the same is true when we let $t \rightarrow \infty$. We have the following theorem.

Theorem 5.2 Let (X_t) be a Brownian motion and $a < b$.

Then almost surely,

$$\lim_{t \rightarrow \infty} \frac{L(t, a)}{C((a, b), t)} = \lim_{t \rightarrow \infty} \frac{E L(t, a)}{E C((a, b), t)} = (b-a)$$

Remark 5.3 The proof of the 2nd equality is immediate from Theorem 4.1 and Theorem 2.1

Corollary 5.4 Let $a < b$, $d < e$. Then almost surely,

$$\lim_{t \rightarrow \infty} \frac{C((a, b), t)}{C((d, e), t)} = \lim_{t \rightarrow \infty} \frac{E C((a, b), t)}{E C((d, e), t)} = \frac{b-a}{e-d}$$

We continue with a Brownian motion (X_t) . The following result gives the average sojourn time in (a, b) per crossing.

Theorem 5.3 If (X_t) is a Brownian motion and $a < b$, then almost surely,

$$\lim_{t \rightarrow \infty} \frac{\int_0^t I_{(a,b)}(X_s) ds}{C((a,b),t)} = \lim_{t \rightarrow \infty} \frac{E \int_0^t I_{(a,b)}(X_s) ds}{E C((a,b),t)} = (b-a)^2$$

We refer to [8] for a proof of this result. The 2nd equality is an immediate consequence of Theorem 1, [5] which is also proved in [11]. We refer to [1] for a more general result in the context of Hunt processes and to [2] for related results involving recurrent diffusions. The following is a different generalization of Theorem 5.3 and can be thought off as a random approximation to the remainder term in a 2nd order Taylor expansion for a C^2 -function. For the proof of this result see [6], [7].

Theorem 5.4 Let (X_t) be a Brownian motion, $a < b$, and f a C^2 -function. Then almost surely,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\int_0^t f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s) ds}{C((a,b),t)} \\ = \lim_{t \rightarrow \infty} \frac{E \int_0^t f''(|X_s - X_{\sigma_s}|_-) I_{(a,b)}(X_s) ds}{E C((a,b),t)} \\ = f(b-a) - f(0) - f'(0)(b-a). \end{aligned}$$

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