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A REMARK ON THE CLASS OF MARTINGALES
WITH BOUNDED QUADRATIC VARIATION

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Let (Ω, \mathcal{F}, P) be a fixed probability space with a filtration (\mathcal{F}_t) satisfying the usual conditions, and consider the class H_∞ of all martingales M adapted to this filtration such that $\langle M \rangle_\infty \in L_\infty$.

The aim of this short note is to prove the following.

PROPOSITION 1. Suppose the existence of a predictable mobile time T such that $P(T > 0) > 0$. Then there exists a bounded continuous martingale which does not belong to the closure \bar{H}_∞ in BMO .

The definition of a mobile time is given in [1]: that is, it is a stopping time T such that for some continuous local martingale X we have $\langle X \rangle_t < \langle X \rangle_T$ on $\{t < T\}$. Recall that a uniformly integrable martingale M is said to be in the class BMO if

$$\|M\|_{BMO} = \sup_T \|E[(M_\infty - M_{T-})^2 | \mathcal{F}_T]^{1/2}\|_\infty < \infty,$$

where the supremum is taken over all stopping times T .

In order to prove Proposition 1, we need the next three lemmas.

LEMMA 1. For $M \in BMO$, let $b(M)$ be the supremum of the set of b for which

$$\sup_T \|E[\exp\{b^2(\langle M \rangle_\infty - \langle M \rangle_{T-})\} | \mathcal{F}_T]\|_\infty < \infty.$$

Then we have

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$$b(M) \geq \frac{1}{\sqrt{2} d(M, H_\infty)},$$

where $d(M, H_\infty)$ is the distance in BMO from M to H_∞ .

PROOF. Recall that, if $\|X\|_{\text{BMO}} < 1$, then

$$E[\exp(\langle X \rangle_\infty - \langle X \rangle_T) | \mathcal{F}_T] \leq (1 - \|X\|_{\text{BMO}}^2)^{-1}$$

for every stopping time T . This is well known as the John-Nirenberg theorem. Let now $0 < b < 1 / \{\sqrt{2} d(M, H_\infty)\}$. Then we have $2b^2 \|M - N\|_{\text{BMO}}^2 < 1$ for some $N \in H_\infty$. Since $\langle N \rangle_\infty \leq C$ for some constant $C > 0$, we find for $s < t$

$$\langle M \rangle_t - \langle M \rangle_{s-} \leq 2(\langle M - N \rangle_t - \langle M - N \rangle_{s-}) + 2C.$$

Then from the John-Nirenberg theorem it follows that

$$\begin{aligned} E[\exp\{b^2(\langle M \rangle_\infty - \langle M \rangle_{T-})\} | \mathcal{F}_T] &\leq e^{2b^2 C} E[\exp\{2b^2(\langle M - N \rangle_\infty - \langle M - N \rangle_{T-})\} | \mathcal{F}_T] \\ &\leq e^{2b^2 C} (1 - 2b^2 \|M - N\|_{\text{BMO}}^2)^{-1}, \end{aligned}$$

which implies that $b \leq b(M)$. Thus the lemma is proved.

LEMMA 2. Let A be an increasing process such that $A_t < A_\infty$ for every finite t . Then there exists a positive continuous increasing process U such that $\int_0^\infty U_s dA_s = \infty$ a.s.

For the proof, see Lemma 2 in [1].

LEMMA 3. Suppose the existence of a predictable mobile time $T > 0$ a.s. Then there exists a continuous local martingale L satis-

fying $\langle L \rangle_\infty = \infty$ a.s.

PROOF. By the definition of a mobile time, for some continuous local martingale X we have $\langle X \rangle_t < \langle X \rangle_T$ on $\{t < T\}$, and, T being predictable, there is a sequence (T_n) of stopping times such that $T_n \uparrow T$ a.s. and $T_n < T$ for every n . Let now $g_n: [n-1, n[\rightarrow [0, \infty[$ be an increasing homeomorphic function, and set

$$\tau_t = \max [T_{n-1}, \min \{ T_n, g_n(t) \}] \quad (t \in [n-1, n[).$$

Then $(\tau_t)_{0 \leq t < \infty}$ is a continuous change of time such that $\tau_0 = 0$, $\tau_k = T_k$ ($k=1, 2, \dots$) and further $\tau_t < T$ for every finite t . So, the process Y defined by $Y_t = X_{\tau_t}$ ($0 \leq t < \infty$) is a continuous local martingale over (F_{τ_t}) , and we have for every finite t

$$\langle Y \rangle_t = \langle X \rangle_{\tau_t} < \langle X \rangle_T = \langle Y \rangle_\infty .$$

Thus from Lemma 2 it follows that $\int_0^\infty U_s d\langle Y \rangle_s = \infty$ a.s. for some positive continuous increasing process $U = (U_t, F_{\tau_t})$.

Next, let $\sigma_t = \inf \{ s : \tau_s > t \}$ and $V_t = U_{\sigma_t}$. As is easily verified, each σ_t is a stopping time with respect to (F_{τ_t}) , so that V_t is $F_{\tau_{\sigma_t}}$ -measurable. However, since $\tau_{\sigma_t} \leq t$ by the definition of σ_t , we find that V_t is in fact F_t -measurable. Thus the stochastic integral $L_t = \int_0^t \sqrt{V_s} dX_s$ is well-defined. Since $L_{\tau_t} = \int_0^{\tau_t} \sqrt{V_s} dY_s$, we find

$$\langle L \rangle_\infty \geq \langle L \rangle_{\tau_\infty} = \int_0^\infty V_{\tau_s} d\langle Y \rangle_s = \int_0^\infty U_{\sigma_{\tau_s}} d\langle Y \rangle_s .$$

Noticing that U is increasing and $\sigma_{\tau_s} \geq s$, we have in conclusion

$$\langle L \rangle_{\infty} \geq \int_0^{\infty} U_s d\langle Y \rangle_s = \infty \text{ a.s.}$$

This completes the proof.

PROOF OF PROPOSITION 1. Let T be a predictable mobile time such that $P(T>0)>0$. We may assume that $T>0$ a.s., because there is no question on $\{T=0\}$. Then by Lemma 3 there exists a continuous local martingale L such that $\langle L \rangle_{\infty} = \infty$ a.s. Let $\theta_t = \inf\{s : \langle L \rangle_s > t\}$ and $W_t = L_{\theta_t}$. As is well known, the process $W = (W_t, F_{\theta_t})$ is a one dimensional Brownian motion. Next, let $\sigma = \inf\{t : |W_t| = 1\}$. Note that $\exp(\pi^2 \sigma / 8)$ is not integrable. It is easy to see that θ_{σ} is a stopping time with respect to (F_t) , and so the process $M_t = L_{t \wedge \theta_{\sigma}}$ is a continuous local martingale over (F_t) . Recalling that L is constant on the stochastic interval $[[t, \theta_{\langle L \rangle_t}]]$, we find

$$M_t = L_{\theta_{\langle L \rangle_t} \wedge \theta_{\sigma}} = W_{\langle L \rangle_t \wedge \sigma},$$

from which it follows that $|M| \leq 1$. On the other hand, we have $\langle M \rangle_{\infty} = \langle L \rangle_{\theta_{\sigma}} = \sigma$, so that $\exp(\pi^2 \langle M \rangle_{\infty} / 8)$ is not integrable. Then $b(M) \leq \pi / \sqrt{8}$ by the definition of $b(M)$ and so we have $d(M, H_{\infty}) \geq 2/\pi$ by Lemma 1. This completes the proof.

We can also verify, under the same assumption as in Proposition 1, that $H_{\infty} \setminus L_{\infty} \neq \emptyset$.

As a corollary to Proposition 1, we shall remark that a change of law gives sometimes rise to a morbid phenomenon. For that, let M be a local martingale such that the solution Z of the equation $Z_t = 1 + \int_0^t Z_{s-} dM_s$ is a uniformly integrable positive martingale, and

let $d\hat{P} = Z_\infty dP$. Then for every continuous local martingale X the process $\hat{X} = \langle X, M \rangle - X$ is a continuous local martingale with respect to $d\hat{P}$ such that $\langle \hat{X} \rangle = \langle X \rangle$ under either probability measure (see [3]). This is better known as transformation of drift or the Girsanov transformation. Note that, if $M \in \text{BMO}$ and $\Delta M \geq -1 + \delta$ for some δ with $0 < \delta \leq 1$, then the mapping $X \rightarrow \hat{X}$ is an isomorphism of BMO onto $\text{BMO}(\hat{P})$ (see [2]). However, we obtain the following interesting result.

PROPOSITION 2. Suppose the existence of a predictable mobile time T such that $P(T > 0) > 0$. Then there is a probability measure \hat{P} equivalent to P such that $\hat{X} \notin H_1(\hat{P})$ for some bounded continuous martingale X .

PROOF. By Proposition 1 there is a bounded continuous martingale X which does not belong to \bar{H}_∞ . As a matter of course, $\langle X \rangle_\infty^{1/2}$ is not bounded. Since the dual of L_1 is L_∞ , there exists a random variable $W > 0$ a.s., $E[W] = 1$, such that $E[W \langle X \rangle_\infty^{1/2}] = \infty$. Then, letting $d\hat{P} = W dP$, the conclusion follows immediately.

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