SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

NICOLE EL KAROUI
IOANNIS KARATZAS

Integration of the optimal risk in a stopping problem with absorption

Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 405-420
<http://www.numdam.org/item?id=SPS_1989__23__405_0>

INTEGRATION OF THE OPTIMAL RISK IN A STOPPING PROBLEM WITH ABSORPTION

NICOLE EL KAROUI
Université Pierre et Marie Curie
Laboratoire de Probabilités
4, place Jussieu - Tour 56
75252 Paris Cedex 05

IOANNIS KARATZAS*
Columbia University
Department of Statistics
619 Mathematics Building
New York, N.Y. 10027

Abstract

Integration with respect to the spatial argument of the optimal risk in a stopping problem with absorption at the origin, yields the value function of the so-called "reflected follower" stochastic control problem and provides a precise description of its optimal policy.

1. INTRODUCTION

In the articles [5], [2] we studied the *Reflected Follower* stochastic control problem with state process

(1.1) $$X_t = x + W_t - \xi_t + K_t \,; \qquad 0 \le t \le \tau \,,$$

where $x \ge 0$ is an initial position, W is a standard Brownian motion and ξ is a nondecreasing process; given (x, W) and ξ, the additional term K in (1.1) represents the smallest among all nondecreasing processes that guarantees

$$X_t \ge 0; \qquad \forall \;\; 0 \le t \le \tau$$

a.s. The control problem is then to choose ξ, so as to minimize the expected cost

(1.2) $$J(\xi; r, x) = E[\int_0^{\tau - r} h(r+t, X_t)dt \; + \int_{[0,\tau-r)} f(r+t)d\xi_t \; + \; g(X_{\tau-r})]$$

for $r \in [0, \tau]$, continuous $f(\,\cdot\,)$, and suitable convex functions $h(r, \,\cdot\,)$, $g(\,\cdot\,)$.

It was shown in [5] that the value function

†Work carried out during a brief visit by the second author at the University Pierre et Marie Curie (Paris VI). The generous support of the CNRS, which made this visit possible, is gratefully acknowledged.

*Supported in part by the U.S. Air Force Office of Scientific Research under grant AFOSR-86-0203.

$$(1.3)\qquad V(r,x)=\inf_{\xi}\ J(\xi;r,x)$$

of this problem is differentiable in the spatial variable x, with gradient equal to

$$(1.4)\qquad u(r,x)\triangleq\inf_{\sigma\epsilon S(\tau-r)}E\Big[\int_0^{\sigma\wedge S(x)}h_x(r+t,x+W_t)dt+f(r+\sigma)\ 1_{\{\sigma<S(x)\wedge(\tau-r)\}}+g'(x+W_{\tau-r})\ 1_{\{\sigma=\tau-r<S(x)\}}\Big]\ ,$$

the optimal risk in a stopping problem for W *with absorption at the origin* at the time $S(x)=\inf\{t\geq 0;\ x+W_t=0\}$. In [2] we studied the finite-fuel version of the reflected follower problem (i.e., under the additional a.s. constraint $\xi_{\tau-r}\leq y$, for given $y>0$), and related it to a family of optimal stopping problems similar to (1.4).

The methodology of [5] (also adopted in [2]) had the control problem of (1.2), (1.3) as its starting point, and used a technique of "switching paths at appropriate random times" to compare expected costs at neighbouring points, to differentiate $V(r,\cdot)$, and finally to obtain the identity

$$(1.5)\qquad V_x(r,x)=u(r,x)\ .$$

We shall follow the opposite approach in the present paper; we shall start by studying in detail the problem (1.5), whose solution is typically given in terms of a moving boundary $s(\cdot)$, in the form: "stop as soon as the absorbed process $(x+W_t)\ 1_{\{t\leq S(x)\}}$ exceeds $s(r+t)$". By *integrating* directly suitable expressions for the optimal risk $u(r,\cdot)$, we arrive at the relation (1.5) and at the representation

$$(1.6)\qquad \begin{aligned} V(r,x)=&E\Big[\int_0^{\tau-r}h(r+t,s(r+t)\wedge|x+W_t|)dt\ +\ g(|x+W_{\tau-r}|)\\ &-\int_0^{\tau-r}f'(r+t)(|x+W_t|-s(r+t))^+dt\ \Big]\end{aligned}$$

for the value function of (1.4). In particular, we evaluate $V(r,\cdot)$ along the moving boundary $s(\cdot)$ as

$$(1.7)\quad V(r,s(r))=\int_r^{\tau}h(\theta,s(\theta))d\theta+g(s(\tau))+f(r)s(r)-f(\tau)s(\tau)+\int_r^{\tau}f'(\theta)s(\theta)d\theta\ ,$$

an expression which coincides, in the case of a moving boundary of bounded variation, with "the cost of a deterministic ride along $s(\cdot)$".

We also prove the *optimality* of the policy that mandates reflection of $x+W$ at the origin and along the moving boundary, with immediate boarding of the latter when to the right of it. The approach is very direct and elementary; it avoids completely the use of analytical tools (such as variational inequalities or free boundary problems), even the use of the change-of-variable formula for semimartingales.

2. THE STOPPING PROBLEM

Consider a finite time-horizon $\tau > 0$ and three continuous functions $f : [0,\tau] \to [0,\infty)$, $g : [0,\infty) \to [0,\infty)$, $h : [0,\tau] \times [0,\infty) \to [0,\infty)$; both f, g are continuously differentiable, and so is the function $h(t,\cdot)$ for every $t \in [0,\tau]$. In addition, we assume that the following conditions are satisfied:

(2.1) $$g'(0) \geq 0, \quad h_x(t,0) \geq 0 \quad ; \qquad \forall\ t \in [0,\tau]$$

(2.2) $$g(\cdot)\ , \quad h(t,\cdot) \quad \text{are convex} \ ; \qquad \forall\ t \in [0,\tau]$$

(2.3) $$g'(x) \leq f(\tau) \quad ; \qquad \forall\ x \in [0,\infty)$$

(2.4) $$h_x(t,x) + g'(x) \leq K\ \exp(\mu x^{\nu}) \quad ; \qquad \forall\ (t,x) \in [0,\tau) \times [0,\infty)$$

for some finite constants $K > 0$, $\mu > 0$ and $\nu \in (0,2)$.

Let us also consider a complete probability space $(\Omega, \mathcal{F}, P)$, rich enough to support a Brownian motion $W = \{W_t;\ 0 \leq t \leq \infty\}$; this process is adapted to a filtration $\{\mathcal{F}_t\}$, which is assumed to satisfy the usual conditions. For any given $(r,x) \in [0,\tau] \times [0,\infty)$, let $\mathcal{S}(\tau - r)$ denote the class of $\{\mathcal{F}_t\}$ - stopping times with values in $[0,\tau - r]$ and

(2.5) $$S(x) = \inf\{t \geq 0;\ \ x + W_t = 0\}$$

denote the hitting time of the origin by the Brownian path started at x. We shall study the optimal risk

(2.6)
$$u(r,x) = \inf_{\sigma \in \mathcal{S}(\tau-r)} E\Big[\int_0^{\sigma \wedge S(x)} h_x(r+t, x+W_t)dt + f(r+\sigma)\ 1_{\{\sigma < S(x) \wedge (\tau - r)\}} + g'(x + W_{\tau-r})\ 1_{\{\sigma = \tau - r < S(x)\}}\Big]$$

of a stopping problem for the Brownian motion $x + W$ with absorption upon hitting the origin, running cost h_x before termination, cost f of stopping before running out of time or being absorbed, and cost g' for exhausting the time-horizon without having hit the origin.

The assumption (2.2) implies, in particular, that $u(r,\cdot)$ is nondecreasing; it will be assumed throughout this paper that the *continuation region*

(2.7) $$\mathcal{C} \triangleq \{(r,x) \in [0,\tau) \times (0,\infty);\quad u(r,x) < f(r)\}$$

for this problem is actually of the form

(2.8) $$\mathcal{C} = \{(r,x);\quad 0 \leq r < \tau, \quad 0 < x < s(r)\}$$

for a continuous function $s : [0,\tau) \to (0,\infty)$, and that the stopping time

(2.9) $$\sigma(r,x) = \inf\{t \in [0,\tau - r);\quad x + W_t \geq s(r+t)\} \wedge (\tau - r)$$

is optimal for the problem of (2.6). We shall denote by $s(\tau)$ the limit $\lim_{r\uparrow\tau} s(r)$.

3. REPRESENTATION OF THE OPTIMAL RISK

In order to cast the optimal stopping problem of (2.6) into a more conventional framework, we introduce the absorbed process

$$A_t(x) = \begin{Bmatrix} x + W_t & ; & 0 \le t < S(x) \\ \Delta & ; & t \ge S(x) \end{Bmatrix}, \tag{3.1}$$

where Δ is a "cemetery state" isolated from $\mathcal{R}^+$; the convention here is that $g'(\Delta) = h_x(t, \Delta) = 0$; $\forall\ 0 \le t \le \tau$. We also introduce the functions

$$G(r,x) \triangleq E\Big[\int_0^{\tau-r} h_x(r+t, A_t(x))dt + g'(A_{\tau-r}(x))\Big] \tag{3.2}$$

$$H(r,x) \triangleq E\Big[\int_0^{\tau-r} h_x(r+t, A_t(x))dt + (g'(A_{\tau-r}(x)) - f(r))\ 1_{\{S(x)\wedge(\tau-r)>0\}}\Big]\ . \tag{3.3}$$

Obviously $H(r,x) = 0$ for $r = \tau$ or $x = 0$, whereas
(3.4)

$$\begin{aligned} H(r,x) &= E\Big[\int_0^{(\tau-r)\wedge S(x)} \{h_x(r+t, A_t(x)) + f'(r+t)\}dt + g'(A_{\tau-r}(x)) - f(\tau \wedge (r + S(x)))\Big] \\ &= G(r,x) - f(r)\ ; \quad \text{for} \quad (r,x) \in [0,\tau) \times (0,\infty)\ . \end{aligned}$$

It develops then, by a use of the strong Markov property, that the function

$$v \triangleq G - u \tag{3.5}$$

admits the representation

$$\begin{aligned} v(r,x) = & \sup_{\sigma \in S(\tau-r)} E\Big[\int_\sigma^{\tau-r} h_x(r+t, A_t(x))\ dt\ + \\ & \qquad \{g'(A_{\tau-r}(x)) - f(r+\sigma)\}\ 1_{\{\sigma<S(x)\wedge(\tau-r)\}}\Big] \\ = & \sup_{\sigma \in S(\tau-r)} EH(r+\sigma,\ A_\sigma(x))\ , \end{aligned} \tag{3.6}$$

as the maximal expected reward in an optimal stopping problem for the process $A(x)$, with payoff function $H(r,x)$.

The function v admits a second representation in terms of the optimal stopping boundary $s(\cdot)$, which we describe now: one starts by defining, by analogy with (3.4) and (2.9), the process

$$\begin{aligned} C_t^{(r,x)} \triangleq & \int_0^{t\wedge S(x)\wedge(\tau-r)} \{h_x(r+\theta, A_\theta(x)) + f'(r+\theta)\}d\theta \\ & + \{g'(A_{\tau-r}(x)) - f(\tau \wedge (r + S(x)))\}\ 1_{\{0<(\tau-r)\wedge S(x)\le t\}} \end{aligned} \tag{3.7}$$

and the family of stopping times

$$\sigma_t(r,x) \triangleq \inf\{\theta \in [t, \tau - r);\ A_\theta(x) \geq s(r+\theta)\} \wedge (\tau - r)\ ; \quad 0 \leq t \leq \tau - r\ . \tag{3.8}$$

Obviously, $\sigma_0(r,x)$ coincides with the optimal stopping time $\sigma(r,x)$ of (2.9), and the expression (3.6) is recast as

$$v(r,x) = E[C^{(r,x)}_{\tau-r} - C^{(r,x)}_{\sigma_0(r,x)}] = E[\tilde{C}^{(r,x)}_{\tau-r}]\ , \tag{3.9}$$

a potential associated with the process of bounded variation

$$\tilde{C}^{(r,x)}_t \triangleq C^{(r,x)}_{\sigma_t(r,x)} - C^{(r,x)}_{\sigma_0(r,x)}$$

which is *not* adapted to the filtration $\{\mathcal{F}_t\}$. However, every excessive function such as $v(r,x)$ is also the potential associated with an *adapted, nondecreasing* process $D^{(r,x)}$, the dual predictable projection of $\tilde{C}^{(r,x)}$ (or, as it is equivalently called, the "balayée prévisible" of $C^{(r,x)}$).

This process can be found explicitly; indeed, using the methodology of section 7 (Appendix) in [3], it can be shown that *the dual predictable projection* $D^{(r,x)}$ *of* $\tilde{C}^{(r,x)}$ *is nondecreasing and is given by*

$$\begin{aligned} D^{(r,x)}_t = \int_0^t \{h_x(r+\theta, A_\theta(x)) + f'(r+\theta)\}\, 1_{\{A_\theta(x)>s(r+\theta)\}}\, d\theta \\ + \{g'(A_{\tau-r}(x)) - f(\tau)\}\, 1_{\{A_{\tau-r}(x)>s(\tau)\}}\, 1_{\{0<\tau-r\leq t\}}\ . \end{aligned} \tag{3.10}$$

3.1 Remark: Because the process $D^{(r,x)}$ of (3.10) is nondecreasing, we deduce that

(i) the region $\{(r,x) \in [0,\tau) \times [0,\infty);\ h_x(r,x) + f'(r) < 0\}$ is included in the continuation region $\mathcal{C}$ of (2.8), and

(ii) $\quad g'(x) = f(\tau);\quad \forall\ x \geq s(\tau)\ .$

This latter conclusion simplifies (3.10) to

$$D^{(r,x)}_t = \int_0^t \{h_x(r+\theta, A_\theta(x)) + f'(r+\theta)\}\, 1_{\{A_\theta(x)>s(r+\theta)\}}\, d\theta\ . \qquad \square$$

From all these considerations, it develops that we can write (3.9) in the form

$$\begin{aligned} v(r,x) &= E[\tilde{C}^{(r,x)}_{\tau-r}] = E[D^{(r,x)}_{\tau-r}] \\ &= E\int_0^{(\tau-r)\wedge S(x)} [h_x(r+t, x+W_t) + f'(r+t)]\, 1_{\{x+W_t>s(r+t)\}}\, dt \end{aligned} \tag{3.11}$$

and from (3.5), (3.2), (3.11) we obtain
(3.12)

$$u(r,y) = E[\int_0^{(\tau-r)\wedge S(y)} h_x(r+t, y+W_t)\, 1_{\{y+W_t \le s(r+t)\}}\, dt + g'(y+W_{\tau-r})\, 1_{\{S(y)>\tau-r\}} - \int_0^{(\tau-r)\wedge S(y)} f'(r+t)\, 1_{\{y+W_t > s(r+t)\}}\, dt\,]$$

for $(r,y) \in [0,\tau] \times [0,\infty)$.

In the next two sections, we shall try to see what happens if we integrate the two expressions (3.12) and
(3.13)

$$u(r,y) = E[\int_0^{\sigma(r,y)\wedge S(y)} h_x(r+t, y+W_t)dt + f(r+\sigma(r,y))\, 1_{\{\sigma(r,y)<S(y)\wedge(\tau-r)\}} + g'(y+W_{\tau-r})\, 1_{\{\sigma(r,y)=\tau-r<S(y)\}}]$$

for the optimal risk $u(r,\,\cdot\,)$, with respect to the spatial variable. In section 6 we shall connect the results of these integrations with a problem of optimal control.

4. FIRST INTEGRATION

We shall integrate first the expression of (3.12) in the variable y.

4.1 Proposition: For every $x \ge 0$, consider the Brownian motion started at x with reflection at the origin:

$$(4.1) \qquad \begin{aligned} R_t(x) &\triangleq x + W_t + L_t(x) = x + W_t + \max[0, \max_{0\le\theta\le t}\{-x - W_\theta\}] \\ &= (x \vee M_t) + W_t\ ; \qquad 0 \le t < \infty\ , \end{aligned}$$

where M is the increasing process

$$(4.2) \qquad M_t \triangleq \max_{0\le s\le t}(-W_s)\ ; \quad 0 \le t \le \infty\ ,$$

and introduce the function

$$(4.3) \qquad \begin{aligned} N(r,x) \triangleq E[\int_0^{\tau-r} & h(r+t, s(r+t)\wedge R_t(x))dt + g(R_{\tau-r}(x)) \\ & - \int_0^{\tau-r} f'(r+t)(R_t(x) - s(r+t))^+ dt\,] \end{aligned}$$

on $[0,\tau] \times [0,\infty)$. We have then

$$(4.4) \qquad \int_{s(r)}^{x} u(r,y)dy = N(r,x) - N(r,s(r))\ .$$

Proof: With the help of the equivalence $S(y) > t \Longleftrightarrow M_t < y$ we obtain back in (3.12) with $z < x$:

$$\int_z^x g'(y+W_{\tau-r})\, 1_{\{S(y)>\tau-r\}}\, dy = \int_{z\vee M_{\tau-r}}^{x\vee M_{\tau-r}} g'(y+W_{\tau-r})dy = g(R_{\tau-r}(x))-g(R_{\tau-r}(z)) ,$$

$$\int_z^x h_x(r+t, y+W_t)\; 1_{\{y+W_t\le s(r+t)\}}\, dy = \int_{z\vee M_t}^{x\vee M_t} h_x(r+t, y+W_t)\; 1_{\{y+W_t\le s(r+t)\}}\, dy$$
$$= h(\{(x\vee M_t)+W_t\}\wedge s(r+t)) - h(\{(z\vee M_t)+W_t\}\wedge s(r+t))$$
$$= h(R_t(x)\wedge s(r+t)) - h(R_t(z)\wedge s(r+t)) \qquad \text{on} \quad \{S(y)>t\},$$

as well as

$$\int_z^x f'(r+t)\; 1_{\{y+W_t>s(r+t)\}}\; 1_{\{S(y)>t\}}dy =$$
$$= f'(r+t)[(x\vee M_t)\vee(s(r+t)-W_t) - (z\vee M_t)\vee(s(r+t)-W_t)]$$
$$= f'(r+t)[(R_t(x)-s(r+t))^+ - (R_t(z)-s(r+t))^+] .$$

The identity (4.4) follows. □

4.2 Corollary: We have

$$(4.5)\quad N(r,s(r)) = \int_r^\tau h(\theta,s(\theta))d\theta + g(s(\tau)) + f(r)s(r) - f(\tau)s(\tau) + \int_r^\tau f'(\theta)s(\theta)d\theta ,$$

and if the function $s(\cdot)$ is of bounded variation:

$$(4.6)\qquad N(r,s(r)) = \int_r^\tau h(\theta,s(\theta))d\theta - \int_r^\tau f(\theta)ds(\theta) + g(s(\tau)) .$$

This is the "cost of a (deterministic) ride along the moving boundary $s(\cdot)$".

Proof: Let us recall that $u(r,y) = f(r)$, for $y \ge s(r)$. It follows then from (4.4) with $x > s(r)$ that

$$N(r,s(r)) = N(r,x) - f(r)\cdot(x-s(r))$$
$$= E[\int_0^{\tau-r} h(r+t, s(r+t)\wedge R_t(x))dt + g(R_{\tau-r}(x)) - f(\tau)(x-s(r))$$
$$- \int_0^{\tau-r} f'(r+t)\{(R_t(x)-s(r+t))^+ - (x-s(r))\}dt\;] .$$

Letting $x\to\infty$ and appealing to the monotone and dominated convergence theorems, as well as to the fact $g(y)-g(s(\tau)) = f(\tau)\cdot(y-s(\tau))$ for $y\ge s(\tau)$, we obtain (4.5).

5. SECOND INTEGRATION

Let us consider the processes $K^{(r,x)}, \Lambda^{(r,x)}$ defined by $K_o^{(r,x)} = \Lambda_o^{(r,x)} = 0$ and by the system of functional equations

$$K_t^{(r,x)} = \max[0, \max_{0\le\theta\le t}\{-x - W_\theta + \Lambda_\theta^{(r,x)}\}] \; ; \qquad 0 \le t \le \tau - r \tag{5.1}$$

$$\Lambda_t^{(r,x)} = \max[0, \max_{0\le\theta\le t}\{x + W_\theta - s(r+\theta) + K_\theta^{(r,x)}\}] \; ; \qquad 0 \le t \le \tau - r \; . \tag{5.2}$$

The solution to this system exists and is unique, for every Brownian path; both $K^{(r,x)}$, $\Lambda^{(r,x)}$ are continuous on $(0, \tau - r]$ and we have $K_{o+}^{(r,x)} = 0$, $\Lambda_{o+}^{(r,x)} = (x - s(r))^+$ (cf. [4], Appendix). Now the process

$$X_t^{(r,x)} \triangleq x + W_t + K_t^{(r,x)} - \Lambda_t^{(r,x)} \; ; \qquad 0 \le t \le \tau - r \tag{5.3}$$

is, for $0 \le x \le s(r)$, a *Brownian motion started at* x *and reflected at the origin and along the moving boundary* $\{s(r+t);\ 0 \le t \le \tau - r\}$; for an initial position $x > s(r)$, the initial jump of $\Lambda^{(r,x)}$ results in $X_{o+}^{(r,x)} = s(r)$, and from then on the situation is the same as described above.

5.1 Proposition: For the function

$$M(r,x) \triangleq E\Big[\int_0^{\tau - r} h(r+t, X_t^{(r,x)})dt + \int_{[0,\tau-r)} f(r+t)d\Lambda_t^{(r,x)} + g(X_{\tau-r}^{(r,x)})\Big] \; , \tag{5.4}$$

we have

$$\int_{s(r)}^{x} u(r,y)dy \;=\; M(r,x) - M(r,s(r)) \; . \qquad \Box \tag{5.5}$$

The validity of (5.5) is obvious for $x > s(r)$; for $x \in [0, s(r)]$, it will follow by *integrating over the interval* $(x, s(r))$ *the expression of (3.13)*. More precisely, we shall take $r = 0$ for simplicity of notation, denote by $(K(x), \Lambda(x), X(x))$ the triple $(K^{(0,x)}, \Lambda^{(0,x)}, X^{(0,x)})$, and establish the following pathwise result:

5.2 Lemma: For every $x \in [0, s(0)]$, define the stopping time

$$\sigma(x) \triangleq \inf\{t \in [0,\tau) \; ; \; x + W_t \ge s(t)\} \wedge \tau \tag{5.6}$$

where $s : [0,\tau) \to (0,\infty)$ is any continuous function. We have then the a.s. identity

$$
\begin{aligned}
(5.7)\qquad & \int_x^{s(0)} [\int_0^{\sigma(y)\wedge S(y)} h_x(t, y+W_t)dt + f(\sigma(y))1_{\{\sigma(y)<\tau\wedge S(y)\}} \\
&\qquad + g'(y+W_\tau)1_{\{\tau=\sigma(y)<S(y)\}}]dy \\
&= [\int_0^\tau h(t, X_t(s(0)))dt + \int_0^\tau f(t)d\Lambda_t(s(0)) + g(X_\tau(s(0)))] \\
&\qquad - [\int_0^\tau h(t, X_t(x))dt + \int_0^\tau f(t)d\Lambda_t(x) + g(X_\tau(x))] .
\end{aligned}
$$

□

Consider the continuous, nondecreasing processes M of (4.2) and

$$
(5.8)\qquad L_t \triangleq \max_{0\le\theta\le t} (W_\theta - s(\theta)) ; \qquad 0 \le t \le \tau ,
$$

with left-continuous inverses given by $S(\cdot), \sigma(\cdot)$ of (2.5), (5.6), respectively:

$$
(5.9)\qquad \{S(x) \le t\} = \{M_t \ge x\} ; \qquad \forall\ 0 \le t < \infty
$$

and

$$
(5.10)\qquad \begin{aligned} &\{\sigma(x) \le t\} = \{L_t \ge -x\} ; \quad \forall\ 0 \le t < \tau \\ &\{\sigma(x) = \tau\} = \{L_\tau \le -x\} . \end{aligned}
$$

We shall work separately on the two events $\{\sigma(x) < S(x)\}$ and $\{\sigma(x) > S(x)\}$.

PROOF OF (5.7) ON $\{\sigma(x) < S(x)\}$: On this event, we have

(5.11)

$$
\Lambda_t(x) = \begin{Bmatrix} 0 & ; & 0 \le t \le \sigma(x) \\ x + L_t & ; & \sigma(x) \le t \le S^*(x) \end{Bmatrix}, \quad X_t(x) = \begin{Bmatrix} x + W_t & ; & 0 \le t \le \sigma(x) \\ W_t - L_t & ; & \sigma(x) \le t \le S^*(x) \end{Bmatrix}
$$

where

$$
(5.12)\qquad S^*(x) \triangleq \inf\{t \in [\sigma(x), \tau) ; \quad W_t \le L_t\} \wedge \tau
$$

is here the first hitting time of the origin by the process $X(x)$. Formulas analogous to (5.11), (5.12) hold for every $y \in [x, s(0)]$, and we have $0 = \sigma(s(0)) \le \sigma(y) \le \sigma(x)$ for such a configuration. In particular, *the processes* $X(y)$ *coincide on* $[\sigma(x), \tau]$ *for every* $x \le y \le s(0)$, and thus

$$
(5.13)\qquad S^*(y) = S^*(s(0)), \quad \sigma(y) < S(y) ; \quad \forall\ y \in [x, s(0)] .
$$

The proof of (5.7) will be complete, as soon as we have established the following a.s. identities:

$$
(5.14)\qquad \int_x^{s(0)} f(\sigma(y))1_{\{\sigma(y)<\tau\wedge S(y)\}}\, dy = \int_0^\tau f(t)d\Lambda_t(s(0)) - \int_0^\tau f(t)d\Lambda_t(x)
$$

$$(5.15)\qquad \int_x^{s(0)} g'(y+W_\tau)\, 1_{\{\tau=\sigma(y)<S(y)\}}\, dy = g(X_\tau(s(0))) - g(X_\tau(x))$$

$$(5.16)\quad \int_x^{s(0)} \left(\int_0^{\sigma(y)\wedge S(y)} h_x(t, y+W_t)dt \right) dy = \int_0^\tau h(t, X_t(s(0)))dt - \int_0^\tau h(t, X_t(x))dt\,.$$

But this is straightforward; thanks to (5.11)-(5.13), the right-hand sides of these expressions are equal to $\int_0^{\sigma(x)} f(t)dL_t$, $[g(W_\tau - L_\tau) - g(x+W_\tau)]\; 1_{\{\sigma(x)=\tau\}}$ and $\int_0^{\sigma(x)} [h(t, W_t - L_t) - h(t, x+W_t)]dt$. On the other hand, by virtue of (5.10), (5.13) the left-hand sides are computed as follows:

$$\int_x^{s(0)} f(\sigma(y))\; 1_{\{\sigma(y)<\tau\}}\, dy = \int_0^{\sigma(x)} f(t)dL_t\,,$$

$$\int_x^{s(0)} g'(y+W_\tau)1_{\{\sigma(y)=\tau\}}\, dy = \int_x^{s(0)} g'(y+W_\tau)1_{\{y\le -L_\tau\}}\, dy$$
$$= 1_{\{\sigma(x)=\tau\}}[g(W_\tau - L_\tau) - g(x+W_\tau)]\,, \qquad \text{and}$$

$$\int_x^{s(0)} dy \int_0^{\sigma(y)} h_x(t, y+W_t)dt = \int_0^\tau dt \int_x^{s(0)} 1_{\{y<-L_t\}}\, h_x(t, y+W_t)dy$$
$$= \int_0^\tau 1_{\{x<-L_t\}}[h(t, W_t - L_t) - h(t, x+W_t)]dt = \int_0^{\sigma(x)} [h(t, W_t - L_t) - h(t, x+W_t)]dt\,.$$

□

5.3 Remark: Using exactly the same procedure as above, one can show that (5.7) is also valid on the event

$$(5.17)\qquad \{\ \sigma(x) > S(x) \quad \text{and} \quad \sigma(y) < S(y)\ ; \quad \forall\ y \in (x, s(0)]\ \}\,.$$

In a realization like this, the Brownian path issued at x just touches the origin at $t = S(x)$ without crossing it, and then goes on to cross the moving boundary (draw a picture).

PROOF OF (5.7) ON $\{\sigma(x) > S(x)\}$: Let us pick a realization that belongs to this event; if it belongs also to the event of (5.17), we are done. If not, we consider for this particular realization the number

$$(5.18)\qquad z \triangleq \inf\{y \in [0, s(0)]\ ;\ \sigma(y) < S(y)\}\,,$$

for which we have

$$(5.19)\qquad \tau \wedge S(y) \le \sigma(y), \qquad \forall\ \ y \in [x, z]\,.$$

In particular, for every such y we have from (5.1)-(5.3):

$$(5.20)\qquad K_t(y) = (M_t - y)^+, \quad X_t(y) = (y \vee M_t) + W_t\ ; \qquad 0 \le t \le \sigma_*(y)\,,$$

where

$$\sigma_*(y) \triangleq \inf\{t \in [0,\tau) \; ; \; X_t(y) \geq s(t)\} \wedge \tau \\ = \inf\{t \in [S(y),\tau) \; ; \; M_t + W_t \geq s(t)\} \wedge \tau \; . \tag{5.21}$$

Quite obviously

$$S(x) \leq S(y) \leq S(z), \;\; \sigma_*(z) = \sigma(z) = \sigma_*(y) \; ; \;\; \forall \;\; y \in [x,z] \tag{5.22}$$

and

$$\text{the processes} \;\; X.(y) \;\; \text{coincide on} \;\; [S(z),\tau] \; ; \;\; \forall \;\; y \in [x,z] \; . \tag{5.23}$$

In view of Remark 5.3, in order to establish (5.7) on $\{\sigma(x) > S(x)\}$, it suffices to show that

$$\int_x^z [\int_0^{\sigma(y)\wedge S(y)} h_x(t, y + W_t)dt + f(\sigma(y))1_{\{\sigma(y)<\tau\wedge S(y))\}} + g'(y + W_\tau)1_{\{\tau=\sigma(y)<S(y)\}}]dy$$

$$= [\int_0^\tau h(t, X_t(z))dt + \int_0^\tau f(t)d\Lambda_t(z) + g(X_\tau(z))] - [\int_0^\tau h(t, X_t(x))dt \\ + \int_0^\tau f(t)d\Lambda_t(x) + g(X_\tau(x)]$$

holds a.s. on this event, or even (thanks to (5.19) - (5.23)) that

$$\int_x^z dy \int_0^{S(y)\wedge\tau} h_x(t, y + W_t)dt = \int_0^{\tau\wedge S(z)} [h(t, X_t(z)) - h(t, X_t(x))]dt \; , \tag{5.24}$$

$$\int_x^z g'(y + W_\tau)1_{\{\tau=\sigma(y)<S(y)\}} \, dy = [g(X_\tau(z)) - g(X_\tau(x))]1_{\{\tau<S(z)\}} \tag{5.25}$$

hold a.s. on $\{\sigma(x) > S(x)\}$. However, a verification of (5.24), (5.25) based on (5.9), (5.20) is straightforward. □

The proof of Lemma 5.2 is now complete.

Comparing the relation (5.5) with (4.3), we see that the functions $M(r, \cdot)$, $N(r, \cdot)$ are both primitives of the optimal stopping risk $u(r, \cdot)$. We shall show in section 7 that these two functions are actually *identical*.

6. THE CONTROL PROBLEM

Consider the class $\mathcal{A}$ of $\{\mathcal{F}_t\}$ - adapted processes $\xi = \{\xi_t; \; 0 \leq t < \infty\}$ with $\xi_0 = 0$ and nondecreasing, left-continuous paths, a.s. Corresponding to any given $x \geq 0$ and $\xi \in \mathcal{A}$, denote by (X, K) the solution to the $RP(x + W - \xi)$, i.e., the Reflection Problem associated with the process $x + W - \xi$:

$$K \in \mathcal{A}, \qquad X = x + W - \xi + K \tag{6.1}$$

$$(6.2) \qquad X_t \geq 0 \; ; \qquad \forall \; 0 \leq t < \infty$$

$$(6.3) \qquad \int_0^\infty X_t dK_t^c = 0$$

$$(6.4) \qquad \Delta K_t \triangleq K_{t+} - K_t = 2X_{t+} \; ; \quad \forall \; t \in [0,\infty) \;\; \text{s.t.} \;\; \Delta K_t > 0$$

hold a.s. Roughly speaking, K represents the "minimal cumulative amount of rightward pushing at the origin that has to be exerted, in order to keep the resulting process X of (6.1) nonnegative".

As shown in [1], for a.e. Brownian path there exists a unique solution to the problem (6.1)-(6.4). Besides, if we donote by $\mathcal{D}(\tau,x)$ the class of processes $\xi \in \mathcal{A}$ for which

$$\Delta\xi_t \leq X_t \; ; \qquad \forall \; 0 \leq t \leq \tau$$

(i.e., processes which never attempt a jump across the origin), then for every $\xi \in \mathcal{D}(\tau,x)$ the corresponding reflection process K is continuous, and is given by $K_t = \max[0, \sup_{0\leq\theta\leq t}\{\xi_\theta - (x + W_\theta)\}]$.

Suppose now that we associate the expected total cost

$$(6.5) \qquad J(\xi;r,x) \triangleq E[\int_0^{\tau-r} h(r+t,X_t)dt + \int_{[0,\tau-r)} f(r+t)d\xi_t + g(X_{\tau-r})]$$

to every $\xi \in \mathcal{A}$, which now we regard as an element of "control", at the disposal of the decision-maker. Here $h(t,\cdot)$ plays the rôle of a running cost on the state X_t, $f(\cdot)$ is the cost per unit time of controlling effort that is exerted, and $g(\cdot)$ is a cost on the state at the terminal time. The so-called *reflected follower* control problem is to choose $\xi \in \mathcal{A}$ that minimizes the expression of (6.5) over this class, and

$$(6.6) \qquad V(r,x) \triangleq \inf_{\xi\in\mathcal{A}} J(\xi;r,x)$$

is the *value function* of this problem.

It can be shown (cf. [5], Proposition 4.1 or [2], Remark 5.7) that the class $\mathcal{D}(\tau - r, x)$ is complete for the problem (6.6), so that

$$(6.7) \qquad V(r,x) = \inf_{\xi\in\mathcal{D}(\tau-r,x)} J(\xi;r,x) \; .$$

Clearly, the process $\Lambda^{(r,x)}$ of (5.2) belongs to $\mathcal{D}(\tau-r,x)$, the pair $(X^{(r,x)}, K^{(r,x)})$ of (5.1), (5.3) is the solution to the $RP(x + W - \Lambda^{(r,x)})$, and from (5.4) we have

$$(6.8) \qquad M(r,x) = J(\Lambda^{(r,x)};\; r,x) \geq V(r,x) \; .$$

Here is then the fundamental result of this paper.

6.1 Theorem: The functions M, N and V of (5.4), (4.3) and (6.6), respectively, are all equal:

(6.9) $$M(r,x) = N(r,x) = V(r,x); \quad \forall \ (r,x) \in [0,\tau] \times [0,\infty) .$$

6.2 Corollary: It follows immediately from (6.8), (6.9) that *the process* $\Lambda^{(r,x)}$ *is optimal for the control problem* of this section.

In other words, as soon as we have the optimal stopping boundary $s(\cdot)$ for the problem of section 2, we can obtain the optimal processes for the control problem by reflecting the Brownian motion W at the origin and along this moving boundary. □

We shall prove Theorem 6.1 in the next two sections, 7 (identity $M \equiv N$) and 8 (identity $N \equiv V$).

7. M ≡ N

It is quite obvious from the defining relations (4.1), (4.3) and (5.4) that the processes

$$\int_0^{t\wedge\sigma(r,x)} h(r+\theta, R_\theta(x))d\theta + N(r+t\wedge\sigma(r,x),\ R_{t\wedge\sigma(r,x)}) \ ; \quad 0 \le t \le \tau - r$$

$$\int_0^{t\wedge\sigma(r,x)} h(r+\theta, R_\theta(x))d\theta + M(r+t\wedge\sigma(r,x),\ R_{t\wedge\sigma(r,x)}) \ ; \quad 0 \le t \le \tau - r$$

are both $\{\mathcal{F}_t\}$ - martingales. On the other hand, the difference

$$D(r) \ \triangleq \ M(r,x) - N(r,x) \ ; \qquad 0 \le r \le \tau$$

is a continuous function of bounded variation (e.g. Theorem 4.3.6 in [6]), independent of the spatial variable by virtue of (4.4), (5.5). It develops that

$$m(t) \ \triangleq \ D(r+t\wedge\sigma(r,x)) = \int_0^t 1_{\{\theta\le\sigma(r,x)\}} \, D'(r+\theta)d\theta, \ \mathcal{F}_t; \ \ 0 \le t \le \tau - r$$

is a continuous martingale with paths of bounded variation (and therefore constant). But $m(0) = D(r)$ and $m(\tau - r) = D(\sigma(r,x))1_{\{\sigma(r,x)<\tau-r\}}$ because $D(\tau - r) = 0$, and thus

$$D(r) = D(\sigma(r,x)) \ 1_{\{\sigma(r,x)<\tau-r\}} \ , \quad \text{a.s.}$$

holds for every $r \in [0,\tau]$, $0 \le x \le s(r)$. This is possible only if $D(r) \equiv 0$.

8. N ≡ V

We begin with an auxiliary result.

8.1 Lemma: The process

$$(8.1) \qquad N(r+t, R_t(x)) + \int_0^t h(r+\theta, R_\theta(x))\, d\theta\ ; \quad 0 \le t \le \tau - r$$

is an $\{\mathcal{F}_t\}$ - submartingale, for every $(r,x) \in [0,\tau] \times [0,\infty)$.

Proof: It suffices to establish

$$(8.2) \qquad E[N(r+\sigma, R_\sigma(x)) + \int_0^\sigma h(r+\theta, R_\theta(x)) d\theta] \ge N(r,x)$$

for any given $\sigma \in S_{0,\tau-r}$ (cf. Problem 1.3.26 in [6]). From (4.3) and the strong Markov property, we have

$$\begin{aligned} EN(r+\sigma, R_\sigma(x)) = E[\int_0^{\tau-r-\sigma} & h(r+\sigma+\theta, s(r+\sigma+\theta) \wedge R_\theta(R_\sigma(x))) d\theta \\ + g(R_{\tau-r-\sigma}(R_\sigma(x))) - \int_0^{\tau-r-\sigma} & f'(r+\sigma+\theta)(R_\theta(R_\sigma(x)) - s(r+\sigma+\theta))^+ d\theta] \\ = E[\int_\sigma^{\tau-r} & h(r+\theta, s(r+\theta) \wedge R_\sigma(x)) d\theta + g(R_{\tau-r}(x)) \\ - \int_\sigma^{\tau-r} & f'(r+\theta)(R_\theta(x) - s(r+\theta))^+ d\theta]\ . \end{aligned}$$

It develops that

$$(8.3) \qquad E[N(r+\sigma, R_\sigma(x)) + \int_0^\sigma h(r+\theta, R_\theta(x)) d\theta] = N(r,x) + \Delta(r,x)\ ,$$

where

$$\begin{aligned} \Delta(r,x) &= E\int_0^\sigma \{h(r+t, R_t(x)) - h(r+t, s(r+t) \wedge R_t(x))\} dt \\ &\qquad + E\int_0^\sigma f'(r+t)(R_t(x) - s(r+t))^+ dt \\ &\ge E\int_0^\sigma \{h_x(r+t, s(r+t)) + f'(r+t)\}(R_t(x) - s(r+t))^+ dt \ge 0\ . \end{aligned}$$

We have used the convexity of $h(r,\cdot)$, as well as the Remark 3.1(i). □

8.2 Remark: Thanks to the Doob-Meyer decompositon and (4.1), (4.3), the continuous and nonnegative submartingale of (8.1) can be written as

$$(8.4) \quad \int_0^t h(r+\theta, R_\theta(x)) d\theta + N(r+t, R_t(x)) = N(r,x) + \int_0^t u(r+\theta, R_\theta(x)) dW_\theta + A_t(r,x)$$

where $A(r,x)$ is a continuous nondecreasing process (cf. Theorems 1.4.10, 1.4.14 and Problem 1.4.13 in [6]).

Here is the fundamental result of this section.

8.3 Proposition: For fixed $(r,x) \in [0,\tau] \times [0,\infty)$, denote by $(X(x,\xi), K(x,\xi))$ the solution to the $RP(x+W-\xi)$ corresponding to any $\xi \in \mathcal{D}(\tau - r, x)$. Then the process

(8.5)
$$Q_t(x,\xi) \triangleq \int_0^t h(r+\theta, X_\theta(x,\xi))d\theta + \int_{[0,t)} f(r+\theta)d\xi_\theta + N(r+t, X_t(x,\xi)) \ ; \quad 0 \le t \le \tau - r$$

is an $\{\mathcal{F}_t\}$ - submartingale.

Proof: The argument will proceed in several steps.

Step 1: $\xi \equiv 0$. This case amounts to Lemma 8.1, because $(X(x,0), K(x,0)) \equiv (R(x), L(x))$ in the notation of (4.1).

Step 2: $\xi_t = \int_0^t z_s ds$ *for a bounded, nonnegative and* $\{\mathcal{F}_t\}$*-progressively measurable process* $z = \{z_t; 0 \le t \le \tau - r\}$. Consider the exponential martingale

$$Z_t = \exp\{-\int_0^t z_s dW_s - \frac{1}{2}\int_0^t z_s^2 ds\} \ ;$$

under the probability measure $\tilde{P}(d\omega) = Z_{\tau-r}(\omega)P(d\omega)$ on $\mathcal{F}_{\tau-r}$, the process

$$\tilde{W}_t \triangleq W_t + \xi_t = W_t + \int_0^t z_s ds \ ; \quad 0 \le t \le \tau - r$$

is a Brownian motion, by virtue of the Girsanov theorem (section 3.5 in [6]). Now (4.1) is written equivalently as

$$R_t(x) = x + \tilde{W}_t - \xi_t + L_t(x), \quad \text{for} \quad 0 \le t \le \tau - r \ ,$$

and because $L(x)$ is flat off $\{t \ge 0 \ ; \ R_t(x) = 0\}$ it develops that

(8.6) $\qquad (R(x), L(x))$ is the solution to the $RP(x + \tilde{W} - \xi)$.

Besides, we have from (8.4):

(8.7)
$$\int_0^t h(r+\theta, R_\theta(x))d\theta + \int_{[0,t)} f(r+\theta)d\xi_\theta + N(r+t, R_t(x)) =$$
$$= N(r,x) + \int_0^t u(r+\theta, R_\theta(x))d\tilde{W}_\theta + \tilde{A}_t(r,x) \ ,$$

where

$$\tilde{A}_t(r,x) \triangleq A_t(r,x) + \int_0^t [f(r+\theta) - u(r+\theta, R_\theta(x))] \, z_\theta \, d\theta$$

is a continuous, nondecreasing process. The assertion follows from this observation, coupled with (8.7) and (8.6).