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MINH DUC NGUYEN
D. NUALART
M. SANZ

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PLANAR SEMIMARTINGALES OBTAINED BY TRANSFORMATIONS OF TWO-PARAMETER MARTINGALES

By

Nguyen Minh Duc[*], *D. Nualart*[**] and *M. Sanz*[**]

(*) Institute of Computer Science and Cybernetics. Hanoi-Vietnam.
(**) Universitat de Barcelona. Barcelona. Spain.

Abstract. In this paper we study the weak local submartingale property and the quasimartingale property of processes obtained by the composition of a two-parameter continuous martingale by $\mathcal{C}^2$-functions whose second derivative is convex.

The work of Nguyen Minh Duc was done during a stay at the University of Barcelona (Spain).

0. INTRODUCTION

In this paper we are concerned with some properties of the process obtained by the composition of a two-parameter martingale with a $\mathcal{C}^2$-class function.

In the one-parameter case the composition of a martingale with a convex function gives a local submartingale. On the other hand, the semimartingale property is preserved under transformation by convex functions, as can be proved by means of a general version of Tanaka's formula ([5]).

Some results in this direction have been given for two-parameter processes. In a previous work ([6], [7]) it has been established that the convexity of f'' implies the weak submartingale property of $f(M)$ under some hypotheses on f and the martingale M.

The notion of semimartingale does not possess up to date an equivalent in the two-parameter theory. For this reason it seems more convenient to deal with the quasi-martingale property (see [1], [4]), which can be expressed in terms of the total variation of the Doléans-Föllmer measure associated to the process. In [7] an explicit expression for the variation of $f(W)$ has been obtained, where f is a $\mathcal{C}^2$-class function with some requirements, and W the Brownian sheet. In particular, we have sufficient conditions for $f(W)$ to possess the quasi-martingale property.

The aim of this paper is to generalize the above mentioned results. One part, section 2, deals with the weak submartingale property of $f(M)$ in the local sense, when f is a $\mathcal{C}^2$-function whose second derivative is convex and positive. This property is obtained using two different methods. The first one consists in proving a suitable two-parameter Itô formula for $\mathcal{C}^2$-functions and continuous martingales bounded in L^2, with path independent variation. The second one is based on the compact Itô formula of [13] and uses a regularization procedure. It requires M to be bounded in L^4, but the martingale may belong to a strictly larger class than before.

In Section 3 we study the quasi-martingale property of $f(M)$, when f is a $\mathcal{C}^2$-function such that f'' is the difference of two convex functions. We prove a formula for the variation of $f(M)$ (see Theorem 3.1) involving the local time, and this allow us to state a necessary and sufficient condition to ensure the quasi-martingale property.

1. PRELIMINARIES AND NOTATION

The parameter space is $T = [0, 1]^2$ endowed with the partial ordering $(s_1,t_1) \le (s_2,t_2)$ if and only if $s_1 \le s_2$, $t_1 \le t_2$; $(s_1,t_1) < (s_2,t_2)$ means $s_1 < s_2$ and $t_1 < t_2$. If f is a map from T to $\mathbb{R}$, the increment of f on a rectangle $(z_1, z_2] = \{z \in T, z_1 < z \le z_2\}$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ is $f((z_1, z_2]) = f(z_2) - f(s_1, t_2) - f(s_2, t_1) + f(z_1)$. For any $z \in T$ we define $R_z = [0, z]$.

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ together with an increasing family of sub-σ-fields of $\mathcal{F}$, $(\mathcal{F}_z)_{z\in T}$ satisfying the usual conditions (F 1) to (F 4) of [2].

Besides the classical notion of martingale, in the two-parameter case, other related definitions can be given. If $M = \{M_z, z \in T\}$ is a real valued, integrable and $\mathcal{F}_z$-adapted process, M is a *i-martingale* (i = 1, 2) if for any $z \le z'$, $z = (s, t)$, $E\{M((z, z'])/\mathcal{F}^i_z\} = 0$, where $\mathcal{F}^1_z$ (resp. $\mathcal{F}^2_z$) is the σ-field $\mathcal{F}_{s1}$ (resp. $\mathcal{F}_{1t}$). M is said to be a *weak martingale* (resp. *weak submartingale*) if $E\{M(z, z']/\mathcal{F}_z\} = 0$ (resp. ≥ 0).

Let $\mathcal{M}^p$ (resp. $\mathcal{M}^p_c$) be the class of two-parameter martingales (resp. continuous martingales) bounded in L^p, $p \ge 1$. It is well known (see [2]) that for $M \in \mathcal{M}^2$, there exists an increasing, predictable process $<M>$, called the *quadratic variation* of M, such that $M^2 - <M>$ is a weak martingale. Moreover, it has been proved in [8] that if $M \in \mathcal{M}^2_c$, $<M>$ has a continuous modification.

If $M \in \mathcal{M}^2_c$, and vanishes on the axes we say that it has *path-independent variation* (p.i.v.) if $<M_{.t}>_s = <M_{s.}>_t = <M>_{st}$.

Consider a grid $\mathcal{S}$ of T given by

$$\mathcal{S} = \{(s_i, t_j) \in T,\ i = 0,\dots, p;\ j = 0,\dots,q;\ 0 = s_o < s_1<\dots<s_p<1,\ 0 = t_o < t_1<\dots<t_q<1\}.$$

For any $(s_i, t_j) = z_{ij} \in \mathcal{S}$ we define

$$\Delta_{ij} = (s_i, s_{i+1}] \times (t_j, t_{j+1}],\ \Delta^1_{ij} = (s_i, s_{i+1}] \times (0, t_j],\ \Delta^2_{ij} = (0, s_i] \times (t_j, t_{j+1}],$$

with the convention $s_{p+1} = t_{q+1} = 1$.

Throughout this paper we will deal with an increasing sequence of grids $\{\mathcal{S}^n, n \ge 1\}$ of T, whose norm tends to zero (i.e. $|\mathcal{S}^n| = \max\{|s_{i+1} - s_i| + |t_{j+1} - t_j|, (s_i, t_j) \in \mathcal{S}^n\} \xrightarrow[n\to\infty]{} 0$), and for any $z \in T$ we define $I^n_z = \{(i,j) \in \mathbb{N}^2, (s_i,t_j) \in \mathcal{S}^n, (s_i, t_j) < z\}$. I^n will denote the set $\{(i,j) \in \mathbb{N}^2, (s_i, t_j) \in \mathcal{S}^n\}$. I_z and I are defined in an analogous way, but referred to $\mathcal{S}$.

In the Doob-Meyer decomposition of a martingale $M \in \mathcal{M}^2_c$ (and therefore in the Itô formulas for f(M)) we encounter a martingale $\tilde{M}$, obtained as the L^1-limit of the sequence

$$\sum_{(i,j)\in I^n_z} M(\Delta^1_{ij})\, M(\Delta^2_{ij})$$

(cf. e.g. [8] and [9]).

For martingales M of $\mathcal{M}^4_c$, null on the axes, it has been proved in [10] that the property p.i.v. implies $<M, \tilde{M}> = 0$. The reciproque is not true in general.

For any integrable, adapted process $X = \{X_z, z \in T\}$ and any rectangle $(z_1, z_2]$, $z_1 \le z_2$, we define

$$\mathrm{Var}^S_{(z_1, z_2]} \; X = \sum_{(i,j)\in I} E \left| E \{X(\Delta_{ij} \cap (z_1, z_2]) / \mathcal{F}_{z_{ij}}\} \right| .$$

When $(z_1, z_2] = (0, 1]^2$ we simply write $\mathrm{Var}^S X$. We also define

$$\mathrm{Var}_{(z_1, z_2]} \; X = \sup_S \; \mathrm{Var}^S_{(z_1, z_2]} \; X,$$

and $\mathrm{Var}\; X = \sup_S \; \mathrm{Var}^S X$.

The total variation of the Doléans-Föllmer measure of the process X on $(z_1, z_2] \times \Omega$ coincides with $\mathrm{Var}_{(z_1, z_2]} X$. Then, following [4] the process $X = \{X_z, z \in T\}$ is said to be a *planar quasimartingale* if $\mathrm{Var}\; X < \infty$.

Sometimes it is convenient to localize the above definitions. This can be done by means of the notion of *stopping domain.* A set $D \subset T \times \Omega$ is a stopping domain if 1_D is a progressively measurable process, and $R_z \subset D(\omega)$, whenever $z \in D(\omega)$. For any class $\mathcal{C}$ of processes, $\mathcal{C}_{loc}$ will denote the class of processes X such that there exists a sequence $\{X^n, n \geq 1\}$, $X^n \in \mathcal{C}$, and an increasing sequence $\{D_n, n \geq 1\}$ of stopping domains, with $\bigcup_n D_n = T$, such that for any n, $X^n_z = X_z$, if $z \in D_n$.

Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ we introduce a sequence of smooth functions $\{f_m, m \geq 1\}$ by means of a sequence of regularization kernels of the form

$$\alpha_m(x) = m \;\; \alpha(m\, x), \tag{1.1}$$

where $\alpha \in \mathcal{C}^\infty_0(\mathbb{R})$ is a nonnegative function whose compact support is contained in the interval $(-\infty, 0]$ and such that $\int_{\mathbb{R}} \alpha(x)\, dx = 1$. We take

$$f_m(x) = (f * \alpha_m)\,(x) = \int_{\mathbb{R}} f(x + y)\, \alpha_m(y)\, dy\, . \tag{1.2}$$

2. WEAK SUBMARTINGALE PROPERTY OF F(M)

In this section we study the weak submartingale property of $f(M)$, f being a real function and M a two-parameter martingale. Two slightly different results in this direction are presented. The first one requires M to be a martingale belonging to $\mathcal{M}^2_c$ and of path independent variation; its proof is based on a special two-parameter Itô's formula for

$\mathcal{C}^2$-functions. The second one assumes that M is in $\mathfrak{M}_c^4$ and $<M, \tilde{M}> = 0$. The proof uses the compact Itô formula of [13] and a regularization procedure.

We first state a deterministic result.

Lemma 2.1. Let X and Y be two functions in $C(T)$ and assume that Y is increasing, in the measure sense. Consider the sequences defined by

$$\varphi_n(z) = \sum_{(i,j)\in I_z^n} (X_{s_i, t_{j+1}\wedge t} - X_{s_i, t_j}) (Y_{s_{i+1}\wedge s, t_j} - Y_{s_i, t_j}), \tag{2.1}$$

$$\psi_n(z) = \sum_{(i,j)\in I_z^n} \int_{s_i}^{s_{i+1}\wedge s} (X_{\sigma,t_{j+1}\wedge t} - X_{\sigma, t_j}) \, d_\sigma Y_{\sigma, t_j}$$

$$= \sum_{\{j,\, t_j<t\}} \int_o^s (X_{\sigma, t_{j+1}\wedge t} - X_{\sigma, t_j}) \, d_\sigma Y_{\sigma, t_j}. \tag{2.2}$$

Then $\{\varphi_n, n \geq 1\}$ and $\{\psi_n, n \geq 1\}$ converge to the same function $X * Y \in C(T)$.

Proof: We first remark that $\varphi_n(z)$ and $\psi_n(z)$ can be written in the following alternative way

$$\varphi_n(z) = \sum_{\{i,\, o\leq s_i<s\}} X_{s_i, t} (X_{s_{i+1}\wedge s, t} - Y_{s_i, t}) - \sum_{\{i,\, o\leq s_i<s\}} X_{s_i, o} (Y_{s_{i+1}\wedge s, o} - Y_{s_i, o})$$

$$- \sum_{\{j,\, o<t_j\leq t\}} \sum_{\{i,\, s_i<s\}} X_{s_i, t_j} \; Y(\Delta_{ij-1}). \tag{2.3}$$

$$\psi_n(z) = \int_o^s X_{\sigma, t} \, d_\sigma Y_{\sigma, t} - \int_o^s X_{\sigma, o} \, d_\sigma Y_{\sigma, o}$$

$$- \sum_{\{j,\, o<t_j\leq t\}} \int_o^s X_{\sigma, t_j} \, d_\sigma (Y_{\sigma, t_j} - Y_{\sigma, t_{j-1}}). \tag{2.4}$$

Therefore, we obtain

$$\sup_{z\in T} | \varphi_n(z) - \psi_n(z) | \leq C \sup_{|u-v| \leq |S^n|} | X_u - X_v | Y_{1,1} . \tag{2.5}$$

On the other hand $\{\psi_n, n \geq 1\}$ is a Cauchy sequence. Indeed, take $m > n$ and suppose that $\mathcal{S}^m = \{(\sigma_{i'}, \tau_{j'}) \in T,\ i' = 0,\dots,p_m,\ j' = 0,\dots,q_m\}$. For every $j = 0,\dots,q_n$ define $J_j = \{j', \tau_{j'} \in (t_j, t_{j+1}]\}$, then, using (2.4) we obtain

$$\sup_{z\in T} | \psi_n(z) - \psi_m(z) |$$

$$= \sup_{z\in T} \Big| \sum_{\{j,\ 0<t_j\le t\}} \sum_{j'\in J_j} \int_o^s [X_{\sigma,t_j} - X_{\sigma,\tau_{j'}}]\, d_\sigma \left(Y_{\sigma,\tau_{j'}} - Y_{\sigma,\tau_{j'-1}}\right) \Big|$$

$$\le C \sup_{|u-v|\le|\mathcal{S}^n|} | X_u - X_v |\ Y_{1,1},$$

which tends to zero as $n \to \infty$, uniformly in m.

This fact together with inequality (2.5) proves the Lemma. □

Let us now prove a particular Itô formula.

Theorem 2.2. Let M be a martingale belonging to $\mathfrak{m}_c^2$ such that $M^2 - \langle M\rangle$ is a martingale. For any real function f of class $\mathcal{C}^2$, and every $z= (s, t) \in T$ we have

$$f(M_z) = f(M_{o,t}) + \int_o^s f'(M_{\sigma,t})\, d_\sigma M_{\sigma,t} + \frac{1}{2} \int_o^s f''(M_{\sigma,o})\, d_\sigma \langle M\rangle_{\sigma,o}$$
$$+ \frac{1}{2} \int_{R_z} f''(M_u)\, d\langle M\rangle_u + \frac{1}{2} \left[f''(M) * \langle M\rangle\right]_z . \tag{2.6}$$

Remark: The assumptions on M are equivalent to the property of p.i.v. if M vanishes on the axes (see [10]).

Proof: To simplify the notation we take $z = (1, 1)$. By Itô's formula we can write

$$f(M_{1,1}) = f(M_{o,1}) + \int_o^1 f'(M_{\sigma,1})\, d_\sigma M_{\sigma,1} + \frac{1}{2} \int_o^1 f''(M_{\sigma,1})\, d_\sigma \langle M\rangle_{\sigma,1} .$$

We have

$$\int_o^1 f''(M_{\sigma,1})\, d_\sigma \langle M\rangle_{\sigma,1} - \int_o^1 f''(M_{\sigma,o})\, d_\sigma \langle M\rangle_{\sigma,o}$$

$$= \sum_{(i,j)\in I^n} \left(\int_{s_i}^{s_{i+1}} f''(M_{\sigma,t_{j+1}})\, d_\sigma <M>_{\sigma,t_{j+1}} - \int_{s_i}^{s_{i+1}} f''(M_{\sigma,t_j})\, d_\sigma <M>_{\sigma,t_j} \right)$$

$$= a_n + b_n + c_n,$$

where:

$$a_n = \sum_{(i,j)\in I^n} f''(M_{z_{ij}}) <M> (\Delta_{ij}) ,$$

$$b_n = \sum_{(i,j)\in I^n} \int_{s_i}^{s_{i+1}} \left[f''(M_{\sigma,t_{j+1}}) - f''(M_{\sigma,t_j}) \right] d_\sigma <M>_{\sigma,t_j} ,$$

and

$$c_n = \sum_{(i,j)\in I^n} \int_{s_i}^{s_{i+1}} \left[f''(M_{\sigma,t_{j+1}}) - f''(M_{z_{ij}}) \right] d_\sigma \left(<M>_{\sigma,t_{j+1}} - <M>_{\sigma,t_j} \right) .$$

The sequence a_n converges a.s. to $\int_T f''(M_u)\, d <M>_u$, as $n \to \infty$.

By Lemma 2.1 applied to the paths of $X = f''(M)$ and $Y = <M>$, we obtain $b_n \longrightarrow (f''(M) * <M>)_{1,1}$ as $n \to \infty$, a.s. Finally, since

$$|c_n| \leq \sup_{|u-v|\leq |S^n|} |f''(M_u) - f''(M_v)| <M> (R_{1,1}) ,$$

$\lim_{n\to\infty} c_n = 0$ a.s., and the theorem is established. □

In order to study the weak submartingale property of $f(M)$ we need the following lemma.

Lemma 2.3. Let φ be a real convex function and M a martingale in $\mathcal{M}_c^2$. The continuous process $\varphi(M) * <M>$ is a local weak submartingale.

Proof: For any positive integer k, define

$$D_k = \left\{ (z, \omega) ,\ \sup_{u\in R_z} |M_u(\omega)| \leq k \right\} , \tag{2.7}$$

and put $N = \varphi(M) * \langle M\rangle$. We have to show the existence of a sequence of weak submartingales $\{N^k,\ k \geq 1\}$ such that, for any $k \geq 1$,

$$1_{D_k}(z)\ N_z = 1_{D_k}(z)\ N_z^k\ . \tag{2.8}$$

Let $z = (s, t)$. For each $i \in N$ such that, $s_i < s$, define

$$T_i^k = \inf\ \{\tau,\ M_{s_i \tau} \notin D_k\}\ ,$$

this is a stopping time with respect to the filtration $\{\mathcal{F}_{1,t},\ t \in [0,1]\}$, and we have

$$1_{D_k}(z) \sum_{(i,j)\in I_z^n} [\ \varphi(M_{s_i, t_{j+1}\wedge t}) - \varphi(M_{s_i, t_j})\ ]\ [\ \langle M\rangle_{s_{i+1}\wedge s,\, t_j} - \langle M\rangle_{s_i, t_j}\ ]$$

$$= 1_{D_k}(z) \sum_{(i,j)\in I_z^n} [\ \varphi(M^{T_i^k}_{s_i, t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i, t_j})\ ]\ [\langle M\rangle_{s_{i+1}\wedge s,\, t_j} - \langle M\rangle_{s_i, t_j}]\ ,$$

where $\{M^{T_i^k}_{s_i, t}\ , t \geq 0\}$ denotes the martingale, $M_{s_i,\ t}$ stopped at T_i^k.

Using the convexity of φ we obtain

$$E\left\{[\varphi(M^{T_i^k}_{s_i, t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i, t_j})\ ]\ [\langle M\rangle_{s_{i+1}\wedge s,\, t_j} - \langle M\rangle_{s_i, t_j}\ ] / \mathcal{F}_{z_{ij}}\right\}$$

$$= E\left\{[\langle M\rangle_{s_{i+1}\wedge s,\, t_j} - \langle M\rangle_{s_i, t_j}]\ E\ \{\varphi(M^{T_i^k}_{s_i, t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i, t_j}) / \mathcal{F}_{1, t_j}\} / \mathcal{F}_{z_{ij}}\right\}$$

$$\geq\ 0.$$

Hence $J_z^{n,k} = \sum_{(i,j)\in I_z^n} [\varphi(M^{T_i^k}_{s_i, t_{j+1}\wedge t}) - \varphi(M^{T_i^k}_{s_i, t_j})]\ [\langle M\rangle_{s_{i+1}\wedge s,\, t_j} - \langle M\rangle_{s_i, t_j}]$ is a weak submartingale.

Notice that $\{J_z^{n,k}, n \geq 1\}$ is uniformly integrable, therefore by Lemma 2.1 $L^1\text{-}\lim_{n\to\infty} J_z^{n,k}$ exists and defines a weak submartingale. Finally since

$$1_{D_k}(z)\, N_z = 1_{D_k}(z)\ L^1\text{-}\lim_{n\to\infty} J_z^{n,k}\ ,$$

(2.8) holds with $N_z^k = L^1\text{-}\lim\ J_z^{n,k}$, and the proof of the Lemma is complete. □

We can now state the main result of this section.

Theorem 2.4. Let f be a $\mathcal{C}^2$ function such that f'' is convex and positive. Then, if M is a martingale in $\mathfrak{M}_c^2$ such that $M^2 - \langle M\rangle$ is a martingale, the process $\{f(M_z), z \in T\}$ is a *local weak submartingale*.

Proof: It is an immediate consequence of Theorem 2.2 and lemma 2.3 applied to $\varphi = f''$. □

The conclusion of Theorem 2.4 can also be obtained by a more direct approach, using the compact Itô formula of [13]. Here the martingale M is supposed to belong to $\mathfrak{M}_c^4$, but the path independent variation property can be replaced by a weaker one. More precisely we can prove the following result:

Theorem 2.5. Let f be a $\mathcal{C}^2$ function whose second derivative is convex and positive. Let $M \in \mathfrak{M}_c^4$, null on the axes and such that $\langle M, \tilde{M}\rangle = 0$. Then the process $\{f(M_z), z \in T\}$ is a *local weak submartingale*.

Proof: (a) Let us first prove the theorem under stronger hypotheses on f : Assume that $f \in \mathcal{C}^4(\mathbb{R})$, f'' convex and positive. By the Itô formula proved in [13] we have

$$
\begin{aligned}
f(M_{s,t}) &= f(0) + \int_{R_{s,t}} f'(M_z)\, dM_z \\
&+ \int_{R_{s,t}} \left[f''(M_z)\, d\tilde{M}_z + f''(M_z)\, dS_z^{(1)} + f''(M_z)\, dS_z^{(2)} + \frac{1}{2} f''(M_z)\, d\langle M\rangle_z\right] \\
&+ \int_{R_{s,t}} \left[\frac{1}{2} f'''(M_z)\, dW_z^{(1)} + \frac{1}{2} f'''(M_z)\, dW_z^{(2)}\right] + \frac{1}{4} \int_{R_{s,t}} f^{iv}(M_z)\, d\langle\tilde{M}\rangle_z \,. \qquad (2.9)
\end{aligned}
$$

We recall that $\{S_z^{(1)}, z \in T\}$ (resp. $\{S_z^{(2)}, z \in T\}$) is a continuous 1-martingale (resp. 2-martingale) obtained as the L^2-limit of the sequence $\{\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^2)\, M(\Delta_{ij}),\ z \in T;\ n \geq 1\}$ (resp. $\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^1)\, M(\Delta_{ij}),\ z \in T;\ n \geq 1\}$), and $\{W_z^{(1)}, z \in T\}$, (resp. $\{W_z^{(2)}, z \in T\}$) is the continuous 1-martingale (resp. 2-martingale) obtained by the L^1-limit of the sequence $\{\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^2)^2 M(\Delta_{ij}^1),\ z \in T;\ n \geq 1\}$ (resp. $\{\sum_{(i,j)\in I_z^n} M(\Delta_{ij}^1)^2\, M(\Delta_{ij}^2),\ z \in T;\ n \geq 1\}$) .These processes are two-parameter stochastic integrators.

The properties on f ensure that the process defined by

$$Z_{s,t} = \int_{R_{s,t}} \left[\frac{1}{2} f''(M_z)\, d\langle M\rangle_z + \frac{1}{4} f^{iv}(M_z)\, d\langle \widetilde{M}\rangle_z \right] ,$$

is increasing in the measure sense.

Consider the increasing sequence $\{D_k, k \geq 1\}$ of stopping domanins defined by 2.7. For $(s, t) \in D_k$, $f(M_{s,t})$ coincides with the weak submartingale given by

$$\int_{R_{s,t}} 1_{D_k}(z) \Big\{ f'(M_z)\, dM_z + f''(M_z)\, d\widetilde{M}_z + f''(M_z)\, dS_z^{(1)} + f''(M_z)\, dS_z^{(2)} + \frac{1}{2} f''(M_z)\, d\langle M\rangle_z$$

$$+ \frac{1}{2} f'''(M_z)\, dW_z^{(1)} + \frac{1}{2} f'''(M_z)\, dW_z^{(2)} + \frac{1}{4} f^{iv}(M_z)\, d\langle \widetilde{M}\rangle_z \Big\} ,$$

and consequently the assertion is proved.

(b) In the general case, we consider a sequence $\{\alpha_n(x),\ n \geq 1\}$ of regularitzation kernels of the form (1.1), and the corresponding $\{f_n,\ n \geq 1\}$.

Each f_n satisfies the hypotheses of part (a), thus, $\{f_n(M_z),\ z \in T\}$ is a local weak submartingale, for any $n \geq 1$.

Consider the martingales $m^k \in \mathcal{M}_c^4$ and $\tilde{m}_k \in \mathcal{M}_c^2$ given by

$$m_{s,t}^k = \int_{R_{s,t}} 1_{D_k}(z)\ dM_z , \quad \tilde{m}_{s,t}^k = \int_{R_{s,t}} 1_{D_k}(z)\ d\widetilde{M}_z .$$

Notice that

$$\langle m^k, \tilde{m}^k\rangle_{\cdot} = \int_{R_{\cdot}} 1_{D_k}(z)\, d\langle M, \widetilde{M}\rangle_z = 0,$$

so, (2.9) still holds with f replaced by f_n and M by m^k, and it follows that $\{f_n(m_z^k),\ z \in T\}$ is, for any $n \geq 1$, a weak submartingale.

Then, since $1_{D_k}(z)\, f_n(M_z) = 1_{D_k}(z)\, f_n(m_z^k)$, taking account of the convergences

$$1_{D_k}(z)\, f_n(M_z) \xrightarrow[n\to\infty]{L^1} 1_{D_k}(z)\, f(M_z) ,$$

and

$$1_{D_k}(z)\, f_n(m_z^k) \xrightarrow[n\to\infty]{L^1} 1_{D_k}(z)\ f(m_z^k) ,$$

we obtain

$$1_{D_k}(z)\, f(M_z) = 1_{D_k}(z)\ f(m_z^k)\ ,$$

where $f(m_z^k)$ is a weak submartingale. This finishes the proof of the Theorem. □

3. QUASIMARTINGALE PROPERTY OF f(M)

The second part of this paper is devoted to study the quasimartingale property of the transformation of a two-parameter martingale by a $\mathcal{C}^2$-function.

In order to carry out our programm we will state a formula for the total variation of the Doléans-Föllmer measure (or conditional variation) of $f(M)$. This formula involves the local time of the martingale M and is obtained by means of the compact Itô's formula proved in [13]. At the same time this provides a necessary and sufficient condition for $f(M)$ to be a quasimartingale. We refer the reader to [14] for analogue results in the one-parameter setting.

In the sequel the martingale M is supossed to be null on the axes.

Assume that $M \in \mathfrak{m}_c^4$ satisfies the following hypothesis:

(H 1) The measure $<M>$ is absolutely continuous with respect to the product of its marginals.

It has been proved in [11] (see Corollary 4.2) that, under (H 1), $<M>$ is absolutely continuous with respect to $<\tilde{M}>$, a.s. Futhermore, since the local time L of the martingale M with respect to the measure $<\tilde{M}>$ always exist (Theorem 3.1, [9]), so does the local time L of M with respect to $<M>$ (cf. Lemma 5.1, [11]), and we have

$$L\,(x, A) = \int_A \phi(u)\ \tilde{L}\,(x, du), \qquad \text{a.s.,} \tag{3.1}$$

ϕ being the Radon-Nikodym derivative $\dfrac{d<M>}{d<\tilde{M}>}$.

Following [14] a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be of class (β) if its second derivative in the distributional sense is a bounded measure ν (i.e. $\int_{\mathbb{R}} |\,\nu\,(dx)\,| < \infty$).

We can now state our main result:

Theorem 3.1. Let M be a martingale belonging to $\mathfrak{m}_c^p$, for some $p > 4$. Assume that M satisfies (H 1), and that $<M, \tilde{M}> = 0$. Let $f \in \mathcal{C}^2$, such that f'' is of class (β). Then, if we denote by ν the second derivative of f'' in the distributional sense, we have, for any $z_1, z_2 \in T$, $z_1 < z_2$,

$$\mathrm{Var}_{(z_1,z_2]} f(M) = E \int_{(z_1,z_2]\times\mathbb{R}} \left\{ \left|\frac{1}{2} f''(a)\ \phi(z) + \frac{1}{4}\ \varphi(a)\right| \tilde{L}(a, dz)\, da + \frac{1}{4}\tilde{L}(a, dz)\, |\nu^s|(da) \right\} \tag{3.2}$$

where $\nu(da) = \varphi(a)\, da + \nu^s(da)$ is the Lebesgue decomposition of ν in its absolutely continuous and singular part, with respect to the Lebesgue measure on $\mathbb{R}$.

Hence, $f(M)$ is a quasimartingale if and only if

$$E \int_T |dA_z| < \infty ,$$

where

$$A_z = \int_{R_z\times\mathbb{R}} \left(\frac{1}{2}\ f''(a)\ \phi(u)\ da + \frac{1}{4}\ \nu(da)\right) \tilde{L}(a, du), \tag{3.3}$$

$z \in T$.

Proof: (1) The hypotheses on f imply the existence of some constants α, β such that

$$f''(x) = \alpha\ x + \beta + \int_0^x dy \int_{(-\infty,\, y]} \nu\,(dz).$$

Consequently f'' is the difference of two convex functions of class (β), φ^1 and φ^2, and the following conditions hold:

$$\begin{aligned} &|f'(x)| \le a_1 |x|^2 + b_1 |x| + c_1 ,\\ &|f''(x)| \le a_2 |x| + b_2 ,\\ &|f'''(x)| \le a_3 . \end{aligned} \tag{3.4}$$

Consider the sequence $\{f_m,\ m \ge 0\}$ defined by (1.2). We remark that f_m'' is the difference of two convex functions (φ^1_m and φ^2_m), and that $(\varphi^i_m)' \uparrow (\varphi^i)'$, $i = 1, 2$. (Here we are dealing with the left derivatives). Furthermore

$$\begin{aligned} &|f_m^{(k)}(x)| \le \sup_{|y|\le 1} |f^{(k)}(x+y)|, &&\text{for}\quad k = 1,2\\ &|f_m^{(k)}(x)| \le C, &&\text{for}\quad k = 3,4 \end{aligned} \tag{3.5}$$

(2) We apply the compact Itô's formula of [13] to $f_m(M)$. Then, we obtain

$$f_m(M_{s,t}) = f_m(0) + \int_{R_{s,t}} f'_m(M_z)\, dM_z$$

$$+ \int_{R_{s,t}} \{f''_m(M_z)\, d\widetilde{M}_z + f''_m(M_z)\, d S^{(1)}_z + f''_m(M_z)\, d S^{(2)}_z + \frac{1}{2} f''_m(M_z)\, d <M>_z\}$$

$$+ \int_{R_{s,t}} \frac{1}{2} f'''_m(M_z)\, dW^{(1)}_z + \frac{1}{2} f'''_m(M_z)\, dW^{(2)}_z$$

$$+ \int_{R_{s,t}} \frac{1}{4} f^{iv}_m(M_z)\, d <\widetilde{M}>_z \,. \tag{3.6}$$

Since $M \in \mathfrak{m}^p_c$ for $p > 4$, and due to conditions (3.5), every term in the right-hand side of (3.6) belongs to L^1. This is obvious for $\int_{R_{s,t}} f''_m(M_z)\, d <M>_z$ and $\int_{R_{s,t}} f^{iv}_m(M_z)\, d <\widetilde{M}>_z$. In order to check this property for the stochastic integrals it suffices to prove that the corresponding sequences of Riemann sums are bounded in $L^{1+\varepsilon}$, for some $\varepsilon > 0$. Consider, for example, the sequence $\{ \sum_{(i,j)\in I^n_{s,t}} f''(M_{z_{ij}})\, M(\Delta^1_{ij})\, M(\Delta^2_{ij}),\ n \geq 1\}$. By Burkholder's inequality we have

$$E \left| \sum_{(i,j)\in I^n_{s,t}} f''_m(M_{z_{ij}})\, M(\Delta^1_{ij})\, M(\Delta^2_{ij}) \right|^{1+\varepsilon}$$

$$\leq C\, E \left(\sum_{(i,j)\in I^n_{s,t}} f''_m(M_{z_{ij}})^2\, M(\Delta^1_{ij})^2 M(\Delta^2_{ij})^2 \right)^{\frac{1+\varepsilon}{2}}$$

$$\leq C \left\{ E \{ \sup_{z\in T} | f''_m(M_z) |^{2(1+\varepsilon)} \}^{1/2} \{ E \{ \sup_i (\sum_j M(\Delta^1_{ij})^2)^{2(1+\varepsilon)} \} \right. .$$

$$. \left. E \{ \sum_i \sup_j M(\Delta^2_{ij})^2 \}^{2(1+\varepsilon)} \right\}^{1/4}$$

$$\leq \{ C_1 + C_2\, E\, | M_{11} |^{2(1+\varepsilon)} \}^{1/2} \{ E\, M_{11}^{4(1+\varepsilon)} \}^{1/2} \,,$$

where the last inequality has been obtained using (3.5), Doob's maximal inequality and Lemma 2.2. of [11].

The same kind of arguments are used for the other terms.

(3) Let $\Delta = (z, z']$, $z \leq z'$. The results proved in part (2) show that

$$E\, [f_m(M)\, (\Delta)/\mathcal{F}_z] = E\, \{ \int_\Delta \frac{1}{2} f''_m(M_u)\, d<M>_u + \frac{1}{4} f^{iv}_m(M_u)\, d<\widetilde{M}>_u / \mathcal{F}_z \}, \tag{3.7}$$

and using the density of occupation formula

$$E\left[f_m(M)(\Delta)/\mathcal{F}_z\right] = E\left\{\int_{R\times\Delta} \frac{1}{2}\, f_m''(a)\, L(a,dz)\, d\,a + \frac{1}{4}\, f_m^{'v}(a)\, \widetilde{L}(a, dz)\; da/\mathcal{F}_z\right\}. \quad (3.8)$$

It is simple to verify that

$$E\left[f_m(M)(\Delta)/\mathcal{F}_z\right] \xrightarrow{L^1} E\left[f(M)(\Delta)/\mathcal{F}_z\right], \quad (3.9)$$

and

$$E\left[\int_{R\times\Delta} \frac{1}{2}\, f_m''(a)\; L(a, dz)\; da/\mathcal{F}_z\right] \xrightarrow[m\to\infty]{L^1} E\left[\int_{R\times\Delta} \frac{1}{2}\, f''(a)\; L(a, dz)\, da/\mathcal{F}_z\right].$$

Denote by ν_i the measure on $(\mathbb{R}, \mathcal{B}, (\mathbb{R}))$ whose distribution function is given by

$$\nu_i\left([a,b)\right) = (\varphi^i)'(b) - (\varphi^i)'(a), \qquad \text{for all } a < b, \text{ and } i = 1, 2.$$

We have

$$(\varphi_m^i)'(b) - (\varphi_m^i)'(a) = \int_a^b (\varphi_m^i)''(x)\, dx \longrightarrow (\varphi^i)'(b) - (\varphi^i)'(a) = \nu_i\left([a, b)\right).$$

Therefore, on any compact set $K \subset \mathbb{R}$, the finite measures $\{\nu_m^i,\ m \geq 1\}$ whose distribution functions are $(\varphi_m^i)'$, $m \geq 1$, converge weakly to ν_i.

By continuity, the path $\{M_z(\omega), z \in T\}$ can only visit the points of a compact set, therefore the integral

$$\int_{\mathbb{R}} f_m^{'v}(a)\; \widetilde{L}(a, \Delta)\, da \quad \left(= \int_{\mathbb{R}} (\varphi_m^1 - \varphi_m^2)''\; \widetilde{L}(a, \Delta)\, da \right)$$

is extended on a compact set K, and the weak convergence of $\{\nu_m^i,\ m \geq 1\}$ to ν^i entails

$$\int_{\mathbb{R}} f_m^{'v}(a)\, \widetilde{L}(a, \Delta)\, da \xrightarrow[m\to\infty]{} \int_{\mathbb{R}} \widetilde{L}(a, \Delta)\, \nu(da).$$

Using this fact and the convergences (3.9) we obtain from (3.8)

$$E\left[f(M)(\Delta)/\mathcal{F}_z\right] = E\left\{\int_{R\times\Delta} \frac{1}{2}\, f''(a)\; L(a, dz)\; da + \frac{1}{4}\, \widetilde{L}(a, dz)\, \nu(da)/\mathcal{F}_z\right\}$$

$$= E \left\{ \int_{R \times \Delta} [\frac{1}{2} f''(a)\ \phi(z)\ \widetilde{L}(a, dz)\, da + \frac{1}{4} \widetilde{L}(a, dz)\ \nu(da) / \mathcal{F}_z \right\}. \qquad (3.10)$$

Thus, if $\{A_z,\ z \in T\}$ is the predictable, bounded variation process defined by (3.3), it follows that

$$\mathrm{Var}_{(z_1, z_2]}\ f(M) = \mathrm{Var}_{(z_1, z_2]}\ A\ . \qquad (3.11)$$

(4) The equality (3.11) allow us to prove formula (3.2). Indeed, it is known (see e.g. [3]) that, if X is a predictable process of bounded variation

$$\mathrm{Var}_{(z_1, z_2]}\ X = E \int_{(z_1, z_2]} |\, d X_z\, |\ .$$

Hence the proof of the theorem is now complete. □

Remark

For the special case of the Brownian sheet, Theorem 3.1 can be paraphrased as follows:

Theorem 3.2. Let $f \in \mathcal{C}^2$ be such that f'' is of class (β). Denote by ν the second derivative of f'' in the distributional sense, and let $\nu(da) = \varphi(a)\, da + \nu^s(da)$ the Lebesgue decomposition of ν with respect to the Lebesgue measure on $\mathbb{R}$.
Then, $f(W)$ is a quasimartingale if and only if

$$E \int_T |\, d A_z\, | < \infty\ ,$$

with

$$A_z = \int_{R_z} \int_R \left(\frac{1}{2} f''(a)\ da + \frac{1}{4}\ x.y\ \ \nu(da) \right) L(a, dx\, dy)\ ,$$

L being the local time of W with respect to the Lebesgue measure. Moreover

$$\mathrm{Var}_{(z_1, z_2]}\ f(W) = E \int_{(z_1, z_2] \times R} \left\{ \left| \frac{1}{2} f''(a) + \frac{1}{4}\ \varphi(a)\ x.y \right| da + \frac{1}{4}\ x.y\ |\nu^s|\,(da) \right\} L(a, dx\, dy),$$

for any z_1, z_2 in T, $z_1 < z_2$.

A result in the same direction has been obtained in [7], where the following formula is proved.

$$\mathrm{Var}_{(o,\,z]}\, f(W) = \frac{1}{2} \int_{o}^{x.y} \frac{1}{q}\, \mathrm{Var}_{(o,\,q]} \left(t\, f''(b_t)\right) dq \quad ,$$

for $z = (x, y)$, and where $\{b_t,\ t \geq 0\}$ is a Brownian motion.

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Nguyen Minh Duc
Institute of Computer Science
and Cybernetics
Nghia Do - Tu Liem
Hanoi R.S. Vietnam

D. Nualart and M. Sanz
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via, 585
08007 Barcelona - Spain