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PLANAR SEMIMARTINGALES OBTAINED BY TRANSFORMATIONS OF TWO-PARAMETER MARTINGALES

By

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Abstract. In this paper we study the weak local submartingale property and the quasimartingale property of processes obtained by the composition of a two-parameter continuous martingale by $\mathcal{C}^2$-functions whose second derivative is convex.

The work of Nguyen Minh Duc was done during a stay at the University of Barcelona (Spain).

0. INTRODUCTION

In this paper we are concerned with some properties of the process obtained by the composition of a two-parameter martingale with a $\mathcal{C}^2$-class function.

In the one-parameter case the composition of a martingale with a convex function gives a local submartingale. On the other hand, the semimartingale property is preserved under transformation by convex functions, as can be proved by means of a general version of Tanaka's formula ([5]).

Some results in this direction have been given for two-parameter processes. In a previous work ([6], [7]) it has been established that the convexity of f'' implies the weak submartingale property of $f(M)$ under some hypotheses on f and the martingale M.

The notion of semimartingale does not possess up to date an equivalent in the two-parameter theory. For this reason it seems more convenient to deal with the quasi-martingale property (see [1], [4]), which can be expressed in terms of the total variation of the Doléans-Föllmer measure associated to the process. In [7] an explicit expression for the variation of $f(W)$ has been obtained, where f is a $\mathcal{C}^2$-class function with some requirements, and W the Brownian sheet. In particular, we have sufficient conditions for $f(W)$ to possess the quasi-martingale property.

The aim of this paper is to generalize the above mentioned results. One part, section 2, deals with the weak submartingale property of $f(M)$ in the local sense, when f is a $\mathcal{C}^2$-function whose second derivative is convex and positive. This property is obtained using two different methods. The first one consists in proving a suitable two-parameter Itô formula for $\mathcal{C}^2$-functions and continuous martingales bounded in L^2, with path independent variation. The second one is based on the compact Itô formula of [13] and uses a regularization procedure. It requires M to be bounded in L^4, but the martingale may belong to a strictly larger class than before.

In Section 3 we study the quasi-martingale property of $f(M)$, when f is a $\mathcal{C}^2$-function such that f'' is the difference of two convex functions. We prove a formula for the variation of $f(M)$ (see Theorem 3.1) involving the local time, and this allow us to state a necessary and sufficient condition to ensure the quasi-martingale property.

1. PRELIMINARIES AND NOTATION

The parameter space is $T = [0, 1]^2$ endowed with the partial ordering $(s_1,t_1) \le (s_2,t_2)$ if and only if $s_1 \le s_2$, $t_1 \le t_2$; $(s_1,t_1) < (s_2,t_2)$ means $s_1 < s_2$ and $t_1 < t_2$. If f is a map from T to $\mathbb{R}$, the increment of f on a rectangle $(z_1, z_2] = \{z \in T, z_1 < z \le z_2\}$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ is $f((z_1, z_2]) = f(z_2) - f(s_1, t_2) - f(s_2, t_1) + f(z_1)$. For any $z \in T$ we define $R_z = [0, z]$.

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ together with an increasing family of sub-σ-fields of $\mathcal{F}$, $(\mathcal{F}_z)_{z\in T}$ satisfying the usual conditions (F 1) to (F 4) of [2].

Besides the classical notion of martingale, in the two-parameter case, other related definitions can be given. If $M = \{M_z, z \in T\}$ is a real valued, integrable and $\mathcal{F}_z$-adapted process, M is a *i-martingale* $(i = 1, 2)$ if for any $z \le z'$, $z = (s, t)$, $E\{M((z, z'])/\mathcal{F}^i_z\} = 0$, where $\mathcal{F}^1_z$ (resp. $\mathcal{F}^2_z$) is the σ-field $\mathcal{F}_{s1}$ (resp. $\mathcal{F}_{1t}$). M is said to be a *weak martingale* (resp. *weak submartingale*) if $E\{M(z, z']/\mathcal{F}_z\} = 0$ (resp. ≥ 0).

Let $\mathfrak{m}^p$ (resp. $\mathfrak{m}^p_c$) be the class of two-parameter martingales (resp. continuous martingales) bounded in L^p, $p \ge 1$. It is well known (see [2]) that for $M \in \mathfrak{m}^2$, there exists an increasing, predictable process $<M>$, called the *quadratric variation* of M, such that $M^2 - <M>$ is a weak martingale. Moreover, it has been proved in [8] that if $M \in \mathfrak{m}^2_c$, $<M>$ has a continuous modification.

If $M \in \mathfrak{m}^2_c$, and vanishes on the axes we say that it has *path-independent variation* (p.i.v.) if $<M_{.t}>_s = <M_{s.}>_t = <M>_{st}$.

Consider a grid $\mathcal{S}$ of T given by

$$\mathcal{S} = \{(s_i, t_j) \in T,\ i = 0,\dots, p;\ j = 0,\dots,q;\ 0 = s_o < s_1 < \dots < s_p < 1,\ 0 = t_o < t_1 < \dots < t_q < 1\}.$$

For any $(s_i, t_j) = z_{ij} \in \mathcal{S}$ we define

$$\Delta_{ij} = (s_i, s_{i+1}] \times (t_j, t_{j+1}],\ \overset{1}{\Delta}_{ij} = (s_i, s_{i+1}] \times (0, t_j],\ \overset{2}{\Delta}_{ij} = (0, s_i] \times (t_j, t_{j+1}],$$

with the convention $s_{p+1} = t_{q+1} = 1$.

Throughout this paper we will deal with an increasing sequence of grids $\{\mathcal{S}^n, n \ge 1\}$ of T, whose norm tends to zero (i.e. $|\mathcal{S}^n| = \max\{|s_{i+1} - s_i| + |t_{j+1} - t_j|, (s_i, t_j) \in \mathcal{S}^n\} \xrightarrow[n\to\infty]{} 0$), and for any $z \in T$ we define $I^n_z = \{(i,j) \in \mathbb{N}^2, (s_i,t_j) \in \mathcal{S}^n, (s_i, t_j) < z\}$. I^n will denote the set $\{(i,j) \in \mathbb{N}^2, (s_i, t_j) \in \mathcal{S}^n\}$. I_z and I are defined in an analogous way, but referred to $\mathcal{S}$.

In the Doob-Meyer decomposition of a martingale $M \in \mathfrak{m}^2_c$ (and therefore in the Itô formulas for f(M)) we encounter a martingale $\tilde{M}$, obtained as the L^1-limit of the sequence

$$\sum_{(i,j)\in I^n_z} M(\overset{1}{\Delta}_{ij})\, M(\overset{2}{\Delta}_{ij})$$

(cf. e.g. [8] and [9]).

For martingales M of $\mathfrak{m}^4_c$, null on the axes, it has been proved in [10] that the property p.i.v. implies $<M, \tilde{M}> = 0$. The reciproque is not true in general.

For any integrable, adapted process $X = \{X_z, z \in T\}$ and any rectangle $(z_1, z_2]$, $z_1 \le z_2$, we define

$$\mathrm{Var}^S_{(z_1, z_2]} \ X = \sum_{(i,j)\in I} E \left| E \{X(\Delta_{ij} \cap (z_1, z_2]) / \mathcal{F}_{z_{ij}}\} \right| .$$

When $(z_1, z_2] = (0, 1]^2$ we simply write $\mathrm{Var}^S X$. We also define

$$\mathrm{Var}_{(z_1, z_2]} \ X = \sup_S \ \mathrm{Var}^S_{(z_1, z_2]} \ X,$$

and $\mathrm{Var}\ X = \sup_S \mathrm{Var}^S X$.

The total variation of the Doléans-Föllmer measure of the process X on $(z_1, z_2] \times \Omega$ coincides with $\mathrm{Var}_{(z_1, z_2]} X$. Then, following [4] the process $X = \{X_z, z \in T\}$ is said to be a *planar quasimartingale* if $\mathrm{Var}\ X < \infty$.

Sometimes it is convenient to localize the above definitions. This can be done by means of the notion of *stopping domain.* A set $D \subset T \times \Omega$ is a stopping domain if 1_D is a progressively measurable process, and $R_z \subset D(\omega)$, whenever $z \in D(\omega)$. For any class $\mathcal{C}$ of processes, $\mathcal{C}_{loc}$ will denote the class of processes X such that there exists a sequence $\{X^n, n \geq 1\}$, $X^n \in \mathcal{C}$, and an increasing sequence $\{D_n, n \geq 1\}$ of stopping domains, with $\bigcup_n D_n = T$, such that for any n, $X^n_z = X_z$, if $z \in D_n$.

Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ we introduce a sequence of smooth functions $\{f_m, m \geq 1\}$ by means of a sequence of regularization kernels of the form

$$\alpha_m(x) = m \ \ \alpha(m\, x), \tag{1.1}$$

where $\alpha \in \mathcal{C}^\infty_0(\mathbb{R})$ is a nonnegative function whose compact support is contained in the interval $(-\infty, 0]$ and such that $\int_{\mathbb{R}} \alpha(x)\, dx = 1$. We take

$$f_m(x) = (f * \alpha_m)\,(x) = \int_{\mathbb{R}} f(x + y)\, \alpha_m(y)\, dy\ . \tag{1.2}$$

2. WEAK SUBMARTINGALE PROPERTY OF F(M)

In this section we study the weak submartingale property of $f(M)$, f being a real function and M a two-parameter martingale. Two slightly different results in this direction are presented. The first one requires M to be a martingale belonging to $\mathcal{M}^2_c$ and of path independent variation; its proof is based on a special two-parameter Itô's formula for

$\mathcal{C}^2$-functions. The second one assumes that M is in $\mathfrak{M}_c^4$ and $\langle M, \tilde{M}\rangle = 0$. The proof uses the compact Itô formula of [13] and a regularization procedure.

We first state a deterministic result.

Lemma 2.1. Let X and Y be two functions in C(T) and assume that Y is increasing, in the measure sense. Consider the sequences defined by

$$\varphi_n(z) = \sum_{(i,j)\in I_z^n} \left(X_{s_i, t_{j+1}\wedge t} - X_{s_i, t_j}\right)\left(Y_{s_{i+1}\wedge s, t_j} - Y_{s_i, t_j}\right), \tag{2.1}$$

$$\begin{aligned}\psi_n(z) &= \sum_{(i,j)\in I_z^n} \int_{s_i}^{s_{i+1}\wedge s} \left(X_{\sigma, t_{j+1}\wedge t} - X_{\sigma, t_j}\right) d_\sigma Y_{\sigma, t_j} \\ &= \sum_{\{j,\, t_j<t\}} \int_0^s \left(X_{\sigma, t_{j+1}\wedge t} - X_{\sigma, t_j}\right) d_\sigma Y_{\sigma, t_j}.\end{aligned} \tag{2.2}$$

Then $\{\varphi_n, n \geq 1\}$ and $\{\psi_n, n \geq 1\}$ converge to the same function $X * Y \in C(T)$.

Proof: We first remark that $\varphi_n(z)$ and $\psi_n(z)$ can be written in the following alternative way

$$\begin{aligned}\varphi_n(z) &= \sum_{\{i,\, o\leq s_i<s\}} X_{s_i, t}\left(X_{s_{i+1}\wedge s, t} - Y_{s_i, t}\right) - \sum_{\{i,\, o\leq s_i<s\}} X_{s_i, o}\left(Y_{s_{i+1}\wedge s, o} - Y_{s_i, o}\right) \\ &\quad - \sum_{\{j,\, o<t_j\leq t\}} \sum_{\{i,\, s_i<s\}} X_{s_i, t_j} \, Y(\Delta_{ij-1}).\end{aligned} \tag{2.3}$$

$$\begin{aligned}\psi_n(z) &= \int_0^s X_{\sigma, t}\, d_\sigma Y_{\sigma, t} - \int_0^s X_{\sigma, o}\, d_\sigma Y_{\sigma, o} \\ &\quad - \sum_{\{j,\, o<t_j\leq t\}} \int_0^s X_{\sigma, t_j}\, d_\sigma\left(Y_{\sigma, t_j} - Y_{\sigma, t_{j-1}}\right).\end{aligned} \tag{2.4}$$

Therefore, we obtain

$$\sup_{z\in T} \left|\varphi_n(z) - \psi_n(z)\right| \leq C \sup_{|u-v|\leq |S^n|} \left|X_u - X_v\right| Y_{1,1}. \tag{2.5}$$

On the other hand $\{\psi_n, n \geq 1\}$ is a Cauchy sequence. Indeed, take $m > n$ and suppose that $\mathcal{S}^m = \{(\sigma_{i'}, \tau_{j'}) \in T,\ \ i' = 0,\dots,p_m,\ \ j' = 0,\dots,q_m\}$. For every $j = 0,\dots,q_n$ define $J_j = \{j', \tau_{j'} \in (t_j, t_{j+1}]\}$, then, using (2.4) we obtain

$$\sup_{z\in T} \left| \psi_n(z) - \psi_m(z) \right|$$

$$= \sup_{z\in T} \left| \sum_{\{j,\ 0<t_j\le t\}} \ \sum_{j'\in J_j} \int_0^s [X_{\sigma,t_j} - X_{\sigma,\tau_{j'}}]\, d_\sigma \left(Y_{\sigma,\tau_{j'}} - Y_{\sigma,\tau_{j'-1}}\right) \right|$$

$$\le C \sup_{|u-v|\le|\mathcal{S}^n|} |X_u - X_v|\, Y_{1,1},$$

which tends to zero as $n \to \infty$, uniformly in m.

This fact together with inequality (2.5) proves the Lemma. □

Let us now prove a particular Itô formula.

Theorem 2.2. Let M be a martingale belonging to $\mathfrak{m}_c^2$ such that $M^2 - \langle M\rangle$ is a martingale. For any real function f of class $\mathcal{C}^2$, and every $z = (s, t) \in T$ we have

$$f(M_z) = f(M_{o,t}) + \int_0^s f'(M_{\sigma,t})\, d_\sigma M_{\sigma,t} + \frac{1}{2} \int_0^s f''(M_{\sigma,o})\, d_\sigma \langle M\rangle_{\sigma,o}$$

$$+ \frac{1}{2} \int_{R_z} f''(M_u)\, d\langle M\rangle_u + \frac{1}{2} \left[f''(M) * \langle M\rangle\right]_z . \tag{2.6}$$

Remark: The assumptions on M are equivalent to the property of p.i.v. if M vanishes on the axes (see [10]).

Proof: To simplify the notation we take $z = (1, 1)$. By Itô's formula we can write

$$f(M_{1,1}) = f(M_{o,1}) + \int_0^1 f'(M_{\sigma,1})\, d_\sigma M_{\sigma,1} + \frac{1}{2} \int_0^1 f''(M_{\sigma,1})\, d_\sigma \langle M\rangle_{\sigma,1} .$$

We have

$$\int_0^1 f''(M_{\sigma,1})\, d_\sigma \langle M\rangle_{\sigma,1} - \int_0^1 f''(M_{\sigma,o})\, d_\sigma \langle M\rangle_{\sigma,o}$$

$$= \sum_{(i,j)\in I^n} \Big(\int_{s_i}^{s_{i+1}} f''(M_{\sigma, t_{j+1}})\, d_\sigma \langle M\rangle_{\sigma, t_{j+1}} - \int_{s_i}^{s_{i+1}} f''(M_{\sigma, t_j})\, d_\sigma \langle M\rangle_{\sigma, t_j} \Big)$$

$$= a_n + b_n + c_n,$$

where:

$$a_n = \sum_{(i,j)\in I^n} f''(M_{z_{ij}}) \langle M\rangle (\Delta_{ij}),$$

$$b_n = \sum_{(i,j)\in I^n} \int_{s_i}^{s_{i+1}} \left[f''(M_{\sigma,t_{j+1}}) - f''(M_{\sigma,t_j}) \right] d_\sigma \langle M\rangle_{\sigma,t_j},$$

and

$$c_n = \sum_{(i,j)\in I^n} \int_{s_i}^{s_{i+1}} \left[f''(M_{\sigma,t_{j+1}}) - f''(M_{z_{ij}}) \right] d_\sigma \left(\langle M\rangle_{\sigma,t_{j+1}} - \langle M\rangle_{\sigma,t_j} \right).$$

The sequence a_n converges a.s. to $\int_T f''(M_u)\, d\langle M\rangle_u$, as $n\to\infty$.

By Lemma 2.1 applied to the paths of $X = f''(M)$ and $Y = \langle M\rangle$, we obtain $b_n \longrightarrow (f''(M) * \langle M\rangle)_{1,1}$ as $n\to\infty$, a.s. Finally, since

$$|c_n| \le \sup_{|u-v|\le|S^n|} |f''(M_u) - f''(M_v)| \langle M\rangle (R_{1,1}),$$

$\lim_{n\to\infty} c_n = 0$ a.s., and the theorem is established. □

In order to study the weak submartingale property of f(M) we need the following lemma.

Lemma 2.3. Let φ be a real convex function and M a martingale in $\mathcal{M}_c^2$. The continuous process $\varphi(M) * \langle M\rangle$ is a local weak submartingale.

Proof: For any positive integer k, define

$$D_k = \Big\{ (z, \omega),\ \sup_{u\in R_z} |M_u(\omega)| \le k \Big\}, \tag{2.7}$$

and put $N = \varphi(M) * \langle M\rangle$. We have to show the existence of a sequence of weak submartingales $\{N^k,\ k \geq 1\}$ such that, for any $k \geq 1$,

$$1_{D_k}(z)\ N_z = 1_{D_k}(z)\ N_z^k\ . \tag{2.8}$$

Let $z = (s, t)$. For each $i \in \mathbb{N}$ such that, $s_i < s$, define

$$T_i^k = \inf\ \{\tau,\ M_{s_i\,\tau} \notin D_k\}\ ,$$

this is a stopping time with respect to the filtration $\{\mathcal{F}_{1t},\ t \in [0,1]\}$, and we have

$$1_{D_k}(z) \sum_{(i,j)\in I_z^n} \Big[\ \varphi(M_{s_i,\,t_{j+1}\wedge t}) - \varphi(M_{s_i,\,t_j})\Big]\Big[\ \langle M\rangle_{s_{i+1}\wedge s,\,t_j} - \langle M\rangle_{s_i,\,t_j}\Big]$$

$$= 1_{D_k}(z) \sum_{(i,j)\in I_z^n} \Big[\ \varphi(M^{T_i^k}_{s_i,\,t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i,t_j})\Big]\Big[\langle M\rangle_{s_{i+1}\wedge s,\,t_j} - \langle M\rangle_{s_i,\,t_j}\Big]\ ,$$

where $\{M^{T_i^k}_{s_i,\,t}\ ,\ t \geq 0\ \}$ denotes the martingale, $M_{s_i,\,t}$ stopped at T_i^k.

Using the convexity of φ we obtain

$$E\left\{\Big[\varphi(M^{T_i^k}_{s_i,\,t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i,\,t_j})\Big]\Big[\langle M\rangle_{s_{i+1}\wedge s,\,t_j} - \langle M\rangle_{s_i,\,t_j}\Big]\ /\mathcal{F}_{z_{ij}}\right\}$$

$$= E\left\{\Big[\langle M\rangle_{s_{i+1}\wedge s,\,t_j} - \langle M\rangle_{s_i,\,t_j}\Big]\ E\ \left\{\varphi(M^{T_i^k}_{s_i,\,t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i,\,t_j})/\mathcal{F}_{1,\,t_j}\right\}/\mathcal{F}_{z_{ij}}\right\}$$

$$\geq\ 0.$$

Hence $J_z^{n,k} = \sum_{(i,j)\in I_z^n} \Big[\varphi(M^{T_i^k}_{s_i,\,t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i,\,t_j})\Big]\Big[\langle M\rangle_{s_{i+1}\wedge s,\,t_j} - \langle M\rangle_{s_i,\,t_j}\Big]$ is a weak submartingale.

Notice that $\{J_z^{n,k},\ n \geq 1\}$ is uniformly integrable, therefore by Lemma 2.1 $L^1\text{-}\lim_{n\to\infty} J_z^{n,k}$ exists and defines a weak submartingale. Finally since

$$1_{D_k}(z)\, N_z = 1_{D_k}(z)\ L^1\text{-}\lim_{n\to\infty} J_z^{n,k}\ ,$$

(2.8) holds with $N_z^k = L^1\text{-}\lim\ J_z^{n,k}$, and the proof of the Lemma is complete. □

We can now state the main result of this section.

Theorem 2.4. Let f be a $\mathcal{C}^2$ function such that f'' is convex and positive. Then, if M is a martingale in $\mathcal{M}_c^2$ such that $M^2 - \langle M\rangle$ is a martingale, the process $\{f(M_z), z \in T\}$ is a *local weak submartingale*.

Proof: It is an immediate consequence of Theorem 2.2 and lemma 2.3 applied to $\varphi = f''$. □

The conclusion of Theorem 2.4 can also be obtained by a more direct approach, using the compact Itô formula of [13]. Here the martingale M is supposed to belong to $\mathcal{M}_c^4$, but the path independent variation property can be replaced by a weaker one. More precisely we can prove the following result:

Theorem 2.5. Let f be a $\mathcal{C}^2$ function whose second derivative is convex and positive. Let $M \in \mathcal{M}_c^4$, null on the axes and such that $\langle M, \tilde{M}\rangle = 0$. Then the process $\{f(M_z), z \in T\}$ is a *local weak submartingale*.

Proof: (a) Let us first prove the theorem under stronger hypotheses on f: Assume that $f \in \mathcal{C}^4(\mathbb{R})$, f'' convex and positive. By the Itô formula proved in [13] we have

$$
\begin{aligned}
f(M_{s,t}) = {} & f(0) + \int_{R_{s,t}} f'(M_z)\, dM_z \\
& + \int_{R_{s,t}} \left[f''(M_z)\, d\tilde{M}_z + f''(M_z)\, dS_z^{(1)} + f''(M_z)\, dS_z^{(2)} + \frac{1}{2} f''(M_z)\, d\langle M\rangle_z \right] \\
& + \int_{R_{s,t}} \left[\frac{1}{2} f'''(M_z)\, dW_z^{(1)} + \frac{1}{2} f'''(M_z)\, dW_z^{(2)} \right] + \frac{1}{4} \int_{R_{s,t}} f^{iv}(M_z)\, d\langle \tilde{M}\rangle_z . \qquad (2.9)
\end{aligned}
$$

We recall that $\{S_z^{(1)}, z \in T\}$ (resp. $\{S_z^{(2)}, z \in T\}$) is a continuous 1-martingale (resp. 2-martingale) obtained as the L^2-limit of the sequence $\{\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^2)\, M(\Delta_{ij}),\ z \in T;\ n \geq 1\}$ (resp. $\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^1)\, M(\Delta_{ij}), z \in T;\ n \geq 1\}$), and $\{W_z^{(1)}, z \in T\}$, (resp. $\{W_z^{(2)}, z \in T\}$) is the continuous 1-martingale (resp. 2-martingale) obtained by the L^1-limit of the sequence $\{\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^2)^2 M(\Delta_{ij}^1),\ z \in T;\ n \geq 1\}$ (resp. $\{\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^1)^2\, M(\Delta_{ij}^2),\ z \in T;\ n \geq 1\}$) .These processes are two-parameter stochastic integrators.

The properties on f ensure that the process defined by

$$Z_{s,t} = \int_{R_{s,t}} \left[\frac{1}{2} f''(M_z)\, d\langle M\rangle_z + \frac{1}{4} f^{iv}(M_z)\, d\langle\widetilde{M}\rangle_z\right],$$

is increasing in the measure sense.

Consider the increasing sequence $\{D_k, k \geq 1\}$ of stopping domanins defined by 2.7. For $(s, t) \in D_k$, $f(M_{s,t})$ coincides with the weak submartingale given by

$$\int_{R_{s,t}} 1_{D_k}(z) \Big\{ f'(M_z)\, dM_z + f''(M_z)\, d\widetilde{M}_z + f''(M_z)\, dS_z^{(1)} + f''(M_z)\, dS_z^{(2)} + \frac{1}{2} f''(M_z)\, d\langle M\rangle_z$$

$$+ \frac{1}{2} f'''(M_z)\, dW_z^{(1)} + \frac{1}{2} f'''(M_z)\, dW_z^{(2)} + \frac{1}{4} f^{iv}(M_z)\, d\langle\widetilde{M}\rangle_z \Big\},$$

and consequently the assertion is proved.

(b) In the general case, we consider a sequence $\{\alpha_n(x), n \geq 1\}$ of regularitzation kernels of the form (1.1), and the corresponding $\{f_n, n \geq 1\}$.

Each f_n satisfies the hypotheses of part (a), thus, $\{f_n(M_z), z \in T\}$ is a local weak submartingale, for any $n \geq 1$.

Consider the martingales $m^k \in \mathfrak{M}_c^4$ and $\widetilde{m}_k \in \mathfrak{M}_c^2$ given by

$$m_{s,t}^k = \int_{R_{s,t}} 1_{D_k}(z)\, dM_z, \quad \widetilde{m}_{s,t}^k = \int_{R_{s,t}} 1_{D_k}(z)\, d\widetilde{M}_z .$$

Notice that

$$\langle m^k, \widetilde{m}^k\rangle_{\cdot} = \int_{R_{\cdot}} 1_{D_k}(z)\, d\langle M, \widetilde{M}\rangle_z = 0,$$

so, (2.9) still holds with f replaced by f_n and M by m^k, and it follows that $\{f_n(m_z^k), z \in T\}$ is, for any $n \geq 1$, a weak submartingale.

Then, since $1_{D_k}(z)\, f_n(M_z) = 1_{D_k}(z)\, f_n(m_z^k)$, taking account of the convergences

$$1_{D_k}(z)\, f_n(M_z) \xrightarrow[n\to\infty]{L^1} 1_{D_k}(z)\, f(M_z),$$

and

$$1_{D_k}(z)\, f_n(m_z^k) \xrightarrow[n\to\infty]{L^1} 1_{D_k}(z)\, f(m_z^k),$$

we obtain

$$1_{D_k}(z)\, f(M_z) = 1_{D_k}(z)\; f(m_z^k) \;,$$

where $f(m_z^k)$ is a weak submartingale. This finishes the proof of the Theorem. □

3. QUASIMARTINGALE PROPERTY OF f(M)

The second part of this paper is devoted to study the quasimartingale property of the transformation of a two-parameter martingale by a $\mathcal{C}^2$-function.

In order to carry out our programm we will state a formula for the total variation of the Doléans-Föllmer measure (or conditional variation) of $f(M)$. This formula involves the local time of the martingale M and is obtained by means of the compact Itô's formula proved in [13].At the same time this provides a necessary and sufficient condition for $f(M)$ to be a quasimartingale. We refer the reader to [14] for analogue results in the one-parameter setting.

In the sequel the martingale M is supossed to be null on the axes.

Assume that $M \in \mathfrak{m}_c^4$ satisfies the following hypothesis:

(H 1) The measure $\langle M\rangle$ is absolutely continuous with respect to the product of its marginals.

It has been proved in [11] (see Corollary 4.2) that, under (H 1), $\langle M\rangle$ is absolutely continuous with respect to $\langle\tilde{M}\rangle$, a.s. Futhermore, since the local time L of the martingale M with respect to the measure $\langle\tilde{M}\rangle$ always exist (Theorem 3.1, [9]), so does the local time L of M with respect to $\langle M\rangle$ (cf. Lemma 5.1, [11]), and we have

$$L\,(x, A) = \int_A \phi(u)\; \tilde{L}\,(x, du), \qquad \text{a.s.,} \tag{3.1}$$

ϕ being the Radon-Nikodym derivative $\dfrac{d\,\langle M\rangle}{d\,\langle\tilde{M}\rangle}$.

Following [14] a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be of class (β) if its second derivative in the distributional sense is a bounded measure ν (i.e. $\int_{\mathbb{R}} |\, \nu\,(dx)\,| < \infty$).

We can now state our main result:

Theorem 3.1. Let M be a martingale belonging to $\mathfrak{m}_c^p$, for some $p > 4$. Assume that M satisfies (H 1), and that $\langle M, \tilde{M}\rangle = 0$. Let $f \in \mathcal{C}^2$, such that $f''$is of class (β). Then, if we denote by ν the second derivative of f'' in the distributional sense, we have, for any $z_1, z_2 \in T$, $z_1 < z_2$,

$$\mathrm{Var}_{(z_1,z_2]}\, f(M) = E \int_{(z_1,z_2]\times\mathbb{R}} \left\{ \left| \frac{1}{2} f''(a)\ \phi(z) + \frac{1}{4}\ \varphi(a) \right| \widetilde{L}(a, dz)\, da + \frac{1}{4}\, \widetilde{L}(a, dz)\, |\nu^s|\,(da) \right\} \tag{3.2}$$

where $\nu(da) = \varphi(a)\, da + \nu^s(da)$ is the Lebesgue decomposition of ν in its absolutely continuous and singular part, with respect to the Lebesgue measure on $\mathbb{R}$.

Hence, $f(M)$ is a quasimartingale if and only if

$$E \int_T |\, d A_z| < \infty ,$$

where

$$A_z = \int_{R_z\times\mathbb{R}} \left(\frac{1}{2}\ f''(a)\ \phi(u)\ da\ +\ \frac{1}{4}\ \nu(da)\right) \widetilde{L}(a, du), \tag{3.3}$$

$z \in T$.

Proof: (1) The hypotheses on f imply the existence of some constants α, β such that

$$f''(x) = \alpha\ x + \beta + \int_0^x dy \int_{(-\infty,\, y]} \nu\,(dz).$$

Consequently f'' is the difference of two convex functions of class (β), φ^1 and φ^2, and the following conditions hold:

$$\begin{aligned} &|\, f'(x)\ | \leq a_1\, |\, x\, |^2 + b_1\, |\, x\, | + c_1\ , \\ &|\, f''(x)\ | \leq a_2\, |\, x\, | + b_2\, , \\ &|\, f'''(x)\, | \leq a_3\ . \end{aligned} \tag{3.4}$$

Consider the sequence $\{f_m,\ m \geq 0\}$ defined by (1.2). We remark that f_m'' is the difference of two convex functions (φ_m^1 and φ_m^2), and that $(\varphi_m^i)' \uparrow (\varphi^i)'$, $i = 1, 2$. (Here we are dealing with the left derivatives). Furthermore

$$\begin{aligned} &|\, f_m^{(k)}\,(x)\, | \leq \sup_{|y|\leq 1}\ |\, f^{(k)}\,(x+y)\, |\, , && \text{for}\quad k = 1,2 \\ &|\, f_m^{(k)}\,(x)\, | \leq C\, , && \text{for}\quad k = 3,4 \end{aligned} \tag{3.5}$$

(2) We apply the compact Itô's formula of [13] to $f_m(M)$. Then, we obtain

$$f_m(M_{s,t}) = f_m(0) + \int_{R_{s,t}} f'_m(M_z)\, dM_z$$

$$+ \int_{R_{s,t}} \left\{ f''_m(M_z)\, d\widetilde{M}_z + f''_m(M_z)\, d S^{(1)}_z + f''_m(M_z)\, d S^{(2)}_z + \frac{1}{2} f''_m(M_z)\, d <M>_z \right\}$$

$$+ \int_{R_{s,t}} \frac{1}{2} f'''_m(M_z)\, dW^{(1)}_z + \frac{1}{2} f'''_m(M_z)\, dW^{(2)}_z$$

$$+ \int_{R_{s,t}} \frac{1}{4} f^{'v}_m(M_z)\, d <\widetilde{M}>_z . \tag{3.6}$$

Since $M \in \mathfrak{m}^p_c$ for $p > 4$, and due to conditions (3.5), every term in the right-hand side of (3.6) belongs to L^1. This is obvious for $\int_{R_{s,t}} f''_m(M_z)\, d <M>_z$ and $\int_{R_{s,t}} f^{'v}_m(M_z)\, d <\widetilde{M}>_z$. In order to check this property for the stochastic integrals it suffices to prove that the corresponding sequences of Riemann sums are bounded in $L^{1+\varepsilon}$, for some $\varepsilon > 0$. Consider, for example, the sequence $\{ \sum_{(i,j)\in I^n_{s,t}} f''(M_{z_{ij}})\, M(\Delta^1_{ij})\, M(\Delta^2_{ij}),\ n \geq 1\}$. By Burkholder's inequality we have

$$E \left| \sum_{(i,j)\in I^n_{s,t}} f''_m(M_{z_{ij}})\, M(\Delta^1_{ij})\, M(\Delta^2_{ij}) \right|^{1+\varepsilon}$$

$$\leq C\, E \left(\sum_{(i,j)\in I^n_{s,t}} f''_m(M_{z_{ij}})^2\, M(\Delta^1_{ij})^2 M(\Delta^2_{ij})^2 \right)^{\frac{1+\varepsilon}{2}}$$

$$\leq C \left\{ E \left\{ \sup_{z\in T} | f''_m(M_z) |^{2(1+\varepsilon)} \right\}^{1/2} \left\{ E \left\{ \sup_i \left(\sum_j M(\Delta^1_{ij})^2 \right)^{2(1+\varepsilon)} \right\} \right. \right. .$$

$$\left. . \, E \left\{ \sum_i \sup_j M(\Delta^2_{ij})^2 \right\}^{2(1+\varepsilon)} \right\}^{1/4}$$

$$\leq \left\{ C_1 + C_2\, E\, | M_{11} |^{2(1+\varepsilon)} \right\}^{1/2} \left\{ E\, M^{4(1+\varepsilon)}_{11} \right\}^{1/2} ,$$

where the last inequality has been obtained using (3.5), Doob's maximal inequality and Lemma 2.2. of [11].

The same kind of arguments are used for the other terms.

(3) Let $\Delta = (z, z']$, $z \leq z'$. The results proved in part (2) show that

$$E \left[f_m(M)(\Delta)/\mathcal{F}_z \right] = E \left\{ \int_\Delta \frac{1}{2} f''_m(M_u)\, d<M>_u + \frac{1}{4} f^{iv}_m(M_u)\, d<\widetilde{M}>_u / \mathcal{F}_z \right\}, \tag{3.7}$$

and using the density of occupation formula

$$E\left[f_m(M)(\Delta)/\mathcal{F}_z\right] = E\left\{\int_{R\times\Delta} \frac{1}{2} f_m''(a) L(a,dz)\, d a + \frac{1}{4} f_m'^{v}(a)\, \tilde{L}(a, dz)\, da/\mathcal{F}_z\right\}. \quad (3.8)$$

It is simple to verify that

$$E\left[f_m(M)(\Delta)/\mathcal{F}_z\right] \xrightarrow{L^1} E\left[f(M)(\Delta)/\mathcal{F}_z\right], \quad (3.9)$$

and

$$E\left[\int_{R\times\Delta} \frac{1}{2} f_m''(a)\; L(a, dz)\; da/\mathcal{F}_z\right] \xrightarrow[m\to\infty]{L^1} E\left[\int_{R\times\Delta} \frac{1}{2} f''(a)\; L(a, dz)\, da/\mathcal{F}_z\right].$$

Denote by ν_i the measure on $(\mathbb{R}, \mathcal{B}, (\mathbb{R}))$ whose distribution function is given by

$$\nu_i\left([a,b)\right) = (\varphi^i)'(b) - (\varphi^i)'(a), \qquad \text{for all } a < b, \text{ and } i = 1, 2.$$

We have

$$(\varphi^i_m)'(b) - (\varphi^i_m)'(a) = \int_a^b (\varphi^i_m)''(x)\, dx \longrightarrow (\varphi^i)'(b) - (\varphi^i)'(a) = \nu_i\left([a, b)\right).$$

Therefore, on any compact set $K \subset \mathbb{R}$, the finite measures $\{\nu^i_m,\ m \geq 1\}$ whose distribution functions are $(\varphi^i_m)'$, $m \geq 1$, converge weakly to ν_i.

By continuity, the path $\{M_z(\omega), z \in T\}$ can only visit the points of a compact set, therefore the integral

$$\int_{\mathbb{R}} f_m'^{v}(a)\; \tilde{L}(a, \Delta)\, da \quad \left(= \int_{\mathbb{R}} (\varphi^1_m - \varphi^2_m)''\; \tilde{L}(a, \Delta)\, da\right)$$

is extended on a compact set K, and the weak convergence of $\{\nu^i_m,\ m \geq 1\}$ to ν^i entails

$$\int_{\mathbb{R}} f_m'^{v}(a) \tilde{L}(a, \Delta)\, da \xrightarrow[m\to\infty]{} \int_{\mathbb{R}} \tilde{L}(a, \Delta)\, \nu(da).$$

Using this fact and the convergences (3.9) we obtain from (3.8)

$$E\left[f(M)(\Delta)/\mathcal{F}_z\right] = E\left\{\int_{R\times\Delta} \frac{1}{2} f''(a)\; L(a, dz)\; da + \frac{1}{4} \tilde{L}(a, dz)\, \nu(da)/\mathcal{F}_z\right\}$$

$$= E \left\{ \int_{R \times \Delta} [\frac{1}{2} f''(a)\ \phi(z)\ \widetilde{L}(a, dz)\, da + \frac{1}{4} \widetilde{L}(a, dz)\ \nu(da) / \mathcal{F}_z \right\}. \qquad (3.10)$$

Thus, if $\{A_z,\ z \in T\}$ is the predictable, bounded variation process defined by (3.3), it follows that

$$\mathrm{Var}_{(z_1, z_2]}\ f(M) = \mathrm{Var}_{(z_1, z_2]}\ A\ . \qquad (3.11)$$

(4) The equality (3.11) allow us to prove formula (3.2). Indeed, it is known (see e.g. [3]) that, if X is a predictable process of bounded variation

$$\mathrm{Var}_{(z_1, z_2]}\ X = E \int_{(z_1, z_2]} |\, d X_z\, |\ .$$

Hence the proof of the theorem is now complete. □

Remark

For the special case of the Brownian sheet, Theorem 3.1 can be paraphrased as follows:

Theorem 3.2. Let $f \in \mathcal{C}^2$ be such that f'' is of class (β). Denote by ν the second derivative of f'' in the distributional sense, and let $\nu(da) = \varphi(a)\, da + \nu^s(da)$ the Lebesgue decomposition of ν with respect to the Lebesgue measure on $\mathbb{R}$.
Then, $f(W)$ is a quasimartingale if and only if

$$E \int_T |\, d A_z\, | < \infty\ ,$$

with

$$A_z = \int_{R_z} \int_R \left(\frac{1}{2} f''(a)\ da + \frac{1}{4}\ x \,.\, y\ \ \nu(da) \right) L(a, dx\, dy)\ ,$$

L being the local time of W with respect to the Lebesgue measure. Moreover

$$\mathrm{Var}_{(z_1, z_2]}\ f(W) = E \int_{(z_1, z_2] \times R} \left\{ \left| \frac{1}{2} f''(a) + \frac{1}{4}\ \varphi(a)\ x.y \right| da + \frac{1}{4}\ x.y\ |\nu^s|\,(da) \right\} L(a, dx\, dy),$$

for any z_1, z_2 in T, $z_1 < z_2$.

A result in the same direction has been obtained in [7], where the following formula is proved.

$$\mathrm{Var}_{(o,\,z]}\, f(W) = \frac{1}{2} \int_o^{x.y} \frac{1}{q}\, \mathrm{Var}_{(o,\,q]} \left(t\, f''(b_t)\right) dq \quad ,$$

for $z = (x, y)$, and where $\{b_t,\ t \geq 0\}$ is a Brownian motion.

REFERENCES

[1] Brennan, M. D. "Planar semimartingales". Journal of multivariate Analysis 9, 465-486 (1979).

[2] Cairoli, R. and Walsh, J. B. "Stochastic integrals in the plane". Acta Math. 134, 111-183 (1975).

[3] Dellacheire, C. et Meyer, P. A. "Probabilités et Potentiel". Vol. 2 Hermann. Paris (1980).

[4] Föllmer, H. " Quasimartingales à deux indices". C. R. Acad. Sc. Paris. Sér 1, t. 288, 61-64 (1979).

[5] Meyer, P. A. "Un cours sur les intégrales stochastiques". Sém. de Probab. X. Lecture Notes in Math. 511. Springer Verlag. Berlin-Heidelberg-New York (1980).

[6] Nguyen Minh Duc and Nguyen Xuan Loc. "On the transformation of a martingale with a two-dimensional parameter set by convex functions". Z. Wahr. un Verw. Gebiete 66, 19-24 (1984).

[7] Nguyen Minh Duc and Nguyen Xuan Loc. "Characterization of functions which transform Brownian sheet into planar semimartingales". Preprint Series n.º 26. Inst. of Math. and Inst. of Computer Scien. and Cybernetics. Hanoi 1985.

[8] Nualart, D. "On the quadratic variation of two-parameter continuous martingales". Annals of Probab. Vol 12, n.º 2, 445-457 (1984).

[9] Nualart, D. "Une formule d'Itô pour les martingales continues à deux indices et quelques applications". Ann. de l'Institut H. Poincaré Vol. 20, 3, 251-275 (1984).

[10] Nualart, D. and Utzet, F. "A property of two-parameter martingales with path-independent variation". Stoch. Proc. and their Appl. 24, 31-49 (1987).

[11] Nualart D., Sanz, M. and Zakai M. "On the relations between increasing functions associated with two-parameter continuous martingales". Tech. Report n.º 190. Center for Stochastic Processes. University of North Carolina at Chapel Hill (1987).

[12] Sanz, M. "Local time for two-parameter continuous martingales with respect to the quadratic variation". Annals of Probab. Vol 16, n.º 2, (1981).

[13] Sanz, M. "r-variations for two-parameter continuous martingales and Itô's formula" Preprint. (1987).

[14] Yoeurp, Ch. "Compléments sur les temps locaux et les quasimartingales". Astérisque 52-53. 197-218 (1978).

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