SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

NICOLE EL KAROUI

IOANNIS KARATZAS

Integration of the optimal risk in a stopping problem with absorption

Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 405-420

http://www.numdam.org/item?id=SPS_1989__23__405_0

© Springer-Verlag, Berlin Heidelberg New York, 1989, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

INTEGRATION OF THE OPTIMAL RISK IN A STOPPING PROBLEM WITH ABSORPTION

NICOLE EL KAROUI

Université Pierre et Marie Curie
Laboratoire de Probabilités
4, place Jussieu - Tour 56
75252 Paris Cedex 05

IOANNIS KARATZAS*

Columbia University
Department of Statistics
619 Mathematics Building
New York, N.Y. 10027

Abstract

Integration with respect to the spatial argument of the optimal risk in a stopping problem with absorption at the origin, yields the value function of the so-called "reflected follower" stochastic control problem and provides a precise description of its optimal policy.

1. INTRODUCTION

In the articles [5], [2] we studied the *Reflected Follower* stochastic control problem with state process

$$(1.1) \quad X_t = x + W_t - \xi_t + K_t; \quad 0 \leq t \leq \tau,$$

where $x \geq 0$ is an initial position, W is a standard Brownian motion and ξ is a nondecreasing process; given (x, W) and ξ , the additional term K in (1.1) represents the smallest among all nondecreasing processes that guarantees

$$X_t \geq 0; \quad \forall \quad 0 \leq t \leq \tau$$

a.s. The control problem is then to choose ξ , so as to minimize the expected cost

$$(1.2) \quad J(\xi; r, x) = E\left[\int_0^{r-r} h(r+t, X_t)dt + \int_{[0, r-r)} f(r+t)d\xi_t + g(X_{r-r})\right]$$

for $r \in [0, \tau]$, continuous $f(\cdot)$, and suitable convex functions $h(r, \cdot)$, $g(\cdot)$.

It was shown in [5] that the value function

[†]Work carried out during a brief visit by the second author at the University Pierre et Marie Curie (Paris VI). The generous support of the CNRS, which made this visit possible, is gratefully acknowledged.

*Supported in part by the U.S. Air Force Office of Scientific Research under grant AFOSR-86-0203.

$$(1.3) \quad V(r, x) = \inf_{\xi} J(\xi; r, x)$$

of this problem is differentiable in the spatial variable x , with gradient equal to

$$(1.4) \quad u(r, x) \triangleq \inf_{\sigma \in S(r-r)} E \left[\int_0^{\sigma \wedge S(x)} h_x(r+t, x+W_t) dt + f(r+\sigma) 1_{\{\sigma < S(x) \wedge (r-r)\}} \right. \\ \left. + g'(x+W_{r-r}) 1_{\{\sigma=r-r < S(x)\}} \right],$$

the optimal risk in a stopping problem for W with absorption at the origin at the time $S(x) = \inf\{t \geq 0; x+W_t = 0\}$. In [2] we studied the finite-fuel version of the reflected follower problem (i.e., under the additional a.s. constraint $\xi_{r-r} \leq y$, for given $y > 0$), and related it to a family of optimal stopping problems similar to (1.4).

The methodology of [5] (also adopted in [2]) had the control problem of (1.2), (1.3) as its starting point, and used a technique of "switching paths at appropriate random times" to compare expected costs at neighbouring points, to differentiate $V(r, \cdot)$, and finally to obtain the identity

$$(1.5) \quad V_x(r, x) = u(r, x).$$

We shall follow the opposite approach in the present paper; we shall start by studying in detail the problem (1.5), whose solution is typically given in terms of a moving boundary $s(\cdot)$, in the form: "stop as soon as the absorbed process $(x+W_t) 1_{\{t \leq S(x)\}}$ exceeds $s(r+t)$ ". By *integrating* directly suitable expressions for the optimal risk $u(r, \cdot)$, we arrive at the relation (1.5) and at the representation

$$(1.6) \quad V(r, x) = E \left[\int_0^{r-r} h(r+t, s(r+t) \wedge |x+W_t|) dt + g(|x+W_{r-r}|) \right. \\ \left. - \int_0^{r-r} f'(r+t)(|x+W_t| - s(r+t))^+ dt \right]$$

for the value function of (1.4). In particular, we evaluate $V(r, \cdot)$ along the moving boundary $s(\cdot)$ as

$$(1.7) \quad V(r, s(r)) = \int_r^r h(\theta, s(\theta)) d\theta + g(s(r)) + f(r)s(r) - f(r)s(r) + \int_r^r f'(\theta)s(\theta) d\theta,$$

an expression which coincides, in the case of a moving boundary of bounded variation, with "the cost of a deterministic ride along $s(\cdot)$ ".

We also prove the *optimality* of the policy that mandates reflection of $x+W$ at the origin and along the moving boundary, with immediate boarding of the latter when to the right of it. The approach is very direct and elementary; it avoids completely the use of analytical tools (such as variational inequalities or free boundary problems), even the use of the change-of-variable formula for semimartingales.

2. THE STOPPING PROBLEM

Consider a finite time-horizon $\tau > 0$ and three continuous functions $f : [0, \tau] \rightarrow [0, \infty)$, $g : [0, \infty) \rightarrow [0, \infty)$, $h : [0, \tau] \times [0, \infty) \rightarrow [0, \infty)$; both f, g are continuously differentiable, and so is the function $h(t, \cdot)$ for every $t \in [0, \tau]$. In addition, we assume that the following conditions are satisfied:

$$(2.1) \quad g'(0) \geq 0, \quad h_x(t, 0) \geq 0 \quad ; \quad \forall t \in [0, \tau]$$

$$(2.2) \quad g(\cdot), \quad h(t, \cdot) \quad \text{are convex} \quad ; \quad \forall t \in [0, \tau]$$

$$(2.3) \quad g'(x) \leq f(\tau) \quad ; \quad \forall x \in [0, \infty)$$

$$(2.4) \quad h_x(t, x) + g'(x) \leq K \exp(\mu x^\nu) \quad ; \quad \forall (t, x) \in [0, \tau] \times [0, \infty)$$

for some finite constants $K > 0$, $\mu > 0$ and $\nu \in (0, 2)$.

Let us also consider a complete probability space $(\Omega, \mathcal{F}, P)$, rich enough to support a Brownian motion $W = \{W_t; 0 \leq t \leq \infty\}$; this process is adapted to a filtration $\{\mathcal{F}_t\}$, which is assumed to satisfy the usual conditions. For any given $(r, x) \in [0, \tau] \times [0, \infty)$, let $\mathcal{S}(\tau - r)$ denote the class of $\{\mathcal{F}_t\}$ -stopping times with values in $[0, \tau - r]$ and

$$(2.5) \quad S(x) = \inf\{t \geq 0; \quad x + W_t = 0\}$$

denote the hitting time of the origin by the Brownian path started at x . We shall study the optimal risk

$$(2.6) \quad u(r, x) = \inf_{\sigma \in \mathcal{S}(\tau - r)} E \left[\int_0^{\sigma \wedge S(x)} h_x(r + t, x + W_t) dt + f(r + \sigma) 1_{\{\sigma < S(x) \wedge (\tau - r)\}} \right. \\ \left. + g'(x + W_{\tau - r}) 1_{\{\sigma = \tau - r < S(x)\}} \right]$$

of a stopping problem for the Brownian motion $x + W$ with absorption upon hitting the origin, running cost h_x before termination, cost f of stopping before running out of time or being absorbed, and cost g' for exhausting the time-horizon without having hit the origin.

The assumption (2.2) implies, in particular, that $u(r, \cdot)$ is nondecreasing; it will be assumed throughout this paper that the *continuation region*

$$(2.7) \quad \mathcal{C} \triangleq \{(r, x) \in [0, \tau] \times (0, \infty); \quad u(r, x) < f(r)\}$$

for this problem is actually of the form

$$(2.8) \quad \mathcal{C} = \{(r, x); \quad 0 \leq r < \tau, \quad 0 < x < s(r)\}$$

for a continuous function $s : [0, \tau] \rightarrow (0, \infty)$, and that the stopping time

$$(2.9) \quad \sigma(r, x) = \inf\{t \in [0, \tau - r]; \quad x + W_t \geq s(r + t)\} \wedge (\tau - r)$$

is optimal for the problem of (2.6). We shall denote by $s(r)$ the limit $\lim_{r \uparrow r} s(r)$.

3. REPRESENTATION OF THE OPTIMAL RISK

In order to cast the optimal stopping problem of (2.6) into a more conventional framework, we introduce the absorbed process

$$(3.1) \quad A_t(x) = \begin{cases} x + W_t & ; \quad 0 \leq t < S(x) \\ \Delta & ; \quad t \geq S(x) \end{cases},$$

where Δ is a "cemetery state" isolated from $\mathcal{R}^+$; the convention here is that $g'(\Delta) = h_x(t, \Delta) = 0$; $\forall \quad 0 \leq t \leq \tau$. We also introduce the functions

$$(3.2) \quad G(r, x) \triangleq E \left[\int_0^{\tau-r} h_x(r+t, A_t(x)) dt + g'(A_{\tau-r}(x)) \right]$$

$$(3.3) \quad H(r, x) \triangleq E \left[\int_0^{\tau-r} h_x(r+t, A_t(x)) dt + (g'(A_{\tau-r}(x)) - f(r)) 1_{\{S(x) \wedge (\tau-r) > 0\}} \right].$$

Obviously $H(r, x) = 0$ for $r = \tau$ or $x = 0$, whereas

$$(3.4) \quad \begin{aligned} H(r, x) &= E \left[\int_0^{(\tau-r) \wedge S(x)} \{h_x(r+t, A_t(x)) + f'(r+t)\} dt + g'(A_{\tau-r}(x)) - f(r \wedge (\tau + S(x))) \right] \\ &= G(r, x) - f(r); \quad \text{for } (r, x) \in [0, \tau) \times (0, \infty). \end{aligned}$$

It develops then, by a use of the strong Markov property, that the function

$$(3.5) \quad v \triangleq G - u$$

admits the representation

$$(3.6) \quad \begin{aligned} v(r, x) &= \sup_{\sigma \in S(r-r)} E \left[\int_{\sigma}^{\tau-r} h_x(r+t, A_t(x)) dt + \right. \\ &\quad \left. \{g'(A_{\tau-r}(x)) - f(r+\sigma)\} 1_{\{\sigma < S(x) \wedge (\tau-r)\}} \right] \\ &= \sup_{\sigma \in S(r-r)} EH(r+\sigma, A_{\sigma}(x)), \end{aligned}$$

as the maximal expected reward in an optimal stopping problem for the process $A(x)$, with payoff function $H(r, x)$.

The function v admits a second representation in terms of the optimal stopping boundary $s(\cdot)$, which we describe now: one starts by defining, by analogy with (3.4) and (2.9), the process

$$(3.7) \quad \begin{aligned} C_t^{(r, x)} &\triangleq \int_0^{t \wedge S(x) \wedge (\tau-r)} \{h_x(r+\theta, A_{\theta}(x)) + f'(r+\theta)\} d\theta \\ &\quad + \{g'(A_{\tau-r}(x)) - f(r \wedge (\tau + S(x)))\} 1_{\{0 < (\tau-r) \wedge S(x) \leq t\}} \end{aligned}$$

and the family of stopping times

$$(3.8) \quad \sigma_t(r, x) \triangleq \inf\{\theta \in [t, \tau - r]; A_\theta(x) \geq s(r + \theta)\} \wedge (\tau - r); \quad 0 \leq t \leq \tau - r.$$

Obviously, $\sigma_0(r, x)$ coincides with the optimal stopping time $\sigma(r, x)$ of (2.9), and the expression (3.6) is recast as

$$(3.9) \quad v(r, x) = E[C_{\tau-r}^{(r, x)} - C_{\sigma_0(r, x)}^{(r, x)}] = E[\tilde{C}_{\tau-r}^{(r, x)}],$$

a potential associated with the process of bounded variation

$$\tilde{C}_t^{(r, x)} \triangleq C_{\sigma_t(r, x)}^{(r, x)} - C_{\sigma_0(r, x)}^{(r, x)}$$

which is *not* adapted to the filtration $\{\mathcal{F}_t\}$. However, every excessive function such as $v(r, x)$ is also the potential associated with an *adapted, nondecreasing* process $D^{(r, x)}$, the dual predictable projection of $\tilde{C}^{(r, x)}$ (or, as it is equivalently called, the "balayée prévisible" of $C^{(r, x)}$).

This process can be found explicitly; indeed, using the methodology of section 7 (Appendix) in [3], it can be shown that *the dual predictable projection $D^{(r, x)}$ of $\tilde{C}^{(r, x)}$ is nondecreasing and is given by*

$$(3.10) \quad D_t^{(r, x)} = \int_0^t \{h_x(r + \theta, A_\theta(x)) + f'(r + \theta)\} 1_{\{A_\theta(x) > s(r + \theta)\}} d\theta \\ + \{g'(A_{\tau-r}(x)) - f(r)\} 1_{\{A_{\tau-r}(x) > s(r)\}} 1_{\{0 < \tau - r \leq t\}}.$$

3.1 Remark: Because the process $D^{(r, x)}$ of (3.10) is nondecreasing, we deduce that

- (i) the region $\{(r, x) \in [0, \tau] \times [0, \infty); h_x(r, x) + f'(r) < 0\}$ is included in the continuation region C of (2.8), and
- (ii) $g'(x) = f(r); \quad \forall x \geq s(r).$

This latter conclusion simplifies (3.10) to

$$D_t^{(r, x)} = \int_0^t \{h_x(r + \theta, A_\theta(x)) + f'(r + \theta)\} 1_{\{A_\theta(x) > s(r + \theta)\}} d\theta. \quad \square$$

From all these considerations, it develops that we can write (3.9) in the form

$$(3.11) \quad v(r, x) = E[\tilde{C}_{\tau-r}^{(r, x)}] = E[D_{\tau-r}^{(r, x)}] \\ = E \int_0^{(\tau-r) \wedge S(x)} [h_x(r + t, x + W_t) + f'(r + t)] 1_{\{x + W_t > s(r + t)\}} dt$$

and from (3.5), (3.2), (3.11) we obtain
(3.12)

$$u(r, y) = E \left[\int_0^{(r-r) \wedge S(y)} h_x(r+t, y+W_t) 1_{\{y+W_t \leq s(r+t)\}} dt \right. \\ \left. + g'(y+W_{r-r}) 1_{\{S(y) > r-r\}} - \int_0^{(r-r) \wedge S(y)} f'(r+t) 1_{\{y+W_t > s(r+t)\}} dt \right]$$

for $(r, y) \in [0, \tau] \times [0, \infty)$.

In the next two sections, we shall try to see what happens if we integrate the two expressions (3.12) and (3.13)

$$u(r, y) = E \left[\int_0^{\sigma(r, y) \wedge S(y)} h_x(r+t, y+W_t) dt + f(r+\sigma(r, y)) 1_{\{\sigma(r, y) < S(y) \wedge (r-r)\}} \right. \\ \left. + g'(y+W_{r-r}) 1_{\{\sigma(r, y) = r-r < S(y)\}} \right]$$

for the optimal risk $u(r, \cdot)$, with respect to the spatial variable. In section 6 we shall connect the results of these integrations with a problem of optimal control.

4. FIRST INTEGRATION

We shall integrate first the expression of (3.12) in the variable y .

4.1 Proposition: For every $x \geq 0$, consider the Brownian motion started at x with reflection at the origin:

$$(4.1) \quad R_t(x) \triangleq x + W_t + L_t(x) = x + W_t + \max[0, \max_{0 \leq \theta \leq t} \{-x - W_\theta\}] \\ = (x \vee M_t) + W_t; \quad 0 \leq t < \infty,$$

where M is the increasing process

$$(4.2) \quad M_t \triangleq \max_{0 \leq s \leq t} (-W_s); \quad 0 \leq t \leq \infty,$$

and introduce the function

$$(4.3) \quad N(r, x) \triangleq E \left[\int_0^{\tau-r} h(r+t, s(r+t) \wedge R_t(x)) dt + g(R_{\tau-r}(x)) \right. \\ \left. - \int_0^{\tau-r} f'(r+t) (R_t(x) - s(r+t))^+ dt \right]$$

on $[0, \tau] \times [0, \infty)$. We have then

$$(4.4) \quad \int_{s(r)}^x u(r, y) dy = N(r, x) - N(r, s(r)).$$

Proof: With the help of the equivalence $S(y) > t \iff M_t < y$ we obtain back in (3.12) with $z < x$:

$$\begin{aligned} \int_z^x g'(y + W_{\tau-r}) 1_{\{S(y) > r-r\}} dy &= \int_{z \vee M_{\tau-r}}^{x \vee M_{\tau-r}} g'(y + W_{\tau-r}) dy = g(R_{\tau-r}(x)) - g(R_{\tau-r}(z)), \\ \int_z^x h_x(r+t, y + W_t) 1_{\{y+W_t \leq s(r+t)\}} dy &= \int_{z \vee M_t}^{x \vee M_t} h_x(r+t, y + W_t) 1_{\{y+W_t \leq s(r+t)\}} dy \\ &= h(\{(x \vee M_t) + W_t\} \wedge s(r+t)) - h(\{(z \vee M_t) + W_t\} \wedge s(r+t)) \\ &= h(R_t(x) \wedge s(r+t)) - h(R_t(z) \wedge s(r+t)) \quad \text{on } \{S(y) > t\}, \end{aligned}$$

as well as

$$\begin{aligned} \int_z^x f'(r+t) 1_{\{y+W_t > s(r+t)\}} 1_{\{S(y) > t\}} dy &= \\ &= f'(r+t)[(x \vee M_t) \vee (s(r+t) - W_t) - (z \vee M_t) \vee (s(r+t) - W_t)] \\ &= f'(r+t)[(R_t(x) - s(r+t))^+ - (R_t(z) - s(r+t))^+]. \end{aligned}$$

The identity (4.4) follows. $\square$

4.2 Corollary: We have

$$(4.5) \quad N(r, s(r)) = \int_r^\tau h(\theta, s(\theta)) d\theta + g(s(r)) + f(r)s(r) - f(\tau)s(r) + \int_r^\tau f'(\theta)s(\theta) d\theta,$$

and if the function $s(\cdot)$ is of bounded variation:

$$(4.6) \quad N(r, s(r)) = \int_r^\tau h(\theta, s(\theta)) d\theta - \int_r^\tau f(\theta) ds(\theta) + g(s(r)).$$

This is the “cost of a (deterministic) ride along the moving boundary $s(\cdot)$ ”.

Proof: Let us recall that $u(r, y) = f(r)$, for $y \geq s(r)$. It follows then from (4.4) with $x > s(r)$ that

$$\begin{aligned} N(r, s(r)) &= N(r, x) - f(r) \cdot (x - s(r)) \\ &= E\left[\int_0^{\tau-r} h(r+t, s(r+t) \wedge R_t(x)) dt + g(R_{\tau-r}(x)) - f(r)(x - s(r))\right. \\ &\quad \left. - \int_0^{\tau-r} f'(r+t)\{(R_t(x) - s(r+t))^+ - (x - s(r))\} dt\right]. \end{aligned}$$

Letting $x \rightarrow \infty$ and appealing to the monotone and dominated convergence theorems, as well as to the fact $g(y) - g(s(r)) = f(r) \cdot (y - s(r))$ for $y \geq s(r)$, we obtain (4.5).

5. SECOND INTEGRATION

Let us consider the processes $K^{(r,x)}, \Lambda^{(r,x)}$ defined by $K_o^{(r,x)} = \Lambda_o^{(r,x)} = 0$ and by the system of functional equations

$$(5.1) \quad K_t^{(r,x)} = \max[0, \max_{0 \leq \theta \leq t} \{-x - W_\theta + \Lambda_\theta^{(r,x)}\}] ; \quad 0 \leq t \leq \tau - r$$

$$(5.2) \quad \Lambda_t^{(r,x)} = \max[0, \max_{0 \leq \theta \leq t} \{x + W_\theta - s(r + \theta) + K_\theta^{(r,x)}\}] ; \quad 0 \leq t \leq \tau - r .$$

The solution to this system exists and is unique, for every Brownian path; both $K^{(r,x)}, \Lambda^{(r,x)}$ are continuous on $(0, \tau - r]$ and we have $K_{o+}^{(r,x)} = 0, \Lambda_{o+}^{(r,x)} = (x - s(r))^+$ (cf. [4], Appendix). Now the process

$$(5.3) \quad X_t^{(r,x)} \triangleq x + W_t + K_t^{(r,x)} - \Lambda_t^{(r,x)} ; \quad 0 \leq t \leq \tau - r$$

is, for $0 \leq x \leq s(r)$, a *Brownian motion started at x and reflected at the origin and along the moving boundary $\{s(r + t); 0 \leq t \leq \tau - r\}$* ; for an initial position $x > s(r)$, the initial jump of $\Lambda^{(r,x)}$ results in $X_{o+}^{(r,x)} = s(r)$, and from then on the situation is the same as described above.

5.1 Proposition: For the function

$$(5.4) \quad M(r, x) \triangleq E \left[\int_0^{\tau-r} h(r+t, X_t^{(r,x)}) dt + \int_{[0, \tau-r)} f(r+t) d\Lambda_t^{(r,x)} + g(X_{\tau-r}^{(r,x)}) \right] ,$$

we have

$$(5.5) \quad \int_{s(r)}^x u(r, y) dy = M(r, x) - M(r, s(r)) . \quad \square$$

The validity of (5.5) is obvious for $x > s(r)$; for $x \in [0, s(r)]$, it will follow by *integrating over the interval $(x, s(r))$ the expression of (3.13)*. More precisely, we shall take $r = 0$ for simplicity of notation, denote by $(K(x), \Lambda(x), X(x))$ the triple $(K^{(0,x)}, \Lambda^{(0,x)}, X^{(0,x)})$, and establish the following pathwise result:

5.2 Lemma: For every $x \in [0, s(0)]$, define the stopping time

$$(5.6) \quad \sigma(x) \triangleq \inf\{t \in [0, \tau) ; x + W_t \geq s(t)\} \wedge \tau$$

where $s : [0, \tau) \rightarrow (0, \infty)$ is any continuous function. We have then the a.s. identity

$$\begin{aligned}
 & \int_x^{s(0)} \left[\int_0^{\sigma(y) \wedge S(y)} h_x(t, y + W_t) dt + f(\sigma(y)) 1_{\{\sigma(y) < \tau \wedge S(y)\}} \right. \\
 & \quad \left. + g'(y + W_\tau) 1_{\{\tau = \sigma(y) < S(y)\}} \right] dy \\
 (5.7) \quad & = \left[\int_0^\tau h(t, X_t(s(0))) dt + \int_0^\tau f(t) d\Lambda_t(s(0)) + g(X_\tau(s(0))) \right] \\
 & \quad - \left[\int_0^\tau h(t, X_t(x)) dt + \int_0^\tau f(t) d\Lambda_t(x) + g(X_\tau(x)) \right].
 \end{aligned}$$

□

Consider the continuous, nondecreasing processes M of (4.2) and

$$(5.8) \quad L_t \triangleq \max_{0 \leq \theta \leq t} (W_\theta - s(\theta)) ; \quad 0 \leq t \leq \tau ,$$

with left-continuous inverses given by $S(\cdot), \sigma(\cdot)$ of (2.5), (5.6), respectively:

$$(5.9) \quad \{S(x) \leq t\} = \{M_t \geq x\} ; \quad \forall \quad 0 \leq t < \infty$$

and

$$\begin{aligned}
 (5.10) \quad & \{\sigma(x) \leq t\} = \{L_t \geq -x\} ; \quad \forall \quad 0 \leq t < \tau \\
 & \{\sigma(x) = \tau\} = \{L_\tau \leq -x\} .
 \end{aligned}$$

We shall work separately on the two events $\{\sigma(x) < S(x)\}$ and $\{\sigma(x) > S(x)\}$.

PROOF OF (5.7) ON $\{\sigma(x) < S(x)\}$: On this event, we have

$$(5.11) \quad \Lambda_t(x) = \begin{cases} 0 & ; \quad 0 \leq t \leq \sigma(x) \\ x + L_t & ; \quad \sigma(x) \leq t \leq S^*(x) \end{cases}, \quad X_t(x) = \begin{cases} x + W_t & ; \quad 0 \leq t \leq \sigma(x) \\ W_t - L_t & ; \quad \sigma(x) \leq t \leq S^*(x) \end{cases}$$

where

$$(5.12) \quad S^*(x) \triangleq \inf\{t \in [\sigma(x), \tau) ; W_t \leq L_t\} \wedge \tau$$

is here the first hitting time of the origin by the process $X(x)$. Formulas analogous to (5.11), (5.12) hold for every $y \in [x, s(0)]$, and we have $0 = \sigma(s(0)) \leq \sigma(y) \leq \sigma(x)$ for such a configuration. In particular, *the processes $X(y)$ coincide on $[\sigma(x), \tau]$ for every $x \leq y \leq s(0)$* , and thus

$$(5.13) \quad S^*(y) = S^*(s(0)), \quad \sigma(y) < S(y) ; \quad \forall \quad y \in [x, s(0)] .$$

The proof of (5.7) will be complete, as soon as we have established the following a.s. identities:

$$(5.14) \quad \int_x^{s(0)} f(\sigma(y)) 1_{\{\sigma(y) < \tau \wedge S(y)\}} dy = \int_0^\tau f(t) d\Lambda_t(s(0)) - \int_0^\tau f(t) d\Lambda_t(x)$$

$$(5.15) \quad \int_x^{s(0)} g'(y + W_\tau) 1_{\{\tau = \sigma(y) < S(y)\}} dy = g(X_\tau(s(0))) - g(X_\tau(x))$$

$$(5.16) \quad \int_x^{s(0)} \left(\int_0^{\sigma(y) \wedge S(y)} h_x(t, y + W_t) dt \right) dy = \int_0^\tau h(t, X_t(s(0))) dt - \int_0^\tau h(t, X_t(x)) dt.$$

But this is straightforward; thanks to (5.11)-(5.13), the right-hand sides of these expressions are equal to $\int_0^{\sigma(x)} f(t) dL_t$, $[g(W_\tau - L_\tau) - g(x + W_\tau)] 1_{\{\sigma(x) = \tau\}}$ and $\int_0^{\sigma(x)} [h(t, W_t - L_t) - h(t, x + W_t)] dt$. On the other hand, by virtue of (5.10), (5.13) the left-hand sides are computed as follows:

$$\begin{aligned} \int_x^{s(0)} f(\sigma(y)) 1_{\{\sigma(y) < \tau\}} dy &= \int_0^{\sigma(x)} f(t) dL_t, \\ \int_x^{s(0)} g'(y + W_\tau) 1_{\{\sigma(y) = \tau\}} dy &= \int_x^{s(0)} g'(y + W_\tau) 1_{\{y \leq -L_\tau\}} dy \\ &= 1_{\{\sigma(x) = \tau\}} [g(W_\tau - L_\tau) - g(x + W_\tau)], \quad \text{and} \\ \int_x^{s(0)} dy \int_0^{\sigma(y)} h_x(t, y + W_t) dt &= \int_0^\tau dt \int_x^{s(0)} 1_{\{y < -L_t\}} h_x(t, y + W_t) dy \\ &= \int_0^\tau 1_{\{x < -L_t\}} [h(t, W_t - L_t) - h(t, x + W_t)] dt = \int_0^{\sigma(x)} [h(t, W_t - L_t) - h(t, x + W_t)] dt. \end{aligned}$$

□

5.3 Remark: Using exactly the same procedure as above, one can show that (5.7) is also valid on the event

$$(5.17) \quad \{ \sigma(x) > S(x) \quad \text{and} \quad \sigma(y) < S(y) ; \quad \forall y \in (x, s(0)) \}.$$

In a realization like this, the Brownian path issued at x just touches the origin at $t = S(x)$ without crossing it, and then goes on to cross the moving boundary (draw a picture).

PROOF OF (5.7) ON $\{\sigma(x) > S(x)\}$: Let us pick a realization that belongs to this event; if it belongs also to the event of (5.17), we are done. If not, we consider for this particular realization the number

$$(5.18) \quad z \triangleq \inf\{y \in [0, s(0)] ; \sigma(y) < S(y)\},$$

for which we have

$$(5.19) \quad \tau \wedge S(y) \leq \sigma(y), \quad \forall y \in [x, z].$$

In particular, for every such y we have from (5.1)-(5.3):

$$(5.20) \quad K_t(y) = (M_t - y)^+, \quad X_t(y) = (y \vee M_t) + W_t; \quad 0 \leq t \leq \sigma_*(y),$$

where

$$(5.21) \quad \begin{aligned} \sigma_*(y) &\triangleq \inf\{t \in [0, \tau] ; X_t(y) \geq s(t)\} \wedge \tau \\ &= \inf\{t \in [S(y), \tau] ; M_t + W_t \geq s(t)\} \wedge \tau . \end{aligned}$$

Quite obviously

$$(5.22) \quad S(x) \leq S(y) \leq S(z), \quad \sigma_*(z) = \sigma(z) = \sigma_*(y) ; \quad \forall y \in [x, z]$$

and

$$(5.23) \quad \text{the processes } X_*(y) \text{ coincide on } [S(z), \tau] ; \quad \forall y \in [x, z] .$$

In view of Remark 5.3, in order to establish (5.7) on $\{\sigma(x) > S(x)\}$, it suffices to show that

$$\begin{aligned} &\int_x^z \left[\int_0^{\sigma(y) \wedge S(y)} h_x(t, y + W_t) dt + f(\sigma(y)) 1_{\{\sigma(y) < \tau \wedge S(y)\}} + g'(y + W_\tau) 1_{\{\tau = \sigma(y) < S(y)\}} \right] dy \\ &= \left[\int_0^\tau h(t, X_t(z)) dt + \int_0^\tau f(t) d\Lambda_t(z) + g(X_\tau(z)) \right] - \left[\int_0^\tau h(t, X_t(x)) dt \right. \\ &\quad \left. + \int_0^\tau f(t) d\Lambda_t(x) + g(X_\tau(x)) \right] \end{aligned}$$

holds a.s. on this event, or even (thanks to (5.19) - (5.23)) that

$$(5.24) \quad \int_x^z dy \int_0^{S(y) \wedge \tau} h_x(t, y + W_t) dt = \int_0^{\tau \wedge S(z)} [h(t, X_t(z)) - h(t, X_t(x))] dt ,$$

$$(5.25) \quad \int_x^z g'(y + W_\tau) 1_{\{\tau = \sigma(y) < S(y)\}} dy = [g(X_\tau(z)) - g(X_\tau(x))] 1_{\{\tau < S(z)\}}$$

hold a.s. on $\{\sigma(x) > S(x)\}$. However, a verification of (5.24), (5.25) based on (5.9), (5.20) is straightforward. $\square$

The proof of Lemma 5.2 is now complete.

Comparing the relation (5.5) with (4.3), we see that the functions $M(r, \cdot)$, $N(r, \cdot)$ are both primitives of the optimal stopping risk $u(r, \cdot)$. We shall show in section 7 that these two functions are actually *identical*.

6. THE CONTROL PROBLEM

Consider the class $\mathcal{A}$ of $\{\mathcal{F}_t\}$ -adapted processes $\xi = \{\xi_t; 0 \leq t < \infty\}$ with $\xi_0 = 0$ and nondecreasing, left-continuous paths, a.s. Corresponding to any given $x \geq 0$ and $\xi \in \mathcal{A}$, denote by (X, K) the solution to the $RP(x + W - \xi)$, i.e., the Reflection Problem associated with the process $x + W - \xi$:

$$(6.1) \quad K \in \mathcal{A}, \quad X = x + W - \xi + K$$

$$(6.2) \quad X_t \geq 0; \quad \forall \quad 0 \leq t < \infty$$

$$(6.3) \quad \int_0^\infty X_t dK_t^c = 0$$

$$(6.4) \quad \Delta K_t \triangleq K_{t+} - K_t = 2X_{t+}; \quad \forall \quad t \in [0, \infty) \quad \text{s.t.} \quad \Delta K_t > 0$$

hold a.s. Roughly speaking, K represents the "minimal cumulative amount of rightward pushing at the origin that has to be exerted, in order to keep the resulting process X of (6.1) nonnegative".

As shown in [1], for a.e. Brownian path there exists a unique solution to the problem (6.1)-(6.4). Besides, if we denote by $\mathcal{D}(\tau, x)$ the class of processes $\xi \in \mathcal{A}$ for which

$$\Delta \xi_t \leq X_t; \quad \forall \quad 0 \leq t \leq \tau$$

(i.e., processes which never attempt a jump across the origin), then for every $\xi \in \mathcal{D}(\tau, x)$ the corresponding reflection process K is continuous, and is given by $K_t = \max[0, \sup_{0 \leq \theta \leq t} \{\xi_\theta - (x + W_\theta)\}]$.

Suppose now that we associate the expected total cost

$$(6.5) \quad J(\xi; r, x) \triangleq E\left[\int_0^{\tau-r} h(r+t, X_t) dt + \int_{[0, \tau-r)} f(r+t) d\xi_t + g(X_{\tau-r})\right]$$

to every $\xi \in \mathcal{A}$, which now we regard as an element of "control", at the disposal of the decision-maker. Here $h(t, \cdot)$ plays the rôle of a running cost on the state X_t , $f(\cdot)$ is the cost per unit time of controlling effort that is exerted, and $g(\cdot)$ is a cost on the state at the terminal time. The so-called *reflected follower* control problem is to choose $\xi \in \mathcal{A}$ that minimizes the expression of (6.5) over this class, and

$$(6.6) \quad V(r, x) \triangleq \inf_{\xi \in \mathcal{A}} J(\xi; r, x)$$

is the *value function* of this problem.

It can be shown (cf. [5], Proposition 4.1 or [2], Remark 5.7) that the class $\mathcal{D}(\tau - r, x)$ is complete for the problem (6.6), so that

$$(6.7) \quad V(r, x) = \inf_{\xi \in \mathcal{D}(\tau-r, x)} J(\xi; r, x).$$

Clearly, the process $\Lambda^{(r, x)}$ of (5.2) belongs to $\mathcal{D}(\tau - r, x)$, the pair $(X^{(r, x)}, K^{(r, x)})$ of (5.1), (5.3) is the solution to the $RP(x + W - \Lambda^{(r, x)})$, and from (5.4) we have

$$(6.8) \quad M(r, x) = J(\Lambda^{(r, x)}; r, x) \geq V(r, x).$$

Here is then the fundamental result of this paper.

6.1 Theorem: The functions M, N and V of (5.4), (4.3) and (6.6), respectively, are all equal:

$$(6.9) \quad M(r, x) = N(r, x) = V(r, x); \quad \forall (r, x) \in [0, \tau] \times [0, \infty).$$

6.2 Corollary: It follows immediately from (6.8), (6.9) that *the process $\Lambda^{(r, x)}$ is optimal for the control problem* of this section.

In other words, as soon as we have the optimal stopping boundary $s(\cdot)$ for the problem of section 2, we can obtain the optimal processes for the control problem by reflecting the Brownian motion W at the origin and along this moving boundary. $\square$

We shall prove Theorem 6.1 in the next two sections, 7 (identity $M \equiv N$) and 8 (identity $N \equiv V$).

7. $M \equiv N$

It is quite obvious from the defining relations (4.1), (4.3) and (5.4) that the processes

$$\begin{aligned} & \int_0^{t \wedge \sigma(r, x)} h(r + \theta, R_\theta(x)) d\theta + N(r + t \wedge \sigma(r, x), R_{t \wedge \sigma(r, x)}); \quad 0 \leq t \leq \tau - r \\ & \int_0^{t \wedge \sigma(r, x)} h(r + \theta, R_\theta(x)) d\theta + M(r + t \wedge \sigma(r, x), R_{t \wedge \sigma(r, x)}); \quad 0 \leq t \leq \tau - r \end{aligned}$$

are both $\{\mathcal{F}_t\}$ - martingales. On the other hand, the difference

$$D(r) \triangleq M(r, x) - N(r, x); \quad 0 \leq r \leq \tau$$

is a continuous function of bounded variation (e.g. Theorem 4.3.6 in [6]), independent of the spatial variable by virtue of (4.4), (5.5). It develops that

$$m(t) \triangleq D(r + t \wedge \sigma(r, x)) = \int_0^t 1_{\{\theta \leq \sigma(r, x)\}} D'(r + \theta) d\theta, \quad \mathcal{F}_t; \quad 0 \leq t \leq \tau - r$$

is a continuous martingale with paths of bounded variation (and therefore constant). But $m(0) = D(r)$ and $m(\tau - r) = D(\sigma(r, x)) 1_{\{\sigma(r, x) < \tau - r\}}$ because $D(\tau - r) = 0$, and thus

$$D(r) = D(\sigma(r, x)) 1_{\{\sigma(r, x) < \tau - r\}}, \quad \text{a.s.}$$

holds for every $r \in [0, \tau]$, $0 \leq x \leq s(r)$. This is possible only if $D(r) \equiv 0$.

8. $N \equiv V$

We begin with an auxiliary result.

8.1 Lemma: The process

$$(8.1) \quad N(r+t, R_t(x)) + \int_0^t h(r+\theta, R_\theta(x)) d\theta ; \quad 0 \leq t \leq \tau - r$$

is an $\{\mathcal{F}_t\}$ - submartingale, for every $(r, x) \in [0, \tau] \times [0, \infty)$.

Proof: It suffices to establish

$$(8.2) \quad E[N(r+\sigma, R_\sigma(x)) + \int_0^\sigma h(r+\theta, R_\theta(x)) d\theta] \geq N(r, x)$$

for any given $\sigma \in S_{0, \tau-r}$ (cf. Problem 1.3.26 in [6]). From (4.3) and the strong Markov property, we have

$$\begin{aligned} EN(r+\sigma, R_\sigma(x)) &= E\left[\int_0^{\tau-r-\sigma} h(r+\sigma+\theta, s(r+\sigma+\theta) \wedge R_\theta(R_\sigma(x))) d\theta\right. \\ &\quad \left.+ g(R_{\tau-r-\sigma}(R_\sigma(x))) - \int_0^{\tau-r-\sigma} f'(r+\sigma+\theta)(R_\theta(R_\sigma(x)) - s(r+\sigma+\theta))^+ d\theta\right] \\ &= E\left[\int_\sigma^{\tau-r} h(r+\theta, s(r+\theta) \wedge R_\sigma(x)) d\theta + g(R_{\tau-r}(x))\right. \\ &\quad \left.- \int_\sigma^{\tau-r} f'(r+\theta)(R_\theta(x) - s(r+\theta))^+ d\theta\right]. \end{aligned}$$

It develops that

$$(8.3) \quad E[N(r+\sigma, R_\sigma(x)) + \int_0^\sigma h(r+\theta, R_\theta(x)) d\theta] = N(r, x) + \Delta(r, x),$$

where

$$\begin{aligned} \Delta(r, x) &= E \int_0^\sigma \{h(r+t, R_t(x)) - h(r+t, s(r+t) \wedge R_t(x))\} dt \\ &\quad + E \int_0^\sigma f'(r+t)(R_t(x) - s(r+t))^+ dt \\ &\geq E \int_0^\sigma \{h_x(r+t, s(r+t)) + f'(r+t)\}(R_t(x) - s(r+t))^+ dt \geq 0. \end{aligned}$$

We have used the convexity of $h(r, \cdot)$, as well as the Remark 3.1(i). $\square$

8.2 Remark: Thanks to the Doob-Meyer decomposition and (4.1), (4.3), the continuous and nonnegative submartingale of (8.1) can be written as

$$(8.4) \quad \int_0^t h(r+\theta, R_\theta(x)) d\theta + N(r+t, R_t(x)) = N(r, x) + \int_0^t u(r+\theta, R_\theta(x)) dW_\theta + A_t(r, x)$$

where $A(r, x)$ is a continuous nondecreasing process (cf. Theorems 1.4.10, 1.4.14 and Problem 1.4.13 in [6]).

Here is the fundamental result of this section.

8.3 Proposition: For fixed $(r, x) \in [0, \tau] \times [0, \infty)$, denote by $(X(x, \xi), K(x, \xi))$ the solution to the $RP(x + W - \xi)$ corresponding to any $\xi \in \mathcal{D}(\tau - r, x)$. Then the process (8.5)

$$Q_t(x, \xi) \triangleq \int_0^t h(r + \theta, X_\theta(x, \xi)) d\theta + \int_{[0, t)} f(r + \theta) d\xi_\theta + N(r + t, X_t(x, \xi)); \quad 0 \leq t \leq \tau - r$$

is an $\{\mathcal{F}_t\}$ - submartingale.

Proof: The argument will proceed in several steps.

Step 1: $\xi \equiv 0$. This case amounts to Lemma 8.1, because $(X(x, 0), K(x, 0)) \equiv (R(x), L(x))$ in the notation of (4.1).

Step 2: $\xi_t = \int_0^t z_s ds$ for a bounded, nonnegative and $\{\mathcal{F}_t\}$ -progressively measurable process $z = \{z_t; 0 \leq t \leq \tau - r\}$. Consider the exponential martingale

$$Z_t = \exp\left\{-\int_0^t z_s dW_s - \frac{1}{2} \int_0^t z_s^2 ds\right\};$$

under the probability measure $\tilde{P}(d\omega) = Z_{\tau-r}(\omega)P(d\omega)$ on $\mathcal{F}_{\tau-r}$, the process

$$\tilde{W}_t \triangleq W_t + \xi_t = W_t + \int_0^t z_s ds; \quad 0 \leq t \leq \tau - r$$

is a Brownian motion, by virtue of the Girsanov theorem (section 3.5 in [6]). Now (4.1) is written equivalently as

$$R_t(x) = x + \tilde{W}_t - \xi_t + L_t(x), \quad \text{for } 0 \leq t \leq \tau - r,$$

and because $L(x)$ is flat off $\{t \geq 0; R_t(x) = 0\}$ it develops that

$$(8.6) \quad (R(x), L(x)) \text{ is the solution to the } RP(x + \tilde{W} - \xi).$$

Besides, we have from (8.4):

$$(8.7) \quad \begin{aligned} \int_0^t h(r + \theta, R_\theta(x)) d\theta + \int_{[0, t)} f(r + \theta) d\xi_\theta + N(r + t, R_t(x)) = \\ = N(r, x) + \int_0^t u(r + \theta, R_\theta(x)) d\tilde{W}_\theta + \tilde{A}_t(r, x), \end{aligned}$$

where

$$\tilde{A}_t(r, x) \triangleq A_t(r, x) + \int_0^t [f(r + \theta) - u(r + \theta, R_\theta(x))] z_\theta d\theta$$

is a continuous, nondecreasing process. The assertion follows from this observation, coupled with (8.7) and (8.6).

Step 3: An arbitrary $\xi \in \mathcal{D}(\tau - r, x)$ can be approximated by a monotonically increasing sequence $\{\xi^{(n)}\}_{n=1}^{\infty}$ of absolutely continuous processes as in Step 2, such that

$$\lim_{n \rightarrow \infty} \xi_t^{(n)} = \xi_t, \quad \lim_{n \rightarrow \infty} K_t(x, \xi^{(n)}) = K_t(x, \xi) \quad \text{and} \\ \lim_{n \rightarrow \infty} X_t(x, \xi^{(n)}) = X_t(x, \xi); \quad \forall \quad 0 \leq t \leq \tau - r,$$

almost surely (cf. [2], Lemmas 5.4, 5.5 and Proposition 5.6). Step 2 shows that every $Q(x, \xi^{(n)})$ is an $\{\mathcal{F}_t\}$ -submartingale, and by the monotone and dominated convergence theorems this property is inherited by the process $Q(x, \xi)$. $\square$

It does not remain now but to put the various results together; the submartingale property of the process in (8.5) gives

$$J(\xi; r, x) = EQ_{\tau-r}(x, \xi) \geq N(r, x); \quad \forall \quad \xi \in \mathcal{D}(\tau - r, x)$$

and from (6.7), (6.8) and section 7 we deduce:

$$V(r, x) \geq N(r, x) = M(r, x) \geq V(r, x).$$

The proof of Theorem 6.1 is complete.

9. REFERENCES

- [1] CHALEYAT-MAUREL, M., EL KAROUI, N. & MARCHAL, B. (1980) Réflexion discontinue et systèmes stochastiques. *Annals of Probability* **8**, 1049-1067.
- [2] EL KAROUI, N. & KARATZAS, I. (1988) Probabilistic aspects of finite-fuel, reflected follower problems. *Acta Applicandae Mathematicae* **11**, 223-258.
- [3] EL KAROUI, N. & KARATZAS, I. (1988) A new approach to the Skorohod problem, and its applications. *Submitted for publication*.
- [4] KARATZAS, I. (1983) A class of singular stochastic control problems. *Advances in Applied Probability* **15**, 225-254.
- [5] KARATZAS, I. & SHREVE, S.E. (1985) Connections between optimal stopping and singular stochastic control II: Reflected Follower problems. *SIAM J. Control & Optimization* **23**, 433-451.
- [6] KARATZAS, I. & SHREVE, S.E. (1987) *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York.