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ON SPECTRAL MEASURES OF STRINGS AND EXCURSIONS OF QUASI DIFFUSIONS

by

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Abstract

The probabilistic counterpart of the theory of strings is the theory of quasi diffusions. The concept of quasi diffusion (generalised diffusion, gap diffusion) generalises the concept of one-dimensional diffusion in that it does not require the speed measure to be strictly positive. This note focuses on some connections between the spectral theory of strings and the excursion theory of quasi diffusions. The main difference in our approach compared with the previous ones is that we are using Krein's theory for "killed" strings as a primary tool instead of dual strings. It is seen that this approach provides a natural setting for various spectral representations for quasi diffusions. In particular, we discuss representations for first hitting time distributions, Lévy measures of inverse local times, and different quantities connected with the Ito excursion law. We consider also the characterisation problem for inverse local times. In fact, it is seen that this is equivalent with the inverse spectral problem for "killed" strings.

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1. Introduction

We use Dym, McKean [3] and Kac, Krein [5] as our basic references. The notation is mainly adopted from [3]. References to Krein's original papers can be found in [3] or [5].

Let lmk be a string in the sense of [3] p. 147, and $X = \{X_t \ ; \ t \geq 0\}$ the corresponding quasi diffusion. In the singular case $l + m(l-) = \infty$ the process X is obtained from a Brownian motion B via a random time change based on the additive functional

$$\alpha_t := \int_{0-}^{+\infty} L_t(x) dm(x),$$

where $(t, x) \to L_t(x)$ is the jointly continuous version of the local time of B (with the Ito-McKean normalization). In the regular case $l + m(l-) < \infty$ X is a random time change of a Brownian motion killed when it hits $l + k$, $0 \leq k \leq +\infty$.

The process X is a Hunt process. Its state space is $E :=$ closure(I_m), where I_m is the set of the points of increase of the function m. It is assumed that $0 \in I_m$. Recall that in this case 0 is always a regular, reflecting point for X.

The infinitesimal generator, G, of X acting on $\mathbf{M} := \mathbf{L}^2([0, l], dm)$ is the generalised second order differential operator $d^2/dm\,dx$. Its domain is a subset of (see [3] p. 151)

$$\mathbf{D}_-(G) := \mathbf{D}_o(G) \cap \{f : f^-(0) = 0\}.$$

The notation $l\hat{m}k$ is used for a string, which is defined as lmk but instead of $\mathbf{D}_-(G)$

we use

$$\hat{\mathbf{D}}_-(G) := \mathbf{D}_o(G) \cap \{f : f(0) = 0\}.$$

For $l\hat{m}k$ it is always assumed that $m(0) = 0$. The corresponding quasi diffusion, $\hat{X}$, is obtained via a random time change as above using a Brownian motion killed when it hits zero. Therefore, the string $l\hat{m}k$ is called a killed string.

For the operator G associated with lmk and $l\hat{m}k$ we introduce as in [3] p. 162–176 the functions A, D and C. For any complex number ω these functions are solutions of the equation $Gu = -\omega^2 u$. In particular, $A(0;\omega) = 1$, $A^-(0;\omega) = 0$, and $C(0;\omega) = 0, C^-(0;\omega) = 1$. For $\omega^2 \in \mathbf{C}^- := \mathbf{C}\backslash[0, +\infty)$

$$D(0;\omega) = \lim_{x \uparrow l} \frac{kC^+(x;\omega) + C(x;\omega)}{kA^+(x;\omega) + A(x;\omega)}. \tag{1.1}$$

In fact, in the singular case $l + m(l-) = \infty$

$$D(0;\omega) = \lim_{x \uparrow l} \frac{C(x;\omega)}{A(x;\omega)}. \tag{1.2}$$

Recall also the Wronskians

$$A^+D - AD^+ = A^-D - AD^- = 1,$$
$$C^+D - CD^+ = C^-D - CD^- = D(0).$$

The Green functions (w.r.t. the measure induced by the function m) for the processes X and $\hat{X}$ are given by ($\omega = ib,\ b > 0$)

$$G_\omega(x,y) = \begin{cases} A(x;\omega)D(y;\omega) & \text{if } x \le y, \\ A(y;\omega)D(x;\omega) & \text{if } x \ge y, \end{cases}$$

and

$$\hat{G}_\omega(x,y) = \begin{cases} \dfrac{C(x;\omega)D(y;\omega)}{D(0;\omega)} & \text{if } x \le y, \\ \dfrac{C(y;\omega)D(x;\omega)}{D(0;\omega)} & \text{if } x \ge y, \end{cases}$$

respectively. For $\omega = ib,\ b > 0$, the functions $x \to A(x;\omega),\ x \to C(x;\omega)$ are increasing, and $x \to D(x;\omega)$ is decreasing.

In the next section we discuss the relationships of the strings lmk and $l\hat{m}k$ to their principal spectral functions (i.e. Krein's correspondence theorems). In fact, for killed strings the principal spectral function alone does *not* determine the string uniquely.

In the third section the principal spectral function of the string $l\hat{m}k$ is used to derive representations for the first hitting time distributions and Lévy measures of the inverse local times for the process X. Moreover, a number of representations connected with the Ito excursion law of X are presented.

Spectral representations for the diffusion hitting times have been considered in Kent [7] and [8]. In fact, in [8] Theorem 1.1 the canonical measure is identified with a spectral measure of a killed process. However, the point we want to make here is that these representations should be seen as a link in a chain of representations - first transition density, then hitting time distribution, and thirdly Lévy measure of the inverse local time.

Representations for Lévy measures of inverse local times have been considered in Knight [9], Kotani, Watanabe [10] p. 248 , and [11]. In [10] and [11] the discussion is based on the concept of dual string (see [3] p. 622), and not on the properties of killed strings. Also in [9] the solution of the characterization problem is obtained without explicit use of killed strings. It is seen below that the spectral theory of killed strings is a natural setting for this problem.

2. Principal spectral functions for lmk and $l\hat{m}k$

We assume that 0 is a point of increase of m and $m(0) = 0$. The latter assumption is necessary for $\hat{\mathbf{D}}(G)$ to be dense in $\mathbf{M}$. Probabilistically this means that the inverse local time at zero of the process X has no drift (see Ito, McKean [4]). A simple condition in terms of the principal spectral function of lmk for this to hold can be found in [3] p. 192, and is given in (2.3)(ii) below.

For the definition of the generalized differential operator G and the domain $\mathbf{D}_+(G)$ see [3] p. 147–149.

2.1 **Theorem** *The operator G acting on the domains (i) $\mathbf{D}(G) := \mathbf{D}_-(G) \cap \mathbf{D}_+(G)$ and (ii) $\hat{\mathbf{D}}(G) := \hat{\mathbf{D}}_-(G) \cap \mathbf{D}_+(G)$ is in both cases self-adjoint and non-positive for each permissible choice of k, $0 \le k \le \infty$.*

Proof of these well known facts can be found for $(G, \mathbf{D}(G))$ in [3] p. 153–158. The case with $(G, \hat{\mathbf{D}}(G))$ can be proved similarly with obvious modifications and using the amplification in [3] p. 167. □

Definitions *(i) The odd non-decreasing function Δ is a principal spectral function of the string lmk if the Green function G_ω, $\omega^2 \in \mathbf{C}^-$, can be represented as*

$$G_\omega(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{A(x;\gamma)A(y;\gamma)}{\gamma^2-\omega^2}\,d\Delta(\gamma) \tag{2.1}$$

for $0 \le x, y \le l, x$ and/or $y = l$ excluded in the singular case $l + m(l-) = \infty$.

(ii) The odd non-decreasing function $\hat{\Delta}$ is a principal spectral function of the string $l\hat{m}k$ if the Green function $\hat{G}_\omega$, $\omega^2 \in \mathbf{C}^-$, can be represented as

$$\hat{G}_\omega(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{C(x;\gamma)C(y;\gamma)}{\gamma^2-\omega^2}\,d\hat{\Delta}(\gamma) \tag{2.2}$$

for $0 \le x$, $y \le l$, x and/or $y = l$ excluded in the singular case $l + m(l-) = \infty$.

The following theorem is a restriction of a deep result of M.G. Krein to the strings lmk with $m(0) = 0$.

2.2 **Theorem** *For lmk there exists a unique principal spectral function Δ. It has the properties*

$$\text{(i)} \quad \int_{-\infty}^{+\infty} \frac{d\Delta(\gamma)}{\gamma^2+1} < \infty, \quad \text{(ii)} \quad \int_{-\infty}^{+\infty} d\Delta(\gamma) = \infty. \tag{2.3}$$

Conversely, for a given odd non-decreasing function Δ with the properties (2.3) there exists a unique string lmk having Δ as the principal spectral function.

Proof is given in [5] or [3] (see p. 176). The condition (2.3)(ii) is due to the assumption $m(0) = 0$ (see [3] p. 192). □

2.3 Theorem *For $l\hat{m}k$ there exists a unique principal spectral function $\hat{\Delta}$ such that*

$$\text{(2.4)} \qquad \text{(i)} \quad \int_{-\infty}^{+\infty} \frac{d\hat{\Delta}(\gamma)}{\gamma^2(\gamma^2+1)} < \infty, \quad \text{(ii)} \quad \int_{-\infty}^{+\infty} \frac{d\hat{\Delta}(\gamma)}{\gamma^2} = \infty.$$

Conversely, for a given odd non-decreasing function $\hat{\Delta}$ with the properties (2.4) there exists a killed string $l\hat{m}k$ having $\hat{\Delta}$ as the principal spectral function.

Proof The existence and the uniqueness of $\hat{\Delta}$ with the properties (2.4) is stated in [5] p. 81–82, and can be proved by modifying the proof for the string lmk in [3] p. 176.

For the converse introduce

$$\text{(2.5)} \qquad \hat{D}(0;\omega) := \int_{-\infty}^{+\infty} \left(\frac{1}{\gamma^2-\omega^2} - \frac{1}{\gamma^2}\right) d\hat{\Delta}(\gamma),$$

where $\omega^2 \in \mathbf{C}^-$. The function $\hat{D}$ is well defined by (2.4)(i). Functions representable as in (2.5) are called S^{-1}- functions in [5] Theorem S1.5.2 . By [5] Lemma S1.5.2 the function

$$\omega^2 \to D^\star(0;\omega) := -\frac{1}{\hat{D}(0;\omega)}$$

is an S-function i.e. there exists an odd non-decreasing function Δ and a constant $c \geq 0$ such that ($\omega^2 \in \mathbf{C}^-$)

$$D^\star(0;\omega) = c + \int_{-\infty}^{+\infty} \frac{d\Delta(\gamma)}{\gamma^2-\omega^2}.$$

Because $D^\star(0;\omega) \geq 0$ for $\omega^2 < 0$ it is easily seen that

$$c = \lim_{\omega^2 \to -\infty} D^\star(0;\omega)$$

(cf. [5] Remark 5.1). Consequently, $c = 0$ because

$$\lim_{\omega^2 \to -\infty} \hat{D}(0;\omega) = \lim_{\omega^2 \to -\infty} \int_{-\infty}^{+\infty} \frac{\omega^2}{\gamma^2(\gamma^2-\omega^2)} d\hat{\Delta}(\gamma) = -\infty$$

by Fatou's lemma and (2.4)(ii). According to Krein's correpondence , i.e. Theorem 2.1 (without (2.3)(ii)), there exists a unique string lmk having Δ as the principal spectral function. We claim that for this string $m(0) = 0$ or equivalently

$$\int_{-\infty}^{+\infty} d\Delta(\gamma) = \infty.$$

Assume this is not the case. Then we obtain by monotone convergence

$$\begin{aligned}\int_{-\infty}^{+\infty} d\Delta(\gamma) &= -\lim_{\omega^2 \to -\infty} \int_{-\infty}^{+\infty} \frac{\omega^2}{\gamma^2 - \omega^2} d\Delta(\gamma) \\ &= -\lim_{\omega^2 \to -\infty} \omega^2 D^\star(0;\omega) = \lim_{\omega^2 \to -\infty} \frac{\omega^2}{\hat{D}(0;\omega)}.\end{aligned}$$

But

$$\lim_{\omega^2 \to -\infty} \frac{\hat{D}(0;\omega)}{\omega^2} = \lim_{\omega^2 \to -\infty} \int_{-\infty}^{+\infty} \frac{d\hat{\Delta}(\gamma)}{\gamma^2(\gamma^2 - \omega^2)} = 0$$

again by monotone convergence using (2.4)(i). Consequently, $m(0) = 0$. Next, we claim that the string lmk has infinite length i.e. $l = \infty$ and/or $l + k = \infty$. By [3] p. 192 this is equivalent to

$$\lim_{\omega^2 \uparrow 0} D^\star(0;\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\Delta(\gamma)}{\gamma^2} = +\infty. \tag{2.6}$$

The claim (2.6) follows from

$$\lim_{\omega^2 \uparrow 0} \hat{D}(0;\omega) = \lim_{\omega^2 \uparrow 0} \int_{-\infty}^{+\infty} \frac{\omega^2}{\gamma^2(\gamma^2 - \omega^2)} d\hat{\Delta}(\gamma) = 0,$$

which is obtained using (2.4)(i) and dominated convergence. From the infinite string lmk, $m(0) = 0$, we construct the corresponding killed string $l\hat{m}k$. This string has $\hat{\Delta}$ as the principal spectral function (see [5] p. 82), and the proof is complete. □

Remarks (i) As is seen from the proof above the function $\hat{\Delta}$ does not determine the string $l\hat{m}k$ uniquely. Note also that the quasi diffusion corresponding to the above constructed string lmk is recurrent. We refer to Knight [9] for some constructions of different killed strings with the same principal spectral function. The terminology in [9] is however different.

(ii) It is seen in the next section that the function $\hat{\Delta}$ can be used to give the spectral representation for the Lévy measure of the inverse local time at zero of the process X. Therefore, Theorem 2.2 gives the solution of the characterization problem for inverse local times.

(iii) Note that the condition (2.4)(ii) quarantees that $0 \in I_m$ for the killed string $l\hat{m}k$ (see [5] p. 82).

3. Spectral representations for quasi diffusions

Let X and $\hat{X}$ be quasi diffusions corresponding to lmk and $l\hat{m}k$, respectively. Recall the assumption $m(0) = 0$. Using the spectral representations for the Green functions it can be proved that the processes X and $\hat{X}$ have symmetric transition densities (w.r.t. m). These are denoted with p and $\hat{p}$, respectively, and are given by

$$p(t; x, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) A(x; \gamma) A(y; \gamma) d\Delta(\gamma),$$

$$\hat{p}(t; x, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) C(x; \gamma) C(y; \gamma) d\hat{\Delta}(\gamma), \tag{3.1}$$

where $0 \leq x, y \leq l$, x and/or $y = l$ excluded in the case $l + m(l-) = \infty$. The functions A, Δ and $C, \hat{\Delta}$ are as in (2.1) and (2.2), respectively.

We use (3.1) to obtain additional spectral representations for the process X (and $\hat{X}$). All our representations are connected with the Ito excursion law for the excursions from zero. Therefore, it seems to be motivated to derive these using the Ito excursion law as a tool. This approach - after some basic facts from the excursion theory are assumed known - leads quite easily to the results.

Let $(\mathbf{U}, U)$ be an appropriate excursion space for excursions of X from zero. Elements in $\mathbf{U}$ are denoted with ξ. The life time of an excursion , denoted with ζ, is infinite if the excursion never returns to 0. In this case there exists a random time point t_o such that $\xi_t = \dagger$ for all $t \geq t_o$. Here, $\dagger$ is the cemetary point. The Ito excursion law is denoted with ν. For sets in U, which we are considering, the following description of ν is sufficient (see Pitman, Yor [12])

$$\nu\{\cdot\} = \lim_{x \downarrow 0} \frac{\hat{\mathbf{P}}_x(\cdot)}{x}, \tag{3.2}$$

where $\hat{\mathbf{P}}_x$ is the probability measure associated with $\hat{X}$ when started from x.

Let $\tau_0 := \inf\{t : X_t = 0\}$, and denote the $\mathbf{P}_x$-density of τ_0 with $n_x(\cdot; 0)$. The notation $\mathbf{P}_x$ is used for the probability measure associated with X when started from x.

3.1 Theorem *The function $n_x(t; 0)$ has the spectral representation*

$$n_x(t; 0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) C(x; \gamma) d\hat{\Delta}(\gamma). \tag{3.3}$$

Proof Recall (see Getoor [2], and Csáki et.al. [1])

$$\nu\{\xi_t \in dx\} = n_x(t;0)dm(x).$$

By (3.2) we have

$$\nu\{\xi_t \in dx\} = \lim_{y\downarrow 0} \frac{\hat{\mathbf{P}}_y(\hat{X}_t \in dx)}{y}.$$

Consequently,

$$\begin{aligned} n_x(t;0) &= \lim_{y\downarrow 0} \frac{\hat{p}(t;x,y)}{y} \\ &= \lim_{y\downarrow 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) C(x;\gamma) \frac{C(y;\gamma)}{y} d\hat{\Delta}(\gamma). \end{aligned} \tag{3.4}$$

The function $x \to C(x;\gamma)$ is for every γ the unique solution of the integral equation $(x \geq 0)$

$$f(x) = x + \gamma^2 \int_0^x dy \int_0^{y+} f(z)dm(z)$$

(see [5] p. 29–30). Therefore, we can take the limit inside the integral sign in (3.4). This completes the proof. □

Remark. The representation (3.3) can also be found - perhaps in a slightly implicit form - in Kent [8].

For $x \in E$ let $L^x = \{L_t^x;\ t \geq 0\}$ be the local time of X at x having the Ito-McKean normalization i.e. for all $A \in B(E)$

$$\int_0^t 1_A(X_s)ds = \int_A L_t^x dm(x) \qquad \text{a.s.}$$

Consider the local time at 0, and denote it with L. Let α be the right continuous inverse of L. Then α is an increasing Lévy process with the Lévy-Khintchin representation

$$\mathbf{E}_0(\exp(-\lambda\alpha_t)) = \exp(-t(c + \int_{0+}^{+\infty} (1 - \exp(-\lambda u))n(du))), \tag{3.5}$$

where $c := l^{-1}$ in the case $l + m(l-) = \infty$, and $c := (l+k)^{-1}$ in the case $l + m(l-) < \infty$, and n is the Lévy measure of α.

3.2 Theorem *The measure n in (3.5) is absolutely continuous w.r.t. the Lebesgue measure, and the density $n(t)$ has the spectral representation*

$$n(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) d\hat{\Delta}(\gamma). \tag{3.6}$$

Proof It is obvious that

$$\nu\{\zeta \in dt\} = n(dt).$$

By (3.2) we have

$$\nu\{\zeta \in dt\} = \lim_{x \downarrow 0} \frac{\hat{\mathbf{P}}_x(\tau_0 \in dt)}{x}.$$

Therefore, consider

$$\begin{aligned} n(t) &= \lim_{x \downarrow 0} \frac{n_x(t;0)}{x} = \lim_{x \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) \frac{C(x;\gamma)}{x} d\hat{\Delta}(\gamma) \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(-\gamma^2 t) d\hat{\Delta}(\gamma) \end{aligned}$$

by (3.3) and a similar argument as in the proof of Theorem 3.1. □

Remarks (i) Combining Theorems 2.3 and 3.2 we have a solution of the characterization problem of inverse local times. Note that the condition (2.4)(ii) implies that the measure n has infinite mass i.e. α is not discrete.

(ii) In Knight [9] a more direct approach is used to solve the characterization problem. This does not seem to give the identification of the representing measure. See also Kotani, Watanabe [10], where the concept of dual string is used.

In the case $l + m(l-) < \infty$ we consider the decomposition

$$\nu\{\zeta > t\} = \nu\{\zeta > t, \xi_t = \dagger\} + \nu\{\zeta > t, \xi_t \in E\}.$$

In fact, $N_1(t) := \nu\{\zeta > t, \xi_t = \dagger\} = \nu\{\xi_t = \dagger\}$ therefore, $t \to N_1(t)$ is non-decreasing. The function $N_2(t) := \nu\{\zeta > t, \xi_t \in E\}$ is non-increasing.

3.3 Proposition *The measures induced by the functions N_1 and N_2 are absolutely continuous w.r.t. the Lebesgue measure, and the densities $n_1(t)$ and $n_2(t)$, respectively, have the spectral representations*

$$n_1(t) = \begin{cases} \dfrac{1}{\pi k} \displaystyle\int_{-\infty}^{+\infty} \exp(-\gamma^2 t) C(l;\gamma) d\hat{\Delta}(\gamma) & \text{if } 0 < k \le \infty, \\ -\dfrac{1}{\pi} \displaystyle\int_{-\infty}^{+\infty} \exp(-\gamma^2 t) C^+(l;\gamma) d\hat{\Delta}(\gamma) & \text{if } \quad k = 0, \end{cases} \tag{3.7}$$

and

$$n_2(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \exp(-\gamma^2 t)(1 - C^+(l;\gamma))d\hat{\Delta}(\gamma). \tag{3.8}$$

Proof We have

$$\nu\{\zeta > t, \xi_t = \dagger\} = \lim_{x\downarrow 0} \frac{\hat{\mathbf{P}}_x(X_{\zeta-} = l, \zeta < t)}{x}, \tag{3.9}$$

where on the right hand side in the parentesis we have simplified the notation by omitting "hat":s. In the case $0 < k < \infty$ we have

$$\hat{\mathbf{P}}_x(X_{\zeta-} = l, \zeta < t) = \int_0^t ds \int_E \kappa(dy)\hat{p}(s;x,y)$$

(see [4] p. 184), where κ is the killing measure of the process $\hat{X}$. Here

$$\kappa(dy) = \frac{1}{k}\epsilon_l(dy),$$

where ϵ_l is Dirac's measure at l. This leads to (3.7). In the case $k = \infty$ the point l is reflecting and the left hand side of (3.9) equals zero. Hence, (3.7) holds even for $k = \infty$. For $k = 0$ we have

$$\hat{\mathbf{P}}_x(X_{\zeta-} = l, \zeta < t) = \hat{\mathbf{P}}_x(\tau_l < t) = \int_0^t n_x(s;l)ds$$

and

$$n_x(s;l) = \lim_{y\uparrow l} -\frac{\hat{p}(s;x,y)}{l-y},$$

which leads to (3.7). For (3.8) consider

$$\begin{aligned} N_2(t) := \nu\{\zeta > t, \xi_t \in E\} &= \int_E n_x(t;0)dm(x) \\ &= \int_E dm(x)\frac{1}{\pi}\int_{-\infty}^{+\infty} d\hat{\Delta}(\gamma)\exp(-\gamma^2 t)C(x;\gamma) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} d\hat{\Delta}(\gamma)\exp(-\gamma^2 t)\int_E dm(x)C(x;\gamma), \end{aligned}$$

where the change of the order of the integration is permitted because $l+m(l-) < \infty$. We may also differentiate under the integral sign to obtain

$$\begin{aligned} n_2(t) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} d\hat{\Delta}(\gamma)\exp(-\gamma^2 t)\int_E dm(x)\gamma^2 C(x;\gamma) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \exp(-\gamma^2 t)(1 - C^+(l;\gamma)d\hat{\Delta}(\gamma), \end{aligned}$$

where we have used the fact that $C(\cdot;\gamma)$ is a solution of $Gu = -\gamma^2 u$. The proof is complete. □

Remark. Making use of the boundary condition

$$kC^+(l;\gamma) + C(l;\gamma) = 0$$

it is seen (cf. (3.6)) that

$$n(t) = n_2(t) - n_1(t),$$

which is quite obvious.

We conclude by presenting a more implicit spectral representation connected with the Ito excursion law of the maximum, M, of an excursion. For this consider the process X killed when it hits a given point $x \in E, x > 0$. Denote this process with X^x and let $X_0^x = y < x$. Further, denote with n^x the density of the Levy measure for the inverse local time at zero of the process X^x.

3.4 **Proposition** *Let $n(t)$ be as in 3.2 Proposition and $n^x(t)$ as above. Then*

$$\nu\{\zeta \in du, M \geq x\} = (n(u) - n^x(u))du,$$

$$\nu\{\zeta = \infty\} = \frac{1}{l+k}, \qquad \nu\{M \geq x\} = \frac{1}{x},$$

and

$$\nu\{\zeta < \infty, M \geq x\} = \frac{1}{x} - \frac{1}{l+k}.$$

Proof We use the formula ($x \in E, x > 0$)

$$\nu\{\zeta \in dt, M \geq x\} = \frac{dt}{x} \int_0^t n_0^{\uparrow}(s;x) n_x(t-s;0)ds, \tag{3.10}$$

where

$$n_0^{\uparrow}(\cdot;x) := \lim_{y \downarrow 0} \frac{x}{y} n_y(\cdot;x)$$

i.e. $n^{\uparrow}$ is the first hitting time density for the process $\hat{X}$ conditioned never to hit 0. The formula (3.10) is a straight forward generalization of the corresponding formula for a Brownian motion (see [13]). Taking the Laplace transforms in u on the both sides of (3.10) we obtain

$$\begin{aligned}\nu\{\exp(-\lambda^2\zeta); M \geq x\} &= \frac{1}{x}\frac{x}{C(x;\mathrm{i}\lambda)}\frac{D(x;\mathrm{i}\lambda)}{D(0;\mathrm{i}\lambda)} \\ &= \frac{A(x;\mathrm{i}\lambda)}{C(x;\mathrm{i}\lambda)} - \frac{1}{D(0;\mathrm{i}\lambda)},\end{aligned}$$

where we have used the definition of D (see [3] p. 175). Further, (see [3] p. 172)

$$-\frac{1}{D^x(0;\mathrm{i}\lambda)} = -\frac{A(x;\mathrm{i}\lambda)}{C(x;\mathrm{i}\lambda)},$$

where D^x is "the function D" for the string lmk, $l = x$, $k = 0$, i.e. for the process X^x. From [5] p. 82

$$\begin{aligned}-\frac{1}{D^x(0;\mathrm{i}\lambda} &= -\frac{1}{x} + \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{\gamma^2+\lambda^2} - \frac{1}{\gamma^2}\right) d\hat{\Delta}^x(\gamma)\\ &= -\frac{1}{x} - \int_0^\infty (1-\exp(-\gamma^2 u))n^x(u)du\end{aligned}$$

by Fubini's theorem and (3.6). Similarly, using $\lim_{\lambda\to 0} D(0;\mathrm{i}\lambda) = l + k$, we obtain

$$-\frac{1}{D(0;\mathrm{i}\lambda)} = -\frac{1}{l+k} - \int_0^\infty (1-\exp(-\gamma^2 u))n(u)du.$$

Consequently,

$$\begin{aligned}\nu\{\exp(-\lambda^2\zeta); M \geq x\} &= \nu\{\exp(-\lambda^2\zeta); \zeta < \infty, M \geq x\}\\ &= -\frac{1}{l+k} + \frac{1}{x} - \int_0^\infty (1-\exp(-\gamma^2 u))(n(u) - n^x(u))du.\end{aligned}$$

Letting $\lambda \to 0$ we obtain

$$\nu\{\zeta < \infty,\ M \geq x\} = \frac{1}{x} - \frac{1}{l+k}$$

because

$$\lim_{\lambda\to 0}\int_0^\infty (1-\exp(-\gamma^2 u))n(u)du = \lim_{\lambda\to 0}\int_0^\infty (1-\exp(-\gamma^2 u))n^x(u)du = 0.$$

Consequently,

$$\nu\{\exp(1-\lambda^2\zeta); \zeta < \infty, M \geq x\} = \int_0^\infty (1-\exp(-\gamma^2 u))(n(u) - n^x(u))du,$$

which gives

$$\nu\{\zeta \in du, M \geq x\} = (n(u) - n^x(u))du.$$

Finally, from the description of X as a random time change of a Brownian motion it follows

$$\nu\{\zeta = \infty\} = \frac{1}{l+k}.$$

The proof is complete. □

Remarks (i) Note also the formulae

$$\nu\{\zeta < \infty,\ M = l\} = \frac{k}{l(l+k)},$$

and

$$\nu\{\zeta \in du,\ x \le M < y\} = (n^y(u) - n^x(u))du, \tag{3.11}$$

where $x, y \in E$. The probabilistic explanation of (3.11) is quite apparent.

(ii) In [11] Cor. 4.6 the above result is proved for recurrent quasi diffusions in terms of dual strings. The well known result $\nu\{M \ge x\} = 1/x$ is due to Williams (see [14]).

References

[1] Csáki, E., Földes, A., Salminen, P.: On the joint distribution of the maximum and its location for a linear diffusion. *Ann. Inst. H. Poincare* Vol. 23(2), 1987, p. 179–194.

[2] Getoor, R.K.: Excursions of a Markov process. *Ann. Prob.* Vol. 7(2), 1979, p. 244–266.

[3] Dym, H., McKean, H.: *Gaussian Processes, Function theory and Inverse Spectral Problem.* Academic Press. New York (1976).

[4] Ito, K., McKean, H.: *Diffusion Processes and their Sample Paths.* Springer Verlag. Berlin (1965).

[5] Kac, I.S., Krein, M.G.: On the spectral functions of the string. *Amer. Math. Soc. Transl.* II Ser. 103, 1974, p. 19–102.

[6] Kac, I.S., Krein, M.G.: R-functions - analytic functions mapping the upper half plane into itself. *Amer. Mat. Soc. Transl.* II Ser. 103, 1974, p. 1–18.

[7] Kent, J.T.: Eigenvalue expansions for diffusion hitting times. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 52, 1980, p. 309–319.

[8] Kent, J.T.: The spectral decomposition of a diffusion hitting time. *Ann. Prob.* 10, 1982, p. 207–219.

[9] Knight, F.B.: Characterization of the Levy measures of inverse local times of gap diffusion. In Seminar on Stochastic Processes, 1981, ed. Cinlar, E., Chung, K.L., Getoor, R.K. *Progress in Probability and Statistics,* Birkhäuser, Boston (1981).

[10] Kotani, S., Watanabe, S.: Krein's spectral theory of strings and generalized diffusion processes. In Functional Analysis in Markov Processes, ed. Fukushima, M. *Lect. notes in math.,* 923, Springer Verlag, Berlin (1981).

[11] Küchler, U.: On sojourn times, excursions and spectral measures connected with quasi diffusions. *J. Math. Kyoto Univ.,* 26(3), 1986, p. 403–421.

[12] Pitman, J., Yor, M.: A decomposition of Bessel bridges. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 59, 1982, p. 425–457.

[13] Salminen, P.: Brownian excursions revisited. In Seminar on Stochastic Processes, 1983, ed Cinlar, E., Chung, K.L., Getoor, R.K. *Progress in Probability and Statistics* 7, Birkhäuser, Boston (1984)

[14] Williams, D.: *Diffusions, Markov Processes and Martingales.* Wiley and Sons, London (1979).