SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

JACQUES NEVEU
JIM PITMAN

Renewal property of the extrema and tree property of the excursion of a one-dimensional brownian motion

Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 239-247
<http://www.numdam.org/item?id=SPS_1989__23__239_0>

© Springer-Verlag, Berlin Heidelberg New York, 1989, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

RENEWAL PROPERTY OF THE EXTREMA AND TREE PROPERTY OF THE EXCURSION OF A ONE-DIMENSIONAL BROWNIAN MOTION

J. NEVEU
Laboratoire de Probabilités
Université P. et M. Curie
Tour 56 - 3ème Etage
75252 PARIS CEDEX 05

J. PITMAN
Department of Statistics
University of California
Berkeley, California 94720
United States

Introduction.

To circumvent the abundance of extrema and excursions of the one dimensional Brownian motion, the following definitions will be seen to be natural.

Let Ω be the space of continuous functions $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ which vanish at the origin, equipped with Wiener measure W, the flow $(\theta_t, t \in \mathbb{R})$ of translations $[\theta_t\omega = \omega(t+\cdot) - \omega(t)]$ and the time reversal ρ $[\rho\omega(\cdot) = \omega(-\cdot)]$. Denoting by B_t $(t \in \mathbb{R})$ the coordinates of Ω, for any real $h > 0$ we shall say that a trajectory ω of the Brownian motion $B = (B_t, t \in \mathbb{R})$ admits an h-minimum at the origin if

$$B_t(\omega) \geq 0 \qquad \text{for} \quad t \in [-T_h(\rho\omega), T_h(\omega)]$$

where $T_h(\omega) = \inf[t : t > 0, B_t(\omega) > h]$. Similarly B will be said to admit an h-mininum (resp. h-maximum) at u $(u \in \mathbb{R})$ if $B \circ \theta_u$ (resp. $-B \circ \theta_u$) admits an h-minimum at 0.

Let Ω_o be the space of excursions, i.e. of real continuous functions ω_o defined on an interval $[0,\zeta(\omega_o)]$ $(0 < \zeta(\omega_o) < \infty)$ which are strictly positive except at the extremities 0, $\zeta(\omega_o)$ where they vanish ; let Π be the Itô measure of Brownian excursions on Ω_o. We shall denote by $\mu(\omega_o)$ the height or maximum of an excursion and for any $h > 0$, those excursions ω_o with heights $\mu(\omega_o) \geq h$ will be called h-excursions. An excursion or h-excursion above x $(x \in \mathbb{R})$ is simply an excursion or h-excursion transposed at level x. Similar notions have already been introduced by Th. Brox [B] and by Tanaka [T] under the name respectively of depressions and valleys.

The two main results of this paper concern the laws of the point process of h-extrema (paragraph 1) and of the tree of h-excursions (paragraph 2). It should be noted that markovian methods play a relatively small role in the presentation of this paper in contrast with the alternative approach offered in the companion paper [NP*] [the figures of that paper may also be helpful to the reader of the present text].

1. The point process of h-extrema.

Let $B_t^* = \max(B_s\ ;\ 0 \le s \le t)$ $(t \in \mathbb{R}_+)$ be the maximal function of the Brownian motion $(B_t, t \in \mathbb{R}_+)$ and for a presently fixed $h > 0$, let

$$\tau := \min(t\ ;\ t \in \mathbb{R}_+\ ,\ B_t^* = B_t + h) \quad \text{and} \quad \beta := B_\tau^*.$$

Then

$$\sigma := \max(s\ ;\ s \le \tau\ ,\ B_s = \beta)$$

is a.s. the unique time $s \in [0,\tau]$ such that $B_s = \beta$ and the following results hold.

Lemma : *The two trajectories* $(B_t, 0 \le t \le \sigma)$ *and* $(\beta - B_{\sigma+t}, 0 \le t \le \tau-\sigma)$ *are independent ; hence* (β,σ) *and* $\tau-\sigma$ *are independent. Furthermore* β *is exponentionally distributed on* $\mathbb{R}_+$ *with mean* h*, the conditional law of* σ *with respect to* β *is such that*

$$E[\exp(-\lambda\sigma)/\beta = x] = \exp[-(x/h)\ \varphi(\lambda h^2)] \qquad (\lambda \in \mathbb{R}_+)$$

whereas

$$E(\exp[-\lambda(\tau-\sigma)]) = \psi(\lambda h^2) \qquad (\lambda \in \mathbb{R}_+)$$

provided φ *and* ψ *are defined on* $\mathbb{R}_+$ *by*

$$\varphi(\lambda^2/2) = \lambda\ \coth(\lambda) - 1\ ,\ \psi(\lambda^2/2) = \frac{\lambda}{\sinh(\lambda)}$$

Hence

$$E(\sigma) = 2h^2/3,\ E(\tau-\sigma) = h^2/3 \quad \textit{and} \quad E(\tau) = h^2.$$

Proof : Considering the passage times T_a $(a > 0)$ of Brownian motion $(B_t; t \in \mathbb{R}_+)$, leads as is well known, to the representation of this Brownian motion in terms of a Poisson point process ν on $\mathbb{R}_+ \times \Omega$ with intensity $da\ d\Pi(\omega)$, by the formula

$$B_t = a - \omega(t-S) \quad \text{for} \quad t \in [S, S+\zeta(\omega)]$$

for each random couple (a,ω) belonging to ν, provided

$$S = \iint 1_{\{a'<a\}}\ \zeta(\omega')\nu(da'd\omega')$$

(moreover $T_a = S + \xi(\omega)$, $T_{a-} = S$).

Then let ν^* be the subprocess of ν consisting of the (a,ω) of ν

for which $\mu(\omega) \geq h$. Since $\Pi(\mu \geq h) = 1/h < \infty$, the ν^* process has a.s. a point (a^*,ω^*) with lowest a^* ; this a^* is exponentially distributed with mean $1/h$ and independent of ω^* which follows the law $\Pi(\cdot/\mu \geq h)$ on Ω. Furthermore $\nu - \nu^*$ (which is a Poisson process with intensity $da\, 1_{(\mu(\omega)<h)}\, \Pi(d\omega)$ on $\mathbb{R}_+ \times \Omega$) is independent of ν^*, hence of (σ^*,ω).

Now, referring to the previous definitions, on the one hand $\beta = a^*$,

$$\sigma = \iint 1_{(a'<a^*)}\, \zeta(\omega')\, \nu(da'd\omega')$$

$$= \left[\iint 1_{(a'<a)}\, \zeta(\omega')\, (\nu-\nu^*)(da'd\omega')\right]_{a=a^*}$$

and by the Poisson representation of B the trajectory $(B_t, 0 \leq t \leq \sigma)$ depends only on ν-N^* and a^*. On the other hand the trajectory $(\beta - B_{\sigma+t}, 0 \leq t \leq \tau-\sigma)$ coïncides with the trajectory ω^* killed at the first time it reaches h (remember that $\mu(\omega^*) \geq h$). The independence property of the lemma is thus proved ; the other statements are standard results. □

Since τ is a stopping time for $(B_t, t \in \mathbb{R}_+)$, the translated motion $((B \circ \theta_\tau)_t, t \in \mathbb{R}_+)$ is a new Brownian motion independent of $(B_t, 0 \leq t \leq \tau)$ on which we still iterate the previous construction after a change of sign. This will directly lead us to the following result.

<u>Proposition</u> : *The times of* h*-extrema of a Brownian motion* $(B_t, t \in \mathbb{R})$ *build a stationary renewal process ; denote them by* $S_n (n \in \mathbb{Z})$ *so that* $\ldots S_{-1} < S_0 \leq 0 < S_1 < S_2 < \ldots.$

More generally the trajectories between h*-extrema*

$$(B_{S_n+t} - B_{S_n}\ ,\ 0 \leq t \leq S_{n-1} - S_n)$$

are independent and for $n \neq 0$, *equidistributed (up to changes of sign). In particular the variables*

$$|B_{S_{n+1}} - B_{S_n}| - h \qquad (n \in \mathbb{Z})$$

are independent and exponentially distributed with mean h *whereas the variables* $S_{n+1} - S_n$ $(n \geq 1)$ *are independent, equidistributed, with Laplace transform* $1/\cosh(h\sqrt{2}\lambda)$, *and mean* h^2.

<u>Proof</u> : Let $\tau_0 = \tau$, $\beta_0 = \beta$, $\sigma_0 = \sigma$ and define recursively the $\tau_n, \beta_n, \sigma_n (n\geq 1)$ so that $(\tau_{n+1} - \tau_n\ ,\ \beta_{n+1}\ ,\ \sigma_{n+1} - \tau_n)$ is the (τ,β,σ)-triplet associated to

the Brownian motion $((-1)^{n-1}(B_{\tau_n+t} - B_{\tau_n}),\ t \in \mathbb{R}_+)$. Then for each $n \geq 1$, σ_n is the time of the first h-extremum, in fact an h-maximum if n is even and an h- minimum if n is odd, after time σ_{n-1} (notice also that σ_o is not necessarily the time of an h-maximum since our definitions only imply that $B_t < B_{\sigma_o}$ on $[0,\sigma_o]$; hence if $B_{\sigma_o} < h$, the behaviour of the Brownian motion to the left of 0 will determine whether σ is or is not the time of an h-maximum).

The preceding lemma shows that the positive trajectories

$$((-1)^{n-1}\ (B_{\sigma_n+t} - B_{\sigma_n})\ ,\ 0 \leq t \leq \sigma_{n+1} - \sigma_n)$$

for $n \leq 1$ are independent and equidistributed. (Each of these trajectories is obtained by gluing together a trajectory of type $(\beta - \beta_{\sigma+t}, 0 \leq t \leq \tau-\sigma)$ and a trajectory of type $(B_t, 0 \leq t \leq \sigma)$ to its right). To obtain the result of the proposition is now easy although in general S_n is not equal to σ_n $(n \geq 1)$ because of a difference in labelling. Indeed translate the origin to $-s$ and the $\sigma_n (n \geq 0)$ accordingly ; then conditionally on $\{\sigma_n \leq 0\}$, an event of limiting probability 1 when s tends to $+\infty$, $S_n = \sigma_{\nu+n-1}$ for all $n \geq 1$ if ν denotes the first $n \geq 1$ for which $\sigma_n > 0$ so that the independence and equidistribution of the trajectories

$$(-1)^{\nu+n}\left(B_{\sigma_{\nu+n-1}+t} - B_{\sigma_{\nu+n-1}}, 0 \leq t \leq \sigma_{\nu+n} - \sigma_{\nu+n-1}\right)$$

imply similar properties for the

$$\left(B_{S_n+t} - B_{S_n}, 0 \leq t \leq S_n - S_{n-1}\right)\ (n \geq 1). \quad \square$$

It follows from the proposition that a.s. the levels of all h-extrema are different.

Let us recall that by definition the Palm measure of any stationary point process on $\mathbb{R}$ defined on $(\Omega, W, (\theta_t, t \in \mathbb{R}))$ is the unique positive measure $\hat{W}$ on Ω such that on $\Omega \times \mathbb{R}$: $\hat{W}(d\omega)\ \lambda(dt) = \Theta[W(d\omega)\ N(\omega,dt]$ for the involution $\Theta(\omega,t) = (\theta_t\omega,-t)$ [λ : Lebesgue measure on $\mathbb{R}$]. The Palm probability $\hat{W}^h(d\omega)/\hat{W}^h(\Omega)$ of the h-extrema process, i.e. "W conditioned by the null event : 0 is an h-extremum", is immediately deduced from the preceding proposition. Let us however first introduce a positive measure on Ω, to be denoted by ΠW and already considered by B. Maisonneuve [M] : ΠW on Ω is the image of $\Pi \times \Pi \times W$ on $\Omega_o \times \Omega_o \times \Omega$ by the mapping $(\omega_o, \omega'_o, \omega) \longrightarrow \tilde{\omega}$ given by

$$\tilde{\omega}(t) = \begin{cases} \omega_o(t) & \text{if } 0 \le t \le \zeta(\omega_o) \text{ , } \omega(t-\zeta(\omega_o)) \text{ if } \zeta(\omega_o) \le t, \\ \omega'_o(-t) & \text{if } -\zeta(\omega'_o) \le t \le 0, \ \omega(t+\zeta'(\omega_o)) \text{ if } t \le -\zeta'(\omega_o). \end{cases}$$

Corollary : *The Palm measure* $\hat{W}^h$ *of the* h-*extrema of Brownian motion* $(B_t, t \in \mathbb{R}_+)$ *has total mass* $1/h^2$ *and with equal mass* $(1/2h^2)$ 0 *is an* h-*minimum or a* h-*maximum for* $\hat{W}^h$

$$1_{\{0 \text{ is an h-minimum}\}} \cdot \hat{W}^h = \frac{1}{2}\, 1_{\{T_h < T_o, T_h o\rho\}} \cdot \Pi W.$$

Proof : The total mass $\hat{W}^h(\Omega)$ is given by the reciprocal $1/h^2$ of the mean distance between the successive h-extrema after 0, as for any renewal process ; since minima and maxima alternate, the first statement is then clear.

By the properties of renewal processes, with respect to the probability $\hat{W}^h(\cdot/0$: h minima), the trajectory ω is a bilateral succession of independent subtrajectories of the two types described in the first lemma and its proof. By the way these subtrajectories have been obtained for Brownian motion, it follows that for $\hat{W}^h(\cdot/0$: h min), $(\omega(t), t \ge 0)$ and $(\omega(-t), t \ge 0)$ are independent and both build out of an excursion higher than h followed by a current trajectory of the Brownian motion. □

When $h \downarrow 0$, the Palm measures $\hat{W}^h$ increase to a limit (since the point processes of h-extrema increase), the Palm measure of all extrema, which is equal to $\frac{1}{2}\,[\Pi W + (\Pi W)']$ where $(\Pi W)'$ is the image of ΠW by $\omega(\cdot) \to -\omega(\cdot)$.

2. The tree of h-excursions.

Let us call "standard binary tree" the random binary tree whose branches 1°) independently either split into two branches or die, each event with probability 1/2, 2°) have independent and exponentially distributed lengths of mean α ($\alpha > 0$). The numbers N_x of branches of this tree at the various levels x ($x \in \mathbb{R}_+$) form a critical Galton-Watson process with continuous parameter and the generating functions $f_x(u) = E(u^{N_x})$ are then explicitly given by

(*) $$1/[1-f_x(u)] = 1/(1-u) + x/2\alpha \quad (x \in \mathbb{R}_+ \ ; \ 0 \le u \le 1).$$

Conversely the standard binary tree is the unique random binary tree such that for each $x > 0$: 1°) N_x is distributed according to the geometrical law with generating function f_x given by preceding formula 2°) conditionally on $N_x = n$, the n sub-tree above x of the considered tree are independent and distributed as the original tree, for every n.

Fix $h > 0$. Consider a Brownian excursion higher than h, i.e. an excursion ω_o distributed according to the probability $\Pi_h = \Pi(\cdot/\mu \geq h)$ which is obtained by conditioning the Itô measure Π by the event $(\mu \geq h)$ of finite Π measure $1/h$. For every $x > 0$, consider also the h-excursions above level x which are contained in ω_o ; let N_x be their number which is necessarily finite ($N_x = 0$ if and only if $x > \mu-h$). Since every h-excursion above level x is part of exactly one h-excursion above any lower level, all these h-excursions above the various levels x can be represented by the points (at the corresponding levels x) of a tree, in such a way that an h-excursion above level x and an h-excursion above level y which contains it $(y < x)$ are represented by two points on an ascending line of the tree.

<u>Proposition</u> : *The tree of the* h*-excursions above the various levels* x *of a Brownian excursion higher than* h *(i.e. following the probability* Π_h*) is a standard binary tree. Hence the numbers* N_x *of* h*-excursions above level* x *of this Brownian excursion form a Galton-Watson process with generating functions given by (*). Here* $\alpha = h/2$.

<u>Proof</u> : Fix an $x > 0$. Let T_o be the first passage time of ω_o through x ; define recursively $T_n (n \geq 1)$ to be the first return time to x of ω_o after having raised to $x + h$ after T_{n-1}. Then each piece $(\omega_o(t), T_{n-1} \leq t \leq T_n)$ $(n \geq 1)$ of ω_o contains exactly one h-excursion above level x provided T_n is well defined and conversely, so that the number N_x of these excursions is also the index n of the last T_n which is well defined.

When $N_x = n$ $(n \geq 1)$, the preceding pieces of ω_o are independent and equidistributed (by the stopping time property of the T_n) ; so are the n h-excursions they contain and so are also the n random binary trees associated to these h-excursions, i.e. the n subtrees above level x of the tree associated to ω_o.

Also

$$\Pi_h(N_x \geq 1) = \Pi(\mu \geq x+h/\mu \geq h) = h/(x+h)$$

whereas for each $n \geq 1$

$$\Pi_h(N_x \geq n+1/N_x \geq n) = x/(x+h)$$

since the first member reduces to the probability that a Brownian motion started at x reaches $x+h$ before 0. An easy computation then shows that $f_x(u) = E(u^{N_x})$ is indeed given by (*) with $\alpha = h/2$.

We have thus proved that under Π_h the tree of h-excursions of ω_o has the characteristic property of the standard binary tree of parameter $h/2$. □

The relations between h-excursions and h-extrema are easily described. If T is the time of an h-maximum for a Brownian excursion ω_o higher that h, this excursion performs around T an h-excursion above level $\omega_o(T)-h$ of height exactly equal to h which thus corresponds to a tip of the tree of h-excursions of ω_o. Conversely to any such tip corresponds an h-excursion of height exactly equal to h whose maximum is an h-maximum of ω_o. Thus h-maxima of the Brownian excursion ω_o correspond to the tips of the tree of h-subexcursions of ω_o, moreover $\omega_o(T) = x+h$ if $\omega_o(T)$ is the value of ω_o at such a maximum and x is the level of the corresponding tip.

Consider next a branching point of this tree and let x be its level in the tree. It corresponds to h-excursions above level $x-\varepsilon$ ($\varepsilon > 0$ sufficiently small) which split into two h-excursions above level x due to the presence of an h-minimum of ω_o at level x which separates these two h-excursions (hence such branching point should be regarded as two points, the two roots of two subtrees above x). Notice here that two h-minima cannot have the same level, so that branching points are indeed binary branching points. Conversely any h-minimum of ω_o (if any) generates a branching point of the tree. Hence the h-minima of the Brownian excursion ω_o correspond to the branching points, if any, of the tree of the h-subexcursions of ω_o, the values of the h-minima being equal to the levels in the tree of the corresponding branching points.

Some simple results follow from these identifications :

a) The probabilities p_n $(n \geq 1)$ that the standard binary tree has n tips (or $2n-1$ branches, or $n-1$ branching points) have the familiar generating function $\tilde{p}(z) = \sum_1^\infty p_n z^n$ given by $\tilde{p}(z) = 1 - \sqrt{1-z}$ since by the branching mechanism $\tilde{p}(z) = [z + \tilde{p}(z)^2]/2$. Hence

$$p_n = \binom{2n}{n} \Big/ (2n-1)2^{2n} \qquad (n \geq 1)$$

and this is also the probability that a Brownian excursion higher than h (i.e. of law Π_h) has n h-maxima and $n-1$ interlaced h-minima.

Similarly it is easily proved that for $0 < h' \leq h$, the number of h'-maxima of a Brownian excursion higher than h is distributed as the number of tips of a standard binary tree with $\alpha = h'/2$ conditioned to be higher than $(h-h')$.

b) Let

$$X_1+h,\ X_1 - Y_1\ ,\ X_1 - Y_1 + X_2 + h, \dots \sum_1^{n-1} (X_i - Y_i) + X_n + h$$

be the value of the $2n-1$ successive h-extrema of a Brownian excursion ω_o higher than h. Proposition 1 then implies that the random variables $X_1, Y_1, X_2, \dots$ are independent and exponentially distributed with mean h, the random integer n being also the first integer for which $\sum_1^h (X_i - Y_i) \leq 0$. but

$$X_1,\ X_1 - Y_1\ ,\ X_1 - Y_2 + X_2\ , \dots \sum_1^{n-1} (X_i - Y_i) + X_n$$

are also the levels of the successive tips and intercalated branching points of the tree of the h-subexcursions of ω_o ; hence the properties of $X_1, Y_1, \dots$ and n are also familiar properties of the standard binary tree.

BIBLIOGRAPHY

[B] **Brox, Th. (1986).** One dimensional diffusion processes in a Wiener medium. Ann. Proba. 14, 1206-1218.

[K] **Kesten, H. (1986).** The limit distribution of Sinaï's walk in random environment. Physica 138A, 299-309.

[M] **Maisonneuve, B. (1975).** Exit systems. Ann. Proba. 3, 399-411.

[N1] **Neveu, J. (1977).** Processus ponctuels in Ecole d'Eté de Probabilités de S^t Flour VI-1976, Lect. Notes in Math. 598, 249-447, Springer.

[N2] **Neveu, J. (1986).** Erasing a branching tree. Supplement to Adv. Applied Proba., Papers in honour of G.E.H. Reuter, 101-108.

[NP*] **Neveu, J. and Pitman, J.** The branching process in a Brownian motion excursion. This volume.

[S] **Sinaï, Y.G. (1982).** The limiting behavior of a one-dimensional random walk in a random medium. Theory Proba. Appl. 27, 256-268.

[T] **Tanaka, H. (1987).** Limit theorem for one dimensional diffusion process in Brownian environment. Stochastic Analysis, Proc. Paris 1987, Ed. Metivier-Watanabe. Lect. Notes in Math. 1322, Springer, pp. 156-172.

[W] **Walsh, J. (1978).** Downcrossings and the Markov property of local time. In : Temps locaux. Astérisque 52-53, 89-115. Soc. Math. France.