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RICHARD F. BASS

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# Using Stochastic Comparison to Estimate Green's Functions

Richard F. Bass

Department of Mathematics, University of Washington  
Seattle, WA 98195 U.S.A.

**1. Introduction** Let  $L$  be the second order elliptic operator on  $\mathbb{R}^d$  defined by

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

The purpose of this note is to illustrate a technique by which very good estimates for the Green's function for  $L$  can be obtained using elementary stochastic calculus, in particular, comparison theorems. This technique could be applied to many other situations as well.

Suppose the  $a_{ij}$  are bounded, uniformly strictly elliptic, and Dini continuous. That is, there exists  $\lambda$  independent of  $x$  such that

$$\lambda \sum_{i=1}^d y_i^2 \leq \sum_{i,j=1}^d y_i y_j a_{ij}(x) \quad \text{if } (y_1, \dots, y_d) \in \mathbb{R}^d$$

and

$$\int_{0+} \frac{\omega(r)}{r} dr < \infty,$$

where  $\omega(r) = \sup_{i,j, |x-y| \leq r} |a_{ij}(x) - a_{ij}(y)|$ . Let  $(P^x, X_t)$  be the strong Markov process corresponding to  $L$  ([5]), let  $D$  be a bounded domain, and define the Green's function for  $L$  to be a function  $g_D(x, y)$  such that

$$E^x \int_0^{\tau_D} f(X_s) ds = \int_D f(y) g_D(x, y) dy$$

for all  $f$  bounded, where

$$\tau_D = \inf\{t : X_t \notin D\}.$$

Our theorem is

**Theorem 1.1.** *Suppose the  $a_{ij}$  satisfy the conditions above. Suppose  $d \geq 3$  and  $\delta > 0$ . Then there exist constants  $c_1, c_2$  (depending on  $\delta$ ) and a version of  $g_D(x, y)$  such that*

$$c_1 |x - y|^{2-d} \leq g_D(x, y) \leq c_2 |x - y|^{2-d}$$

*whenever  $|x - y| < \delta$  and  $\text{dist}(x, \partial D), \text{dist}(y, \partial D) > 2\delta$ .*

By a slight refinement of our proof, one can show

$$g_D(x, y) \sim c |x - y|^{2-d} \quad \text{as } |x - y| \rightarrow 0.$$

Both the statement and proof of Theorem 1.1 go through for  $d = 2$  provided  $|x - y|^{2-d}$  is replaced by  $-\log |x - y|$ .

Theorem 1.1 was first proved (in a partial differential equations formulation) by Gilbarg and Serrin [2] as a consequence of some results of theirs about extended maximum principles. The extended maximum principles of [2] can also be obtained as consequences of Theorem 1.1. Gilbarg and Serrin also gave some examples to show that the estimates of Theorem 1.1 need not hold if the assumption of Dini continuity is removed.

Some notation: if  $Y$  is any continuous one dimensional process, let

$$T_{Y,a} = \inf\{t : Y_t = a\},$$

and  $T_{Y,a,b} = T_{Y,a} \wedge T_{Y,b}$ . The letter  $c$ , with or without subscripts, denotes constants whose value may change from line to line.

**2. Some facts about one dimensional diffusions.** We prove some simple facts about certain one dimensional diffusions. These could also be obtained as consequences of more general results about diffusions [4].

**Lemma 2.1.** *Suppose  $R > 0$ ,  $r < R/2$ , and  $Y_t$  solves*

$$dY_t = dW_t + \frac{d-1-\beta(Y_t)}{2Y_t}dt, \quad Y_0 = r,$$

where  $W_t$  is one dimensional Brownian motion,  $\beta$  is bounded, nonnegative, and  $\int_0^1 y^{-1}\beta(y)dy < \infty$ . Then

- (i)  $P^r(T_{Y,0} < T_{Y,R}) = 0$ , and
- (ii) if  $\varepsilon < r/2$ ,  $E^r \int_0^{T_{Y,R}} 1_{[0,\varepsilon]}(Y_s)ds \leq c(R)\varepsilon^d r^{2-d}$ .

**Proof.** Define a function  $s$  by

$$s(x) = \int_x^1 y^{1-d} \exp\left(-\int_y^1 z^{-1}\beta(z)dz\right)dy.$$

(This is the scale function.) Note that  $s'(x) < 0$ , hence  $s$  is decreasing,

$$(2.1) \quad c_1 x^{1-d} \leq |s'(x)| \leq c_2 x^{1-d} \quad \text{for } x < R/2,$$

$$(2.2) \quad c_3 x^{2-d} \leq s(x) \leq c_4 x^{2-d} \quad \text{for } x < R/2,$$

and

$$(2.3) \quad \frac{1}{2}s''(x) + \frac{d-1-\beta(x)}{2x}s'(x) = 0.$$

Let  $Z_t = s(Y_t)$ . By Ito's lemma,  $Z_t$  solves

$$(2.4) \quad dZ_t = \sigma(Z_t) d\widetilde{W}_t, \quad Z_0 = s(r),$$

for  $t < T_{Y,0}$ , where  $\widetilde{W}_t$  is a Brownian motion and  $\sigma(y) = s'(s^{-1}(y))$ .

Now  $Z_t$  is a continuous martingale, hence a time change of Brownian motion, and so if  $a < r < R$ ,

$$P^y(T_{Y,a} < \infty, T_{Y,a} < T_{Y,R}) = P^{s(y)}(T_{Z,s(a)} < \infty, T_{Z,s(a)} < T_{Z,s(R)}) \leq \frac{s(y) - s(R)}{s(a) - s(R)}.$$

Letting  $a \rightarrow 0$  and observing  $s(a) \rightarrow \infty$  by (2.2) gives (i).

If  $W_t$  is a Brownian motion, then it is well-known ([1], page 363) that if  $a \leq x \leq b$ ,

$$E^x \int_0^{T_{W,a,b}} f(W_t) dt = \int f(y) G_{a,b}(x, y) dy,$$

where  $G_{a,b}(x, y) = \begin{cases} 2(x-a)(b-y)/(b-a), & x \leq y \\ 2(y-a)(b-x)/(b-a), & x \geq y \end{cases}$  if  $a \leq x \leq y \leq b$ , 0 otherwise.

If  $B_t = \int_0^t \sigma^2(Z_s) ds$  and  $A_t = B_t^{-1}$ , then  $Z_{A_t}$  is a continuous martingale with quadratic variation  $t$ , hence  $Z_{A_t}$  is a Brownian motion. Therefore

$$E^x \int_0^{T_{Z,a,b}} h(Z_t) dt = E^x \int_0^{T_{W,a,b}} h\sigma^{-2}(W_t) dt = \int G_{a,b}(x, y) (h\sigma^{-2})(y) dy.$$

And then

$$\begin{aligned} E^x \int_0^{T_{Y,a,b}} h(Y_t) dt &= E^{s(x)} \int_0^{T_{Z,s(a),s(b)}} h \circ s^{-1}(Z_t) dt \\ &= \int G_{s(b),s(a)}(s(x), y) (h\sigma^{-2}) \circ s^{-1}(y) dy \\ &= \int G_{s(b),s(a)}(s(x), s(y)) \frac{h(y)}{s'(y)} dy, \end{aligned}$$

the last equality following by a change of variables and the definition of  $\sigma$ .

As  $a \rightarrow 0$ ,  $s(a) \rightarrow \infty$ , and

$$G_{s(b),s(a)}(s(x), s(y)) \rightarrow \begin{cases} 2(s(x) - s(b)) & s(y) \geq s(x) \text{ or } y \leq x \\ 2(s(y) - s(b)) & s(y) \leq s(x) \text{ or } y \geq x \end{cases}$$

boundedly and pointwise.

Therefore

$$E^x \int_0^{T_{Y,0,R}} 1_{[0,\varepsilon]}(Y_t) dt = \int_0^\varepsilon \frac{2(s(r) - s(R))}{s'(y)} dy \leq cs(r)\varepsilon^d,$$

which proves (ii).  $\square$

The same proof shows

**Corollary 2.2.** If “ $\beta$  nonnegative” is replaced by “ $\beta$  nonpositive” and “ $\leq$ ” in (ii) is replaced by “ $\geq$ ”, then Lemma 2.1 still holds.

**3. Proof of Theorem 1.1** By a change of coordinates, suppose  $y = 0 \in D$  and  $a(0)$  is the identity matrix. If  $q(x) > 0$  and  $\tilde{L}$  is defined by  $\tilde{L}f(x) = q(x)Lf(x)$ , it is well-known ([5]) that the Green’s function for  $\tilde{L}$  is  $\tilde{g}_D(x, y) = g_D(x, y)/q(y)$ . So using the strict ellipticity of  $a(x)$ , we may assume without loss of generality that  $x^*a(x)x/|x|^2 \equiv 1$ , where  $*$  denotes transpose.

Pick  $\sigma$  so that  $\sigma\sigma^*(x) = a(x)$ , and let  $X_t$  be a solution to

$$(3.1) \quad dX_t = \sigma(X_t)d\widehat{W}_t, \quad X_0 = x,$$

where  $\widehat{W}$  is some  $d$  dimensional Brownian motion.

Let  $r = |x|$ ,  $R = 2 \operatorname{diam}(D)$ ,  $V_t = |X_t|$ . Then by Ito’s lemma,  $V_t$  solves

$$(3.2) \quad dV_t = V_t^{-1}X_t^*\sigma(X_t)d\widehat{W}_t + (\operatorname{trace}(a(X_t)) - 1)/(2V_t)dt, \quad V_0 = r,$$

for  $t < T_{V,0}$ . Since  $x^*a(x)x/|x|^2 = 1$ , the martingale part of  $V_t$  has quadratic variation  $t$ , hence is a Brownian motion  $W_t$ .

Let  $\beta$  be a function that is Lipschitz on compact subintervals of  $(0, \infty)$ , bounded, and satisfies

$$\beta(u) \geq \sup_{|z|=u} |\operatorname{trace} a(z) - d| + 2(\log u)^{-2}, \quad u \leq R,$$

and  $\int_0^1 u^{-1}\beta(u)du < \infty$ , (This is possible by the Dini continuity of  $a$ .) Let  $Y_t$  be the solution to

$$dY_t = dW_t + (d - 1 - \beta(Y_t))/(2Y_t)dt, \quad Y_0 = r, \quad t < T_{Y,0}.$$

If we modify the drift coefficients for  $V_t$  and  $Y_t$  to something smooth and bounded when  $V_t \leq b$ ,  $Y_t \leq b$ ,  $b < r$ , we do not change  $V_t$  or  $Y_t$  for  $t < T_{Y,b} \wedge T_{V,b}$ .

Since  $\operatorname{trace}(a(x)) \geq d - \beta(|x|) + (\log(|x|))^{-2}$  for  $|x| \leq R$ , we can apply [3], p.352, to get  $V_t \geq Y_t$ , a.s. for  $t \leq T_{V,b,R} \wedge T_{Y,b,R}$ . Since  $Y_t$  does not hit 0 before time  $T_{Y,R}$  by Lemma 2.1 (i), we can let  $b \rightarrow 0$  to conclude  $V_t \geq Y_t$  for all  $t \leq T_{Y,R}$ .

If  $B_\varepsilon$  is the ball of radius  $\varepsilon$  about 0, and  $\varepsilon < |x|/2$ , we then have, using Lemma 2.1 (ii),

$$(3.3) \quad E^x \int_0^{T_D} 1_{B_\varepsilon}(X_t)dt \leq E^x \int_0^{T_{V,R}} 1_{[0,\varepsilon]}(V_t)dt \leq E^x \int_0^{T_{Y,R}} 1_{[0,\varepsilon]}(Y_t)dt \leq c\varepsilon^d |x|^{2-d}.$$

Since  $y$  was arbitrary as long as  $\operatorname{dist}(y, \partial D) > \delta$ , the estimate (3.3) both tells us that  $g_D(x, y)$  exists and gives the upper bound.

To get the lower bound, let  $R = 2\delta$ . Note that we have  $P^r(T_{V,0} < \infty, T_{V,0} < T_{V,R}) = 0$  from above. We compare the process  $V_t$  with the solution  $\widehat{Y}_t$  to

$$d\widehat{Y}_t = dW_t + (d - 1 + \beta(\widehat{Y}_t))/(2\widehat{Y}_t)dt, \quad \widehat{Y}_0 = r.$$

We argue just as in the upper bound, using Corollary 2.2 in place of Lemma 2.1.  $\square$

1. L. Breiman, *Probability*. Addison-Wesley, Reading, Mass., 1968.
2. D. Gilbarg and J. Serrin, On isolated singularities of solutions of second order elliptic differential equations. *J. Analyse Math.* 4 (1955/56) 309–340.
3. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*. North Holland/Kodansha, Amsterdam, 1981.
4. K. Itô and H.P. McKean, Jr., *Diffusion Processes and their Sample Paths*. Springer, New York, 1974.
5. D.W. Stroock and S.R.S. Varadhan, *Multidimensional Diffusion Processes*. Springer, New York, 1979.