Séminaire de probabilités (Strasbourg)

Eugene Wong
Moshe Zakai

Spectral representation of isotropic random currents

Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 503-526
<http://www.numdam.org/item?id=SPS_1989__23__503_0>

Spectral Representation of Isotropic Random Currents[1]

Eugene Wong and Moshe Zakai [2]

Department of Electrical Engineering
University of California at Berkeley
Berkeley, CA 94720

Department of Electrical Engineering
Technion, Israel Institute of Technology
Haifa, 32000, ISRAEL

1. Introduction

The theory of random vector fields originated in the statistical theory of turbulence (cf. the references in [2], [13]) and led to the study of random fields with second order properties that are invariant under shift (homogeneity) and under rotation (isotropy). In 1955, K. Ito [2] generalized these notions by considering random differential r-forms in $\mathbb{R}^N$, which for $r = 1$ are vector fields, and by considering differential forms with coefficients that are random Schwartz distributions, namely a random version of the currents introduced by deRham [5].

Roughly speaking, the theory of homogeneous random r-currents in $\mathbb{R}^N$ is similar to that of vector-valued stationary random processes on $\mathbb{R}^N$, namely processes parametrized by n-tuples of test functions, with $n = \frac{N!}{(N-r)!\,r!}$. Once isotropy is added, the notion of random currents and the associated operations (e.g., exterior products, exterior derivatives and the Hodge star operation) become essential.

In [2], K. Ito established a general theory of homogeneous and isotropic random currents and gave a complete characterization of the spectral measure associated with such processes, generalizing earlier results of S. Ito [3] for the case $r = 1$. K. Ito's characterization shows that, regardless of the dimension of the space N and of the order of the current r $(0 < r < N)$, the spectral measure of a homogeneous and isotropic current is uniquely determined by two slowly

1 Work supported by the Army Research Office, Grant No. DAAG29-85-K-0233.
2 Work done while at University of California at Berkeley.

increasing, scalar-valued measures on $(0, \infty)$ and one real number. For $r = 0$ or N, one such measure suffices. The case of random vector fields ($r = 1$) was also treated independently by Yaglom ([13], cf. also [14]).

The notion of random currents has recently played an important role in establishing up a framework for the analysis of Markov fields [10]-[12]. Let X_t, $t \in \mathbb{R}^N$ be a random field on $\mathbb{R}^N$. A natural definition of the Markov property for X_t is the conditional independence of $\sigma\{ X_t, t \in D \}$ and $\sigma\{ X_t, t \in D^C \}$, given $\sigma\{ X_t, t \in \partial D \}$. It turns out, however, that the class of random fields having this Markov property is quite restricted. Consequently a modified definition of the Markov property (viz; germ-field Markov) has been introduced and studied [4] [6]. This property is considerably weaker than the classical Markov property. Recently, we have returned to a more direct notion of the Markov property for random fields, as well as generalized fields, by introducing the σ-fields associated with the boundary data. This is done for random currents that can be "localized" to (N−1) dimensional subsets by defining the Markov property as to require the (N−1) dimensional boundary field to be a splitting field (cf. [12] for further details).

The purpose of this paper is two-fold: The first is to present a new and complete proof of Ito's characterization of the spectral measure of homogeneous and isotropic currents, of which only a sketch was given in [2]. In so doing, we shall also present an exposition of Ito's paper. The second (section 7 of this paper) is to present a spectral representation for the samples of homogeneous and isotropic random currents. This representation (see (7.4)) is new and simple, and an application of this representation yields some new results on the structure of the spectral measure associated with random currents (proposition 7.2). The representation (7.4) was used recently to prove a Markov property for certain homogeneous and isotropic currents [12].

Nonrandom differential forms and deRham currents are briefly reviewed in the next section. Random currents are introduced in section 3. A special class of random currents —

random measures that are related to processes of orthogonal increments is also introduced in section 3. Homogeneous currents (i.e., currents for which the first two moments are shift invariant) are introduced in section 3 and their spectral representation analyzed. Every (smooth, non-random) vector field (i.e. $r = 1$) in $N = 3$ dimensional space is the sum of a constant (space-independent) vector field, the gradient of a scalar potential and the curl of a vector potential. This result was generalized to nonrandom currents and for $r > 1$ or $N \neq 3$ by Hodges, Kodaira, and deRham [5]. In correspondence with this decomposition of non-random currents, Ito derived a decomposition of homogeneous random currents into the sum of an invariant component, an irrotational component, and a solenoidal component. This is presented in section 4. Ito's characterization of the spectral measure associated with homogeneous and isotropic currents is discussed in section 6. The proof in [2] is sketched very briefly, the case $r = 1$ was derived independently by Yaglom [13], [14]. We give in section 6 a proof of Ito's characterization.

Section 7 deals with the sample representation of homogeneous and isotropic currents. It is shown that every isotropic and homogeneous current X with zero invariant part can be decomposed into the sum of the exterior derivative of an (r−1) current Y and the co-derivative of an $(N - r + 1)$ current Z ($X = dY + {*dZ}$), and that the spectral measures associated with Y and Z in any Euclidean coordinate system both have orthogonal components.

2. Differential forms and currents

We begin with a short review of alternating multilinear r-covectors in $\mathbb{R}^N$. Let $e_1, \cdots, e_N$ denote an orthonormal basis in $\mathbb{R}^N$ and $e_1, \cdots, e_N$ in this order will denote a positive orientation. An alternating multilinear r-covector, or just an r-covector for short, a_r is expressed by

$$\begin{aligned} a_r &= \sum_{[i]} a_i \cdot e_{i_1} \wedge \cdots \wedge e_{i_r} \\ &= \sum_{[I]} a_I \cdot e_I \end{aligned} \tag{2.1}$$

where $\mathbf{i}$ is the multi-index $\mathbf{i} = (\mathbf{i}_1, \cdots, \mathbf{i}_r)$, the summation is over all the *ordered* r-tuples $[\mathbf{i}]$ and for any r vectors $v_1, \cdots, v_r$ in $\mathbb{R}^N$.

$$a_r(v_1, \cdots, v_r) = \sum_{[\mathbf{i}]} a_{\mathbf{i}} \det\{ (e_{\mathbf{i}_j} v_k) \} \tag{2.2}$$

where $(e_{\mathbf{i}_j} v_k)$ is the scalar product of the two vectors and is the (j,k) entry in an $r \times r$ matrix.

Remark: Whenever convenient, we shall use a (Cartesian) coordinate system. It should, however, be noted that the objects and operations with which we are dealing (differential forms, currents, double currents) are *intrinsic* and the results are independent of the coordinate system. Thus the left hand sides of equations (2.1) and (2.2) are intrinsic while the right-hand sides are their representations in a given coordinate system.

The multiple Kronecker δ: $\delta(i_1, \cdots i_p\,;\, j_1, \cdots, j_q)$ is equal to $+1$ if $p = q$ and $\{ j_1, \cdots, j_q \}$ is an even permutation of $\{ i_1, \cdots, i_q \}$, it is equal to -1 if the permutation is odd and equal to zero otherwise. Let $(\mathbf{i}, \mathbf{j})$ denote the concatenation of $\mathbf{i}$ and $\mathbf{j}$; let $\mathbf{k} = [(\mathbf{i}, \mathbf{j})]$ then

$$a_p \wedge b_q = \sum_{[\mathbf{i}], [\mathbf{j}]} a_{\mathbf{i}} b_{\mathbf{j}} e_{\mathbf{i}} \wedge e_{\mathbf{j}} \tag{2.3}$$

and

$$e_{\mathbf{i}} \wedge e_{\mathbf{j}} = \sum_{[\mathbf{k}]} \delta(([\mathbf{i}], [\mathbf{j}])\,;\, \mathbf{k})\, e_{\mathbf{k}} \tag{2.4}$$

Let $|\mathbf{i}|$ denote the cardinality of $\mathbf{i}$ (if $\mathbf{i} = (i_1, \cdots i_r)$, $|\mathbf{i}| = r$. Let $\mathbf{i}^*$ denote the $(N-|\mathbf{i}|)$ multi-index complementary to $\mathbf{i}$ in increasing order, and a^c will denote the complex conjugate of a. The Hodge star operation transforms an r-covector into an (N - r) covector by

$$*a_r = \sum_{[\mathbf{i}]} a_{\mathbf{i}}^c\, \delta\Big((\mathbf{i}, \mathbf{i}^*)\,;\, [(\mathbf{i}, \mathbf{i}^*)] \Big) e_{\mathbf{i}^*} \tag{2.5}$$

The interior product of two covectors is defined by

$$(a_r \vee b_r) = (a_r, b_r) = \sum_{[\mathbf{i}]} a_{\mathbf{i}} b_{\mathbf{i}}^c = *(a_r \wedge (*b_r))$$

and more generally, for $r \neq p$, the interior product is defined by

$$(a_r \vee b_p) = (-1)^{(p-r)(N-p)} *(a_r \wedge (*b_p)) \tag{2.6}$$

Let e denote a unit covector, consider first a coordinate system with e as one of the basis vectors $(e_1, e_2, \cdots, e_N)$. Then for any r, $1 \leq r \leq N-1$, $e \wedge (e_{i_1} \wedge \cdots \wedge e_{i_r})$ will vanish if one of the vectors e_{i_j} is e. It follows that for any a_r, $1 \leq r \leq N-1$

$$a_r = e \wedge (e \vee a_r) + e \vee (e \wedge a_r) \tag{2.7}$$

and

$$((e \wedge (e \vee a_r)), e \vee (e \wedge a_r))) = 0 \tag{2.8}$$

which is an orthogonal decomposition of the covector a_r into an r-covector "in the e-direction" and another r-covector, which is "perpendicular to the e-direction." Furthermore,

$$((e \wedge (e \vee a_r)), e \wedge (e \vee b_r))) = ((e \vee a_r), (e \vee b_r)) \tag{2.9}$$

and a similar result with $\vee$ and $\wedge$ interchanged (cf. proposition 2.23 p. 169 of [8]). Note that the assumption that e is one of the basis vectors was done for the purpose of the exposition only; equations (2.7), (2.8) and (2.9) are coordinate-free. A little bookkeeping: the number of terms in the right-hand side of (2.1) is at most $N!\left((r!(N-r)!)\right)^{-1}$. If e is one of the basis vectors, then the number of components in the first and second terms of (2.7) is easily evaluated to be $(N-1)!/((r-1)!(N-r)!$ and $(N-1)!/(r!(N-r-1)!)$ respectively.

With each $t \in \mathbb{R}^N$ we associate now an r-covector

$$\phi_r(t) = \sum_{[I]} \phi_I(t) e_I \tag{2.10}$$

if $\phi_r(t)$ are C^∞ (i.e., differentiable of all order) functions on $\mathbb{R}^N$ then (2.10) is said to be a differential form. For a differential r-form ϕ_r, the exterior derivative $d\phi$ is the differential (r+1) form

$$(d\phi_r)(t) = \sum_{(k, [I])} \frac{\partial \phi_I(t)}{\partial t_k} e_k \wedge e_I \tag{2.11}$$

and the codifferential is the (r−1) form

$$\delta\phi_r = (-1)^{Nr+N+1} *(d*\phi) \tag{2.12}$$

and

$$< \phi_r, \psi_r > = \int_{\mathbb{R}^N} (\phi_r(t), \psi_r(t))\, dt_1, \cdots, dt_n \tag{2.13}$$

Let S denote the Schwartz space of real-valued fast decreasing functions on $\mathbb{R}^N$ i.e. $\phi \in S$ if $\phi \in C^\infty$ and all its partial derivatives of all order multiplied by any polynomial (of any order) converge to zero as $|t| \to \infty$. A sequence $\phi^{(k)}$, $k = 1, 2, \cdots \phi^{(k)} \in S$ converges to zero if, for any partial differential operator Q (of any order) and any polynomial in t (of any order) P(t), $P(t) \cdot Q\phi^{(k)}(t)$ converges to zero uniformly in $\mathbb{R}^N$. The space S^r will denote the space of differential forms with $\phi_i \in S$ for all i.

A deRham *r-current* is, roughly speaking, a form of type (2.10) in which the coefficients $\phi_I(\cdot)$ are Schwartz distributions. More precisely, an r-current U_r is a continuous, real valued linear functional $U_r(\phi)$ on $\phi \in S^{N-r}$ ([5] and chapter IX of [7]). It is, therefore, natural to define the exterior derivative, the Hodge star operation and the interior differential for currents, as follows:

$$dU_r(\phi) = (-1)^{r+1}\, U_r(d\phi)\,, \quad \phi \in S^{N-r-1} \tag{2.14}$$

$$*U_r(\phi) = (-1)^{r(N-r)}\, (U_r(*\phi))^c\,, \quad \phi \in S^{N-r} \tag{2.15}$$

$$\delta U_r = (-1)^{N\cdot r + N + 1}\, (*d*U_r) \tag{2.16}$$

The exterior and interior product of currents with differential forms is defined by

$$(a_q(\cdot) \wedge U_r)(\phi) = (-1)^{qr}\, U_r(a_q \wedge \phi)\,,\ \phi \in S^{N-r-q}\,,\ a_q \in S^q \tag{2.17}$$

$$(a_q(\cdot) \vee U_r)(\phi) = (-1)^{q(r-q)}\, U_r(a_q \vee \phi)\,,\ \phi \in S^{N-r+q}\,,\ a_q \in S^q \tag{2.18}$$

We conclude this section by introducing the Fourier transform of a current. For any $\phi \in S^r$, $\phi = \sum_{[I]} \phi_I e_I$, $|i| = r$ and $\phi_I \in S$, hence ϕ_I possesses a Fourier transform $\hat{\phi}_I$. Define the Fourier transform $\hat{\phi}_r$ of an r-form ϕ as

$$\hat{\phi}_r(\nu) = \sum_{[I]} \hat{\phi}_I(\nu) e_I$$

$$\hat{\phi}_I(\nu) = \int_{\mathbb{R}^N} \phi_I(t) e^{-i(\nu, t)} dt \tag{2.19}$$

Remark: Our definition of the Fourier transform of a current follows that of [2] which is different from the definition of [7].

Returning to (2.19), recall that $\phi \in S$ implies $\hat{\phi} \in S$. For any r-current U_r define its Fourier transform $\hat{U}_r$ as the r-current satisfying

$$\hat{U}_r(\phi) = U_r(\hat{\phi})$$

for every $\phi \in S^{N-r}$. Therefore $\hat{U}_r$ is also a current and every r-current is the Fourier transform of an r-current. Note that

$$(dU_r)\hat{} = i(\nu \wedge \hat{U}_r) \tag{2.20}$$

$$(\delta U_r)\hat{} = (-1)^r \cdot i \cdot (\nu \vee \hat{U}_r) \tag{2.21}$$

where $i = \sqrt{-1}$ and for any point $\nu \in \mathbb{R}^N$ the one-form ν is just the vector from the origin to the point ν. (We are using here ν to denote both a point in $\mathbb{R}^N$ and a one form, since ν as one-form is used only in conjunction with the $\vee$ or $\wedge$ operations and there is no danger of ambiguity).

3. Random currents and random measures

Let H denote the Hilbert space of zero mean random variables on some fixed probability space. A random current U_r is an H-valued deRham current, namely a continuous linear mapping from S^{N-r} to H. Note that the elements of S^{N-r} are nonrandom and for every $\phi \in S^{N-r}$, $U_r(\phi)$ is a zero mean L^2 random variable and moreover $U_r(\phi^k)$ $k = 1, 2, \ldots$ converges in L^2 to $U_r(\phi^\infty)$ whenever ϕ^k converges to ϕ^∞ in S^{N-r} (cf. § 8 of [5]). A sequence of random currents $\{ U_r^k, k = 1, 2, \ldots \}$ is said to converge in L_2 to a random current U_r^∞ if $U_r^k(\phi)$ converges in

L^2 to $U_r^\infty(\phi)$ for every ϕ in S^{N-r}. The definition of the operations d, *, δ for random currents is the same as for deRham currents and is given by equations (2.14), (2.15), and (2.16). The exterior product of currents with non-random differential forms is as defined by (2.17) and (2.18).

The class of random measures, which is a special class of random currents, will now be considered. The random currents to be considered in this paper — homogeneous and isotropic currents — are in general not random measures; however, their spectral representation is a random measure.

A random Schwartz distribution is just a zero random current. A random r-current U_r and a fixed (non-random and independent of $t \in \mathbb{R}^N$) N-r covector a_{N-r} induce a random Schwartz distribution $\left(U_r\, ,\, a_{N-r}\right)$ by setting for every $\phi \in S$

$$\left(U_r\, ,\, a_{N-r}\right)(\phi) = U_r(\phi \cdot a_{N-r}) \tag{3.1}$$

Let $(M(\phi),\ \phi \in S)$ be a random Schwartz distribution, namely a zero current. $(M(\phi),\ \phi \in S)$ is called a random measure with respect to a σ-finite measure $m(dt_1, \cdots, dt_N)$ on $\mathbb{R}^N$ if for any ϕ and ψ in S.

$$E(M(\phi)M^c(\psi)) = \int_{\mathbb{R}^N} \phi(t)\, \psi^c(t)\, m(dt) \tag{3.2}$$

In this case, it is known that

$$\int_{\mathbb{R}^N} \frac{m(dt)}{(1+|t|^2)^k} < \infty$$

for some integer k (cf. p. 242 of [7]). Note that if $M(\phi)$, $\phi \in S$ is a random measure, then $M(\phi)$ can be extended beyond S by continuity to all measurable functions f(t) for which

$$\int_{\mathbb{R}^N} |f(t)|^2\, m(dt) < \infty$$

Let E be a bounded set in $\mathbb{R}^N$, set $M(E) = M(\chi_E)$ where χ_E is the indicator function of the set

E. This yields the representation

$$M(\phi) = \int_{\mathbb{R}^N} \phi(t) M(dt) \tag{3.3}$$

A random r-current U_r is said to be a random measure of degree r, if there exist σ-finite measures $m_{ij}(dt)$ such that for every i, j, $|i| = |j| = r$ and $\phi, \psi \in S$

$$E\{ U_r(\phi \wedge e_i) \cdot U^c(\psi \wedge e_j) \} = \int_{\mathbb{R}^N} \phi(t)\, \psi^c(t)\, m_{ij}(dt) \tag{3.4}$$

Remark: m_{ij}, $|i| = |j| = r$, define a double current in the sense of deRham (cf. sections 12, 13 of [5]).

4. Homogeneous Currents

Consider $k \in \mathbb{R}^N$. For $\phi \in S$ define the shift $\left(\tau_h \phi\right)(t) = \phi(t+h)$ for every $t \in \mathbb{R}^N$. For $\phi \in S^r$, $\phi = \sum_{|i|} \phi_i e_i$ set $\tau_h \phi = \sum \phi_i(t+h) e_i$ and

$$\tau_h U_r(\phi) = U_r(\tau_h \phi)$$

Definition: A random current is said to be homogeneous if for every $h \in \mathbb{R}^N$, $\phi, \psi \in S^{N-r}$,

$$E\{ \tau_h U_r(\phi) \cdot \tau_h U_r^c(\psi) \} = E\{ U_r(\phi) U_r^c(\psi) \} \tag{4.1}$$

If U_r is a homogeneous current, then for i fixed, $U_r(\phi \cdot e_i)$ $\phi \in S$ defines a generalized wide-sense stationary random field on $\mathbb{R}^N$, indexed by $\phi \in S$. That is, setting

$$\{ V^i(\phi) = U_r(\phi e_i); \phi \in S, |i| = r \}$$

yields a generalized wide-sense stationary vector-valued field with $N!/(r!(N-r)!)$ components. By well-known results for generalized wide-sense stationary processes, we have the following:

Theorem 4.1: If U_r is an homogeneous current, then there exist random measures $M_i(d\nu)$

with $EM_i(d\nu) = 0$ and

$$E\{\ M_i(d\nu)\, M_j^c(d\nu')\ \} = M_{i,j}(d\nu \cap d\nu') \tag{4.2}$$

such that for any $\phi \in S$

$$U_r(\phi \cdot e_i) = \int_{\mathbf{R}^N} \hat{\phi}(\nu) M_i(d\nu)$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ (cf. equation (2.19)), $\phi \in S$. For $\psi \in S^{N-r}$ define the random r-measure M_r by

$$M_r(\psi) = M_r(\sum_{[i]} \psi_{i*}\, e_{i*})$$

$$= \sum_{[i]} M_i\,(\psi_{i*})\ ,\ \ |i| = r$$

Then for all $\phi \in S_{N-r}$

$$U_r(\phi) = M_r(\hat{\phi}) \tag{4.3}$$

Moreover, if ϕ, $\psi \in S$, then

$$EU_r(\phi \cdot e_i) \cdot U_r^c(\psi \cdot e_j) = \int_{\mathbf{R}^N} \hat{\phi}(\nu) \hat{\psi}^c(\nu) m_{i,j}(d\nu) \tag{4.4}$$

and there exists a finite integer k, such that

$$\int_{\mathbf{R}^N} \frac{m_{i,j}\,(d\nu)}{(1 + |\nu|^2)^k} < \infty \tag{4.5}$$

for all i, $|i| = r$. The measures $m_{i,j}(d\nu)$ will be called the spectral measures associated with U_r.

Remark: If U_r is homogeneous and $m_{i,j}(d\nu)$ $|i| = |j| = r$ is the spectral measure associated with U_r, then $*U_r$ is also homogeneous and $\tilde{m}_{i,j}\,(d\nu)$, the spectral measure associated with $*U_r$, satisfies for $i = [i]$, $j = [j]$, $|j| = |j| = N-r$,

$$\tilde{m}_{i,j} = m_{i*,j*} \tag{4.6}$$

Let U_r be an homogeneous r-current and $M_r(d\nu)$ the corresponding random measure defined by (4.3). We now consider a decomposition of $M_r(d\nu)$, which will play a key role later in this paper. First define $M_r^{(0)}(d\nu) = M_j^{(0)}(d\nu)$ by $M_j^{(0)}(d\nu \wedge \{ 0 \})$, i.e., $M_r^{(0)}(d\nu)$ is the part of the random measure $M_r^{(0)}(d\nu)$ supported by the point $\nu = 0$ and it corresponds to the "D.C." part of U_r. Let e_ν denote the unit vector in the direction from 0 to ν, $e_\nu = (\nu_1/|\nu|, \nu_2/|\nu|, \ldots, \nu_N/|\nu|)$. Set

$$M_r^{(u)}(d\nu) = M_r(d\nu) - M_r^{(0)}(d\nu) ,$$

which corresponds to the "A.C." part of U_r, and decompose $M_r^{(u)}(d\nu)$ into a component in the radial direction and one perpendicular to the radial direction (cf. (2.7)) as follows:

$$M_r^{(i)}(d\nu) = e_\nu \wedge (e_\nu \vee M_r^{(u)}(d\nu)) \tag{4.7}$$

$$M_r^{(s)}(d\nu) = e_\nu \vee (e_\nu \wedge M_r^{(u)}(d\nu) . \tag{4.8}$$

(i) stands for irrotational and (s) stands for solenoidal, as will be clarified later. By (2.7)

$$M_r(d\nu) = M_r^{(0)}(d\nu) + M_r^{(i)}(d\nu) + M_r^{(s)}(d\nu) , \tag{4.9}$$

corresponding to (4.9), define

$$U_r = U_r^{(0)} + U_r^{(i)} + U_r^{(s)} \tag{4.10}$$

Since for any vector a, $a \wedge a = 0$, $a \vee a = 0$, it follows by (2.20) that the solenoidal part $U_r^{(i)}$ characterized by $\left(U_r^{(i)}\right)^{(0)} = 0$ and $dU_r^{(i)} = 0$. Similarly by (2.21), the solenoidal part is characterized by $\left(U_r^{(s)}\right)^{(0)} = 0$ and $\delta U_r^{(s)} = 0$. $U_r^{(0)}$ is called the invariant part and is characterized by $\delta U_r^{(0)} = 0$, $dU_r^{(0)} = 0$.

The decomposition (4.10) was derived via (4.9), namely by a spectral decomposition argument, and it is this approach that will be needed later. We conclude this section by showing that there exists a random current ω_r such that

$$U_r - U_r^{(0)} = d\delta\omega_r + \delta d\omega_r \tag{4.11}$$

namely $U_r - U_r^{(0)} = \Delta\omega_r$, where Δ is the generalized Laplacian

$$\Delta = d\delta + \delta d \tag{4.12}$$

since $dd(\ ) = 0$, $\delta\delta(\ ) = 0$, (4.11) implies that $U_r^{(i)} = d\delta\omega_r$ and $U_r^{(s)} = \delta d\omega_r$. Taking the Fourier transform of both sides of the equation $U_r = \Delta\omega_r$ (assuming $U_r^{(0)} = 0$) yields via (2.20), (2.21), (4.11), and (4.12)

$$\hat{U}_r = (-1)^{r+1}\,|\nu|^2\,\hat{\omega}_r \tag{4.13}$$

If $|\nu|^{-2}\,\hat{U}_r$ is a current then equation (4.13) yields a solution for $\hat{\omega}_r$ by inversion; however, $|\nu|^{-2}\,\hat{U}_r$ need not be a current, because of the singularity at $|\nu| = 0$. A solution to (4.11) can still be derived as follows [2].

Let $q(\nu) = 1$ if $|\nu| \leq 1$ and zero otherwise. Let

$$G(t, \nu) = |\nu|^{-2}\,[e^{-i(\nu, t)} - (1 - i(\nu, t))\,q(\nu)]$$

Set

$$G(\phi_r, \nu) = \sum_{[I]} \left(\int_{\mathbf{R}^N} G(t, \nu)\,\phi_I(t)dt \right) e_I \tag{4.14}$$

and

$$\omega_r\left(\phi_{N-r}\right) = \int_{\mathbf{R}^N} G(\phi_{N-r}, \nu) \wedge M(d\nu) \tag{4.15}$$

where M is the random measure associated with U. Then

$$(d\omega_r)(\psi) = \omega_r(d\psi) = \int_{\mathbf{R}^N} G(d\psi, \nu) \wedge M(d\nu) \tag{4.16}$$

By letting $\theta(t) \in S$, integration by parts yields

$$\int_{\mathbf{R}^N} G(t, \nu)\frac{\partial\theta(t)}{\partial t_j}\,dt = -\int_{\mathbf{R}^N} \theta(t)\,\frac{\partial G(t, \nu)}{\partial t_j}\,dt$$

$$= -i \frac{\nu_j}{|\nu|} \int_{\mathbb{R}^N} \theta(t) \frac{e^{-i(\nu,t)} - q(\nu)}{|\nu|} dt \tag{4.17}$$

Repeated differentiation and substitution into (4.16) that ω_r as defined by (4.15) solves (4.11). Note that ω_r satisfying equation (4.11) is not unique since if ω_r satisfies (4.11) so does $\omega_r + \eta_r$ where η_r is a solution to $\Delta \eta_r = 0$ and Δ is as defined by equation (4.12).

5. Isotropic Currents

We start with a few words regarding the transformation of differential forms and currents induced by a rotation. Let A and B be replicas of $\mathbb{R}^N$, a rotation g defines a transformation from A onto B and every tangent vector in A is transformed into a tangent vector in B. Every differential form in B, $\phi = \sum \phi_I \cdot e_I$ defines a differential form on A as follows: let $i = (i_1, i_2, \cdots, i_r)$, and let $\nu_1, \nu_2, \cdots, \nu_r$ be vectors in A. Define $\sigma_g(\phi)$ by

$$\left(\sigma_g(\sum_{[I]} \phi_I(t) \cdot e_I)\right)\left(\nu_1, \nu_2, \cdots, \nu_r\right) = \left(\sum_{[I]} \phi_i(g \cdot t) \cdot e_i\right)\left(g\nu_1, g\nu_2, \cdots, g\nu_r\right)$$

$$= \left(\sum_{[I]} \phi_I(g \cdot t)(g' e_{I_1} \wedge g' e_{I_2} \wedge \cdots \wedge g' e_{I_r}\right)\left(\nu_1, \nu_2, \cdots, \nu_r\right) \tag{5.1}$$

where g' is the adjoint of g. Until this point g could have been any non-singular transformation. In particular, when g is a rotation or reflection $g' = g^{-1}$. This defines σ_g for differential forms, for currents set

$$\sigma_g U_r \left(\phi_{N-r}\right) = U_r\left(\sigma_{g'} \phi_{N-r}\right) \tag{5.2}$$

Let G denote the whole group of orthogonal transformations (rotations and reflections) in $\mathbb{R}^N$.

Definition: A random current U_r is said to be isotropic if for all ϕ,ψ in S^{N-r} and all $g \in G$

$$E\left(U_r(\phi)\, U_r^c(\psi)\right) = E\left(\sigma_g U_r(\phi) \sigma_g U_r^c(\psi)\right) \tag{5.3}$$

Lemma 5.1:

(a) If a random current U_r is homogeneous, then so is dU, $*U$ (and consequently δU).

(b) If a random current is isotropic, so is dU, $*U$, δU.

(c) If $U_r = dU_{r-1}$ and U_r is homogeneous (isotropic), then U_{r-1} is not necessarily homogeneous (isotropic).

Proof: The proof of (a) follows directly from the definitions, since d and * commute with τ_h and σ_g. Turning to (c), by letting $r = N = 2$, $\nu_1 = at_2\, dt_1$ where a is a zero mean Gaussian random variable, then we see $d\nu_1$ is both homogeneous and isotropic, but ν_1 is neither.

6. The characterization of the spectral measure associated with homogeneous and isotropic random currents

A (scalar-valued) σ-finite measure $m(d\nu)$ on $\mathbb{R}^N$ is said to be spherically invariant, if $m(A) = m(gA)$ for every Borel set A in $\mathbb{R}^N$ and every g in G. Let S_ρ^{N-1} denote the sphere of radius ρ in $\mathbb{R}^N$ (i.e. $S_\rho^{N-1} = \{ \nu : |\nu| = \rho \}$) and let $\eta_\rho(d\theta)$ denote the uniform measure on S_ρ^{N-1} where $\theta = \rho \cdot \nu/|\nu|$ and $\eta_\rho\left(S_\rho^{N-1}\right) = N \cdot \pi^{N/2} \cdot \rho^{N-1}\left(\Gamma(N + ½)\right)^{-1}$. Then $m(d\nu)$ is spherically invariant if and only if there exists a measure $\mu(d\rho)$ on $(0,\infty)$

$$\int_C m(d\nu) = \int_{[0,\infty)} \eta_\rho \; (C \cap \{\nu : |\nu| = \rho\})\, \mu(d\rho) \tag{6.1}$$

for every compact C in $\mathbb{R}^N$. Assume from now on that $m(C \cap \{ 0 \}) = 0$. Let ν be a point in $\mathbb{R}^N$, $|\nu| \neq 0$, and (λ, θ), $\lambda \in (0, \infty)$, $\theta \in S_1^{N-1}$, the coordinates of ν in a $(0,\infty) \times S_1^{N-1}$ coordinate system, then $\lambda = |\nu|$, θ is a point on the unit sphere corresponding to $(\nu_1/\lambda, \nu_2/\lambda, \cdots, \nu_N/\lambda)$. If $m(d\nu)$ is spherically invariant then for any bounded and measurable $f(\nu)$, define $\psi_f(\lambda, \theta) = f(\nu)$ and the following equality holds:

$$\int_{\mathbb{R}^N} f(\gamma)\, m(d\nu) = \int_{(0,\infty)\times S_1^{N-1}} \psi_f(\lambda,\theta)\, \eta(d\theta)\, F(d\lambda) \tag{6.2}$$

where

$$\eta(d\theta) = \eta_{\rho - 1}(d\theta)\,,$$

$$F(d\lambda) = \lambda^{N-1}\,\mu(d\lambda) \tag{6.3}$$

and $\mu(d\lambda)$ is as defined by (6.1).

A double (non-random) r-current is a continuous bilinear map from $S^{N-r} \times S^{N-r}$ to the reals (cf. [5], in our case $r_1 = r_2 = r$). A double r-current L is said to be spherically invariant if

$$L(g\phi, g\psi) = L(\phi, \psi) \tag{6.4}$$

for all $g \in G$, $\phi, \psi \in S^{N-r}$. Define $\sigma_g L$ through

$$(\sigma_g L)(\phi, \psi) = L(\sigma_g \phi, \sigma_g \psi) \tag{6.5}$$

then L is spherically invariant if and only if $\sigma_g L = L$.

Theorem 6.1: Let L be a spherically invariant double r-current, such that there exist slowly increasing measures $m_{i,j}(dt)$, for which

$$L\left(\phi_i \cdot e_{i*}, \psi_j \cdot e_{j*}\right) = \int_{\mathbb{R}^N} m_{i,j}(dt)\,\phi_i(t)\,\psi_j^c(t)$$

for all $|i| = |j| = r$, $\phi_i, \psi_j \in S$, then there exist two scalar-valued spherically invariant measures on $\mathbb{R}^N - \{0\}$ $m^{(i)}(dt)$, $m^{(s)}(dt)$ with $m^{(i)}(dt \cap \{0\}) = 0$, $m^{(s)}(dt \cap \{0\}) = 0$ and a constant $F^{(0)}$ such that for all i, j, $|i| = |j| = r$

$$m_{i,j}(dt) = (e_t \vee e_i, e_t \vee e_j)\, m^{(i)}(dt) + (e_t \wedge e_i, e_t \wedge e_j)\, m^{(s)}(dt) + (e_i, e_j)\, F^0 \cdot \delta(dt) \tag{6.6}$$

where $\delta(dt)$ is the Dirac unit measure supported by $t = 0$.

Remarks:

(a) Recall that if $m^{(i)}$ $(m^{(s)})$, is a spherically invariant measure on $\mathbb{R}^N - \{0\}$, then it is uniquely determined by a measure on $(0, \infty)$ and if $q = i$ or s, then

$m^{(q)}(dt) = \mu^{(q)}(d|t|)\,\eta_{|t|}(d\theta) = \mu^{(q)}(d|t|)\,|t|^{N-1}\eta(d\theta) = F^{(q)}(d|t|)\,\eta(d\theta)$.

(b) For the case of $r = 1$, $e_t = (t_1/|t|, t_2/|t|, \cdots t_N/|t|)$ and $(e_t \vee e_i)$ is the zero covector $t_i/|t|$, hence $(e_t \vee e_i, e_t \vee e_j) = t_i t_j/|t|^2$. Turning to $(e_t \wedge e_i, e_t \wedge e_j)$, note that $e_t \wedge e_i = (e_t - e_i) \wedge e_i$. Therefore for $i = j$,

$$(e_t \wedge e_i, e_t \wedge e_i) = \sum_{q=1}^{N} \left(t_q/|t|\right)^2 - \left(t_i^2/|t|^2\right) = 1 - \left(t_i^2/|t|^2\right)$$

For $i \neq j$, let $\tilde{e} = e_t - \left(t_i/|t|\right)e_i - \left(t_i/|t|\right)e_j$. Then

$$(e_t \wedge e_i, e_t \wedge e_j) = \left(\left(\tilde{e} + (t_j/|t|)e_j\right) \wedge e_i, \left(\tilde{e} + (t_j/|t|)e_i\right) \wedge e_j\right)$$

$$= -\left(t_j t_i/|t|^2\right)\left(e_i \wedge e_j, e_i \wedge e_j\right)$$

$$= -t_j t_i/|t|^2 \tag{6.7}$$

Hence

$$m_{i,j}(dt) = \frac{t_j t_i}{|t|^2}\, m^{(i)}(dt) + \left(\delta_{ij} - \frac{t_j t_i}{|t|^2}\right) m^{(s)}(dt) + \delta_{ij}\, F^0\, \delta(dt) \tag{6.8}$$

(c) Let e_0 be one of the basis vectors $e_1, e_2, \cdots, e_N$. Then for $|i| = |j| = r$,

$$\left((e_0 \vee e_i), (e_0 \vee e_j)\right) = \begin{cases} \pm 1 \text{ if } [i] = [j] \text{ and } e_0 \text{ is one of the vectors forming } e_i\ (e_0 \wedge e_i = 0) \\ 0 \text{ otherwise} \end{cases} \tag{6.9}$$

and

$$\left((e_0 \wedge e_i), (e_0 \wedge e_j)\right) = \begin{cases} \pm 1 \text{ if } [i] = [j] \text{ and } e_0 \text{ is not included in } e_i\ (e_0 \wedge e_{i*} = 0) \\ 0 \text{ otherwise} \end{cases} \tag{6.10}$$

and the sum of the two products is $\pm\delta_{ij}$ ($= (e_i, e_j)$)

Proof: Let $m(dt)$ denote the double r-current induced by the spectral measure $\{ m_{i,j}(dt),\ |i| = |j| = r \}$, i.e.

$$m(dt) = \sum_{[i],[j]} e_i \cdot m_{i,j}(dt) \cdot e_j \tag{6.11}$$

and for ϕ, ψ in S^{N-r},

$$m(\phi, \psi) = \int_{\mathbb{R}^N} \phi \wedge m(dt) \wedge \psi \tag{6.12}$$

For $g \in G$, define

$$\sigma_g m(dt) = \sum_{[i],[j]} g(e_i) \cdot m_{i,j}(g \cdot dt) \cdot g(e_j)$$

where

$$g\left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}\right) = g^{-1} e_{i_1} \wedge g^{-1} e_{i_2} \wedge \cdots g^{-1} e_{i_r} \tag{6.14}$$

Then, since L was assumed invariant,

$$m(\phi, \psi) = m\left(\sigma_g \phi, \sigma_g \psi\right)$$

$$= \left(\sigma_{g^{-1}} m\right)\left(\phi, \psi\right)$$

where the second inequality follows from (6.12) by a change of variables. Consequently

$$\sigma_g m(dt) = m(dt) \tag{6.15}$$

Now, let $\phi \in S^r$, consider $\mathbb{R}^N - 0$; let e_t denote the unit vector in the t direction; set (cf. (2.7))

$$\phi^{\perp} = \sum_{[i]} e_t \vee \left(e_t \wedge e_i\right) \phi_i(t)$$

and

$$\phi^{-} = \phi - \phi^{\perp}$$

Let t_0 be fixed, and consider an orthonormal basis with e_{t_0} as one of the basis vectors. Assume that the support of ϕ, $\psi \in S^{N-r}$ is in the vicinity of t_0. Let g be the reflection transforming e_t into $-e_t$ with all other basis vectors unchanged. Then $g\phi^{\perp} = -\phi^{\perp}$, $g\phi^{-} = \phi^{-}$. Hence (assuming first that $m_{i,j}(dt)$ have continuous densities and then approximating the $m_{i,j}(dt)$ by such measures,

$$m\left(\phi^{\perp}, \psi^{-}\right) = m\left(g\phi^{\perp}, \psi^{-}\right)$$

$$= m\left(-\phi^{\perp}, \psi^{-}\right)$$

$$= -m\left(\phi^{\perp}, \psi^{-}\right)$$

$$= 0 \tag{6.16}$$

Let $v_1, \cdots, v_N$ be an orthonormal basis with $e_{t_0} = v_1$ and $dt = (d|t| \times d\theta)$ where $(|t|, \theta)$ is the spherical representation of t. Then by the reflection argument above, with respect to any of the basis vector, we have

$$v_i \wedge m(d|t|, d\theta) \wedge v_j = 0 , [i] \neq [j] \tag{6.17}$$

Let t_1 ($\neq 0$) be a point in $\mathbb{R}^N$ and $v_1, v_2, \cdots, v_N$ an orthonormal basis with $v_i = e_{t_1}$. Let t_2 be another point in $\mathbb{R}^N$ with $|t_1| = |t_2|$ and $w_1, w_2, \cdots, w_N$ another orthonormal basis with $e_{t_2} = w_1$. Now, let $v_i = v_{i_1} \wedge v_{i_2} \wedge v_{i_r}$, $i = [i]$ and $w_j = w_{j_1} \wedge w_{j_2} \wedge w_{j_r}$, $j = [j]$ be r-covectors. Is there a $g \in G$, such that $gt_1 = t_2$ (namely, $gv_1 = w_1$) and $\sigma_g w_j = v_i$? Since $gt_1 = t_2$, it is necessary that either the index is in both **i** and **j** or in both **i*** and **j***. This condition is also sufficient: we define g as follows: let $\alpha = (\mathbf{i}, \mathbf{i}^*)$ be the concatenation of **i** with **i*** and similarly $\beta = (\mathbf{j}, \mathbf{j}^*)$. $\alpha = \left(\alpha_1, \alpha_2, \cdots, \alpha_N\right)$; $\beta = \left(\beta_1, \beta_2, \cdots, \beta_N\right)$. Set $g\, v_{\alpha_k} = w_{\beta_k}$, $k = 1, 2, \cdots, N$, then $\sigma_g w_j = w_i$ (equation 5.1) and the necessary condition assures that $gv_1 = w_1$. Let Q_1 be the collection of multi-indices **i** with the index 1 in **i** and Q_2 the collection of those with 1 in **i**. Then, in view of the above observation (with $t_1 = t_2$) and the spherical invariance of m, there exist two spherically invariant measures,

$\Phi_1(d\lambda \times d\theta) = F^{(1)}(d\lambda)\,\eta(d\theta)$, $\Phi_2(d\lambda \times d\theta) = F^{s}(d\lambda)\,\eta(d\theta)$, such that

$$v_i \wedge m(d\,|t| \times d\theta) \wedge v_i = \begin{cases} F^{(1)}(d\,|t|)\,\eta(d\theta)\,, & i \in Q_1 \\ F^{(s)}(d\,|t|)\,\eta(d\theta)\,, & i \in Q_2 \end{cases} \tag{6.18}$$

Equations (6.17) and (6.18) characterize $m(d\nu)$ in terms of local coordinates $\left(v_1(t)\,, v_2(t)\,,\ldots, v_N(t)\right)$, with $v_1(t) = e_t$. In order to restate these results in a fixed coordinate system $e_1, \cdots, e_N)$ in $\mathbb{R}^N$, consider $e_i \wedge m(dt) \wedge e_j$. Then

$$e_i = \sum_{k \in Q_1} \alpha_k v_k + \sum_{k \in Q_2} \alpha_k v_k$$

$$e_j = \sum_{k \in Q_1} \beta_k v_k + \sum_{k \in Q_2} \beta_k v_k$$

Hence, by (6.18) and (6.17)

$$e_i \wedge m(dt) \wedge e_j = \sum_{k \in Q_1} \alpha_k \beta_k F^{(1)}(d\,|t|)\,\eta(d\theta) + \sum_{k \in Q_2} \alpha_k \beta_k F^{(s)}(d\,|t|)\,\eta(d\theta) \tag{6.19}$$

By (2.7) to (2.9)

$$\sum_{k \in Q_1} \alpha_k \beta_k = \left(e_t \wedge (e_t \vee e_i),\, e_t \wedge (e_t \vee e_j)\right)$$

$$= (e_t \vee e_i\,,\ e_t \vee e_j)$$

and

$$\sum_{k \in Q_2} \alpha_k \beta_k = \left(e_t \vee (e_t \wedge e_i),\, e_t \vee (e_t \wedge e_j)\right)$$

$$= (e_t \wedge e_i\,,\ e_t \wedge e_j)$$

Substituting the last two equations into (6.19) yields (6.6).

Theorem 6.2 (*Ito's representation theorem):* Let U_r be an homogeneous random r-current and $\{ m_{I,J}(d\nu), |i| = |j| = r \}$ the spectral measure induced by U_r. A necessary and sufficient condition for U_r to be isotropic is that there exist two spherically symmetric measures, $\eta(d\theta)F^{(i)}(d\lambda)$ and $\eta(d\theta)F^{(s)}(d\lambda)$ $(\theta_i = \nu_i/\lambda, \lambda = |\nu|)$ where $F^{(i)}(d\lambda)$, $F^{(s)}(d\lambda)$ are slowly increasing measures on $(0, \infty)$ and a measure supported by $\nu = 0$, $F_0\delta(\nu)$ such that

$$m_{I,J}(d\nu) =$$
$$\Big((e_\nu \vee e_I), (e_\nu \vee e_J)\Big)\eta(d\theta)F^{(i)}(d\lambda) + \Big((e_\nu \wedge e_I), (e_\nu \wedge e_J)\Big)\eta(d\theta)F^{(s)}(d\lambda) + (e_I, e_J)F_0\,\delta(\nu) \tag{6.20}$$

Proof: Let

$$L_1(\phi, \psi) = E\Big(U_r(\phi)\, U_r^c(\psi)\Big),\ \phi, \psi \in \mathcal{S}^{N-r}.$$

Now set

$$L(\hat{\phi}, \hat{\psi}) = L_1(\phi, \psi)$$

where $\hat{\phi}$ is the Fourier transform of ϕ. In view of theorem 6.1, all we have to prove is that U_r is isotropic if and only if L is spherically invariant. From the definition of L, it follows that

$$L(\phi, \psi) = L_1(\hat{\phi}, \hat{\psi})$$

hence,

$$L(g\phi, g\psi) = L_1\Big((g\phi)\hat{}, (g\psi)\hat{}\Big)$$

Since g and the Fourier transform commute,

$$\begin{aligned} L(g\phi, g\psi) &= L_1(g\cdot\hat{\phi}, g\cdot\hat{\psi}) \\ &= L_1(\hat{\phi}, \hat{\psi}) \\ &= L(\phi, \psi) \end{aligned}$$

which completes the proof.

7. A sample function representation for U_r

It will be assumed throughout this section that $U_r^{(0)}$, the invariant part of U_r, is zero.

Theorem 7.1: Let $\widetilde{U}_r$ be an homogeneous and isotropic r-current, with spectral measure $m^{(i)}(d\nu), m^{(s)}(d\nu)$. Then there exist an (r-˜1) random measure $\hat{Y}(d\nu)$ satisfying

$$E\,\hat{Y}_i\,(d\nu)\cdot\hat{Y}_j^c\,(d\nu') = \begin{cases} 0\,, & [i] \neq [j] \\ m^{(i)}\,(d\nu \cap d\nu')\,, & i = j \end{cases} \tag{7.1}$$

and an (N—r—1) random measure $\hat{Z}(d\nu)$ satisfying

$$E\,\hat{Z}_i\,(d\nu)\,\hat{Z}_j^c\,(d\nu') = \begin{cases} 0\,, & [i] \neq [ji] \\ m^{(s)}\,(d\nu \cap d\nu')\,, & i = j \end{cases} \tag{7.2}$$

with $\hat{Y} = 0$ if $r = 0$, $\hat{Z} = 0$ if $r = N$ and

$$E\,\hat{Z}_i\,(d\nu)\,\hat{Y}_j^c\,(d\nu) = 0\,, \tag{7.3}$$

such that

$$U_r\,(\phi_{N-r}) \doteq \int_{\mathbf{R}^N} \hat{\phi}_{N-r}(\nu) \wedge \left(\frac{i\nu}{|\nu|} \wedge \hat{Y}\,(d\nu)\right) + \int_{\mathbf{R}^N} {}^*\hat{\phi}_{N-r}(\nu) \wedge \left(\frac{i\nu}{|\nu|} \wedge \hat{Z}\,(d\nu)\right) \tag{7.4}$$

Proof: Set

$$\hat{Y}(d\nu) = e_\nu \vee M_r(d\nu)a + e_\nu \wedge W^a(d\nu)$$

$$\hat{Z}(d\nu) = {}^*\left(e_\nu \wedge M_r(d\nu)\right) + {}^*\left(e_\nu \vee W^b(d\nu)\right) \tag{7.5}$$

where M_r is the random measure associated with U_r, $W^a(d\nu)$ is a random (r —2) measure independent of M_r and satisfying

$$E\,W_i^a(d\nu)\,W_j^a(d\nu') = \begin{cases} 0, & [i] \neq [j] \\ m^{(i)}(d\nu \cap d\nu'), & i = j,\ |i| = (r-2) \end{cases} \tag{7.6}$$

and $W^b(d\nu)$ is a random (r+2) measure independent of M_r and W^a, satisfying

$$E\, W_i^b(d\nu) W_j^b(d\nu') = \begin{cases} 0, & [i] \neq [j] \\ m^{(s)}(d\nu \cap d\nu'), & i = j,\ |i| = (r+2) \end{cases} \tag{7.7}$$

Substituting (7.5) into the right-hand side of (7.4) and using the identity $*(e_1 \vee M) = (e_1 \wedge *M)$ yield for the right-hand side of (7.4):

$$\int_{\mathbf{R}^N} \hat{\phi}_{N-r} \wedge \Big(e_\nu \wedge (e_\nu \vee M(d\nu) \Big) + \int_{\mathbf{R}^N} \hat{\phi}_{N-r} \wedge \Big(e_\nu \vee (e_\nu \wedge M(d\nu) \Big)$$

which, by (2.7) is just the left-hand side of (7.3). It remains, therefore, to be shown that $\hat{Y}(d\nu)$ and $\hat{Z}(d\nu)$ as defined by (7.4) satisfy equations (7.1), (7.2), and (7.3). This follows directly from (6.20) by considering at point ν a coordinate system with $e_\nu = e_1$, which completes the proof.

From the previous results, it is clear that if U_r is homogeneous and isotropic, then $U(\phi \cdot e_i)$ and $U(\psi \cdot e_j)$ are, in general, not orthogonal. However, as an application to the sample function representation it will be shown now that "if **i** differs considerably from **j**," then they are orthogonal. For this purpose we define $\Delta(\mathbf{i},\mathbf{j})$ to denote the number of indices in i which are not in j; thus $i = 1, 2, 3$, $j = 2, 3, 5$, $\Delta(i, j) = 1$. Note that $\Delta(j, i) = \Delta(i, j) = \Delta(i^*, j^*)$ for $|i| = |j|$. Hence $\Delta \leq \min(r, N-r)$.

Proposition 7.2: Let $\phi = \phi_a e_{i_a}$, $\psi = \psi_b e_{i_b}$, then

$$EU(\phi)\, U^c(\psi) = 0 \quad \text{whenever} \quad \Delta(i_a, i_b) \geq 2\,.$$

Proof: Consider first $m^{(s)}(d\nu) = 0$, then $\hat{Z}(d\nu) = 0$ and by theorem 7.1:

$$EU(\phi)U^c(\psi) = E \int_{\mathbf{R}^N \times \mathbf{R}^N} \Big(\hat{\phi}(\nu) \wedge \frac{i\nu}{|\nu|} \wedge \hat{Y}(d\nu) \Big) \Big(\hat{\psi}(\eta) \wedge \frac{i\eta}{|\eta|} \wedge \hat{Y}(d\eta) \Big)^c$$

Since $\hat{Y}$ is of orthogonal increments,

$$EU(\phi)U^c(\psi) = E \int_{\mathbf{R}^N} \Big(\hat{\phi}(\nu) \wedge \frac{i\nu}{|\nu|} \wedge \hat{Y}(d\nu) \Big) \Big(\hat{\psi}(\eta) \wedge \frac{i\nu}{|\nu|} \wedge \hat{Y}(d\nu) \Big)^c$$

Let $\mathbf{i_a}$, $\mathbf{i_b}$ be two r-multi-indices with $[\mathbf{i_a}] = \mathbf{i_a}$, $[\mathbf{i_b}] = \mathbf{i_b}$, set $\phi = \phi_a e_{i_a}, \psi = \phi_b e_{i_b}$, then

$$EU(\phi)\overline{U}(\psi) = E \int_{\mathbf{R}^N} \phi_a(\nu)\psi_b^c(\nu) \cdot \left(\sum_{(k,j) \in A_a} \frac{\nu_k}{|\nu|} \hat{Y}_j(d\nu) \right) \left(\sum_{(p,m) \in A_b} \frac{\nu_p}{|\nu|} \hat{Y}_m(d\nu) \right)^c \quad (7.8)$$

where A_a is the set of pairs (k,j) with $|k| = 1$, $|j| = r-1$, $|j| = \mathbf{j}$ and $[k, \mathbf{j}] = \mathbf{i_a}$ and A_b is defined similarly by $A_b = \{ (p, m):[p,m] = i_b \}$. By the orthogonality of Y_j, Y_m, $[m] \neq [j]$ and (2.6)

$$EU(\phi)U^c(\psi) = \int_{\mathbf{R}^N} \hat{\phi}_a(\nu)\hat{\psi}_b^c(\nu) \cdot \sum_{(k,p)} \frac{\nu_k \nu_p}{|\nu|^2} \left(\left(e_k \vee e_{i_a} \right), \left(e_p \vee e_{i_b} \right) \right) m^{(i)}(d\nu) \quad (7.9)$$

Note that, at most, one term in the above sum will be different from zero, cf. (2.6), which proves proposition 7.2 for $m^{(s)}(d\nu) = 0$. Turning to the case where $m^{(i)}(d\nu) = 0$, let $\phi = \phi_a e_{i_a^*}$, $\psi = \phi_b e_{i_b^*}$

$$U(\phi) = \int_{\mathbf{R}^N} \hat{\phi}_a(\nu) e_{i_a} \wedge \frac{i\nu}{|\nu|} \wedge \hat{Z}(d\nu)$$

Hence,

$$EU(\phi)U^c(\psi) = E\int_{\mathbf{R}^N} \hat{\phi}_a(\nu)\hat{\psi}_b^c(\nu) \left(\sum_{(k,j) \in B_a} \frac{\nu_k}{|\nu|} \hat{Z}_j \, d(\nu) \right) \cdot \left(\sum_{(p,m) \in B_b} \frac{\nu_p}{|\nu|} \hat{Z}_m \, d(\nu) \right)^c (7.10)$$

where B_a is the set of pairs (k, j), $|k| = 1$, $|j| = N - r - 1$ and such that $[k, \mathbf{j}] = \mathbf{i}_a^*$, and B_b is defined similarly with p, $\mathbf{m}] = \mathbf{i}_b^*$.

Hence,

$$U(\phi)U^c(\psi) = \int_{\mathbf{R}^N} \hat{\phi}_a(\nu)\hat{\phi}_b^c(\nu) \sum_{(k,p)} \frac{\nu_k \nu_p}{|\nu|^2} \cdot \left(\left(e_k \vee e_{i_a^*} \right), \left(e_p \vee e_{i_b^*} \right) \right) m^{(s)}(d\nu) \quad (7.11)$$

which completes the proof.

References

[1] H. Flanders, *Differential Forms,* Academic Press, 1963.

[2] K. Ito, "isotropic random current," *Proceedings,* Third Berkeley Symp. on Math. Stat. and Prob., 1956, pp. 125-32.

[3] S. Ito, "On the canonical form of turbulence," *Nagoya Math. J.,* vol. 2, 1951, pp. 83-92.

[4] V. Mandrekar, "Markov properties of random fields," *Probabilistic Analysis and Related Topics,* vol. 3, pp. 161-193, A.T. Bharucha-Reid (ed.), Academic Press, 1983.

[5] G. deRham, *Differentiable Manifolds,* Springer-Verlag, 1982.

[6] Y.A. Rozanov, *Markov Random Fields,* Academic Press, 1982.

[7] L. Schwartz, *Theory des distributions,* Herman, 1966.

[8] C. von Westenholtz, *Differential Forms in Mathematical Physics,* North Holland, 1981.

[9] E. Wong and B. Hajek, *Stochastic Processes in Engineering Systems,* Springer-Verlag, 1985.

[10] E. Wong and M. Zakai, "Markov processes on the plane," *Stochastics,* vol. 15, 1985, pp. 311-333.

[11] E. Wong and M. Zakai, "Martingale differential forms," *Prob. Th. Rel. Fields,* vol. 74, 1987, pp. 429-453.

[12] E. Wong and M. Zakai, "Isotropic Gauss Markov currents," to appear in *Prob. Th. Rel. Fields.*

[13] A.M. Yaglom, "Some classes of random fields in n-dimensional space related to stationary random processes," *Theory of Probability and Its Applications,* vol. 2, 1957, pp. 273-320.

[14] A.M. Yaglom, *Correlation theory of stationary and related random functions.* Vol. I, II. Springer-Verlag, New York, 1987