RICHARD F. BASS

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Using Stochastic Comparison to Estimate Green's Functions

Richard F. Bass
Department of Mathematics, University of Washington
Seattle, WA 98195 U.S.A.

1. Introduction Let L be the second order elliptic operator on $\mathbb{R}^d$ defined by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

The purpose of this note is to illustrate a technique by which very good estimates for the Green's function for L can be obtained using elementary stochastic calculus, in particular, comparison theorems. This technique could be applied to many other situations as well.

Suppose the a_{ij} are bounded, uniformly strictly elliptic, and Dini continuous. That is, there exists λ independent of x such that

$$\lambda \sum_{i=1}^{d} y_i^2 \le \sum_{i,j=1}^{d} y_i y_j a_{ij}(x) \quad \text{if } (y_1, \dots, y_d) \in \mathbb{R}^d$$

and

$$\int_{0+} \frac{\omega(r)}{r} dr < \infty,$$

where $\omega(r) = \sup_{i,j,|x-y|\le r} |a_{ij}(x) - a_{ij}(y)|$. Let (P^x, X_t) be the strong Markov process corresponding to L ([5]), let D be a bounded domain, and define the Green's function for L to be a function $g_D(x,y)$ such that

$$E^x \int_0^{\tau_D} f(X_s)ds = \int_D f(y) g_D(x,y) dy$$

for all f bounded, where

$$\tau_D = \inf\{t : X_t \notin D\}.$$

Our theorem is

Theorem 1.1. *Suppose the a_{ij} satisfy the conditions above. Suppose $d \ge 3$ and $\delta > 0$. Then these exist constants c_1, c_2 (depending on δ) and a version of $g_D(x,y)$ such that*

$$c_1|x-y|^{2-d} \le g_D(x,y) \le c_2|x-y|^{2-d}$$

whenever $|x-y| < \delta$ and dist $(x, \partial D)$, dist $(y, \partial D) > 2\delta$.

By a slight refinement of our proof, one can show

$$g_D(x,y) \sim c|x-y|^{2-d} \quad \text{as } |x-y| \to 0.$$

Both the statement and proof of Theorem 1.1 go through for $d = 2$ provided $|x-y|^{2-d}$ is replaced by $-\log|x-y|$.

Theorem 1.1 was first proved (in a partial differential equations formulation) by Gilbarg and Serrin [2] as a consequence of some results of theirs about extended maximum principles. The extended maximum principles of [2] can also be obtained as consequences of Theorem 1.1. Gilbarg and Serrin also gave some examples to show that the estimates of Theorem 1.1 need not hold if the assumption of Dini continuity is removed.

Some notation: if Y is any continuous one dimensional process, let

$$T_{Y,a} = \inf\{t : Y_t = a\},$$

and $T_{Y,a,b} = T_{Y,a} \wedge T_{Y,b}$. The letter c, with or without subscripts, denotes constants whose value may change from line to line.

2. Some facts about one dimensional diffusions. We prove some simple facts about certain one dimensional diffusions. These could also be obtained as consequences of more general results about diffusions [4].

Lemma 2.1. *Suppose $R > 0$, $r < R/2$, and Y_t solves*

$$dY_t = dW_t + \frac{d-1-\beta(Y_t)}{2Y_t}dt, \quad Y_0 = r,$$

where W_t is one dimensional Brownian motion, β is bounded, nonnegative, and $\int_0^1 y^{-1}\beta(y)dy < \infty$. Then

(i) $P^r(T_{Y,0} < T_{Y,R}) = 0$, and

(ii) if $\varepsilon < r/2$, $E^r \int_0^{T_{Y,R}} 1_{[0,\varepsilon]}(Y_s)ds \leq c(R)\varepsilon^d r^{2-d}$.

Proof. Define a function s by

$$s(x) = \int_x^1 y^{1-d} \exp(-\int_y^1 z^{-1}\beta(z)dz)dy.$$

(This is the scale function.) Note that $s'(x) < 0$, hence s is decreasing,

$$c_1 x^{1-d} \leq |s'(x)| \leq c_2 x^{1-d} \quad \text{for } x < R/2, \tag{2.1}$$

$$c_3 x^{2-d} \leq s(x) \leq c_4 x^{2-d} \quad \text{for } x < R/2, \tag{2.2}$$

and

$$\frac{1}{2}s''(x) + \frac{d-1-\beta(x)}{2x}s'(x) = 0. \tag{2.3}$$

Let $Z_t = s(Y_t)$. By Ito's lemma, Z_t solves

$$dZ_t = \sigma(Z_t)d\widetilde{W}_t, \quad Z_0 = s(r), \tag{2.4}$$

for $t < T_{Y,0}$, where $\widetilde{W}_t$ is a Brownian motion and $\sigma(y) = s'(s^{-1}(y))$.

Now Z_t is a continuous martingale, hence a time change of Brownian motion, and so if $a < r < R$,

$$P^y(T_{Y,a} < \infty, T_{Y,a} < T_{Y,R}) = P^{s(y)}(T_{Z,s(a)} < \infty, T_{Z,s(a)} < T_{Z,s(R)}) \le \frac{s(y) - s(R)}{s(a) - s(R)}.$$

Letting $a \to 0$ and observing $s(a) \to \infty$ by (2.2) gives (i).

If W_t is a Brownian motion, then it is well-known ([1], page 363) that if $a \le x \le b$,

$$E^x \int_0^{T_{W,a,b}} f(W_t)dt = \int f(y)G_{a,b}(x,y)dy,$$

where $G_{a,b}(x,y) = \begin{cases} 2(x-a)(b-y)/(b-a), & x \le y \\ 2(y-a)(b-x)/(b-a), & x \ge y \end{cases}$ if $a \le x \le y \le b$, 0 otherwise.

If $B_t = \int_0^t \sigma^2(Z_s)ds$ and $A_t = B_t^{-1}$, then Z_{A_t} is a continuous martingale with quadratic variation t, hence Z_{A_t} is a Brownian motion. Therefore

$$E^x \int_0^{T_{Z,a,b}} h(Z_t)dt = E^x \int_0^{T_{W,a,b}} h\sigma^{-2}(W_t)dt = \int G_{a,b}(x,y)(h\sigma^{-2})(y)dy.$$

And then

$$\begin{aligned} E^x \int_0^{T_{Y,a,b}} h(Y_t)dt &= E^{s(x)} \int_0^{T_{Z,s(b),s(a)}} h \circ s^{-1}(Z_t)dt \\ &= \int G_{s(b),s(a)}(s(x),y)(h\sigma^{-2}) \circ s^{-1}(y)dy \\ &= \int G_{s(b),s(a)}(s(x),s(y))\frac{h(y)}{s'(y)}dy, \end{aligned}$$

the last equality following by a change of variables and the definition of σ.

As $a \to 0$, $s(a) \to \infty$, and

$$G_{s(b),s(a)}(s(x),s(y)) \to \begin{cases} 2(s(x)-s(b)) & s(y) \ge s(x) \text{ or } y \le x \\ 2(s(y)-s(b)) & s(y) \le s(x) \text{ or } y \ge x \end{cases}$$

boundedly and pointwise.

Therefore

$$E^x \int_0^{T_{Y,0,R}} 1_{[0,\varepsilon]}(Y_t)dt = \int_0^\varepsilon \frac{2(s(r)-s(R))}{s'(y)}dy \le cs(r)\varepsilon^d,$$

which proves (ii). □

The same proof shows

Corollary 2.2. *If "β nonnegative" is replaced by "β nonpositive" and "$\leq$" in (ii) is replaced by "$\geq$", then Lemma 2.1 still holds.*

3. Proof of Theorem 1.1 By a change of coordinates, suppose $y = 0 \in D$ and $a(0)$ is the identity matrix. If $q(x) > 0$ and $\widetilde{L}$ is defined by $\widetilde{L}f(x) = q(x)Lf(x)$, it is well-known ([5]) that the Green's function for $\widetilde{L}$ is $\widetilde{g}_D(x,y) = g_D(x,y)/q(y)$. So using the strict ellipticity of $a(x)$, we may assume without loss of generality that $x^*a(x)x/|x|^2 \equiv 1$, where $*$ denotes transpose.

Pick σ so that $\sigma\sigma^*(x) = a(x)$, and let X_t be a solution to

$$dX_t = \sigma(X_t)d\widehat{W}_t, \quad X_0 = x, \tag{3.1}$$

where $\widehat{W}$ is some d dimensional Brownian motion.

Let $r = |x|, R = 2\,\mathrm{diam}(D), V_t = |X_t|$. Then by Ito's lemma, V_t solves

$$dV_t = V_t^{-1}X_t^*\sigma(X_t)d\widehat{W}_t + (\mathrm{trace}\,(a(X_t)) - 1)/(2V_t)dt, \quad V_0 = r, \tag{3.2}$$

for $t < T_{V,0}$. Since $x^*a(x)x/|x|^2 = 1$, the martingale part of V_t has quadratic variation t, hence is a Brownian motion W_t.

Let β be a function that is Lipschitz on compact subintervals of $(0,\infty)$, bounded, and satisfies

$$\beta(u) \geq \sup_{|z|=u} |\ \mathrm{trace}\, a(z) - d| + 2(\log u)^{-2}, \quad u \leq R,$$

and $\int_0^1 u^{-1}\beta(u)du < \infty$, (This is possible by the Dini continuity of a.) Let Y_t be the solution to

$$dY_t = dW_t + (d - 1 - \beta(Y_t))/(2Y_t)dt, \quad Y_0 = r, \quad t < T_{Y,0}.$$

If we modify the drift coefficients for V_t and Y_t to something smooth and bounded when $V_t \leq b, Y_t \leq b, b < r$, we do not change V_t or Y_t for $t < T_{Y,b} \wedge T_{V,b}$.

Since trace $(a(x)) \geq d - \beta(|x|) + (\log(|x|))^{-2}$ for $|x| \leq R$, we can apply [3], p.352, to get $V_t \geq Y_t$, a.s. for $t \leq T_{V,b,R} \wedge T_{Y,b,R}$. Since Y_t does not hit 0 before time $T_{Y,R}$ by Lemma 2.1 (i), we can let $b \to 0$ to conclude $V_t \geq Y_t$ for all $t \leq T_{Y,R}$.

If B_ε is the ball of radius ε about 0, and $\varepsilon < |x|/2$, we then have, using Lemma 2.1 (ii),

$$E^x \int_0^{\tau_D} 1_{B_\varepsilon}(X_t)dt \leq E^r \int_0^{T_{V,R}} 1_{[0,\varepsilon]}(V_t)dt \leq E^r \int_0^{T_{Y,R}} 1_{[0,\varepsilon]}(Y_t)dt \leq c\varepsilon^d|x|^{2-d}. \tag{3.3}$$

Since y was arbitrary as long as dist $(y, \partial D) > \delta$, the estimate (3.3) both tells us that $g_D(x,y)$ exists and gives the upper bound.

To get the lower bound, let $R = 2\delta$. Note that we have $P^r(T_{V,0} < \infty, T_{V,0} < T_{V,R}) = 0$ from above. We compare the process V_t with the solution $\widehat{Y}_t$ to

$$d\widehat{Y}_t = dW_t + (d - 1 + \beta(\widehat{Y}_t))/(2\widehat{Y}_t)dt, \quad \widehat{Y}_0 = r.$$

We argue just as in the upper bound, using Corollary 2.2 in place of Lemma 2.1. □

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