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L.C.G. ROGERS

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Multiple points of Markov processes in a complete metric space

by
L.C.G. Rogers

1. Introduction.

Let (S, d) be a complete metric space with Borel σ -field \mathcal{S} , and let $(X_t)_{t \geq 0}$ be an S -valued strong Markov process whose paths are right continuous with left limits. We ask

(Q) Is $P(X_{t_1} = \cdots = X_{t_k} \text{ for some } 0 < t_1 < \cdots < t_k) > 0$?

This is equivalent to the question

(Q') Is $P(X(I_1) \cap \cdots \cap X(I_k) \neq \emptyset) > 0$ for some disjoint compact intervals I_1, \dots, I_k ?

We shall find conditions sufficient to ensure that X has k -multiple points with positive probability, and we will apply this to Lévy processes, providing another proof of a result of LeGall, Rosen and Shieh [6], and its improvement due to Evans [3]. However, it is advantageous to begin with the easier question

(\bar{Q}) Is $P(\bar{X}(I_1) \cap \cdots \cap \bar{X}(I_k) \neq \emptyset) > 0$ for some disjoint compact intervals I_1, \dots, I_k ?

Here, $\bar{X}(I_j) \equiv \text{closure}(\{X_s : s \in I_j\})$, a compact subset of S . In recent years, much effort has been devoted to a study of (Q), usually in the form of constructing some non-trivial random measure on the set $\{(t_1, \dots, t_k) : X_{t_1} = \cdots = X_{t_k}\}$ from which the existence of common points in the ranges $X(I_j)$ follows immediately. We mention only the work of Dynkin [1] and Evans [2] on symmetric Markov processes, of Rosen [8], [9], Geman, Horowitz and Rosen [4], LeGall, Rosen and Shieh [6] and Evans [3] on more concrete Markov processes in \mathbb{R}^n , as a sample of recent activity. Typically, one studies the random variables

$$(1) \quad Z_\varepsilon \equiv \int_C I_U(X_{t_1}) F_\varepsilon(X_t) dt,$$

where $C = I_1 \times \cdots \times I_k$, with the I_j disjoint compact intervals in \mathbb{R}^+ , $U \in \mathcal{S}$, and

$$(2) \quad F_\varepsilon(x_1, \dots, x_k) \equiv \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}),$$

(where f_ε is some 'spike' function such that $f_\varepsilon(x,y) = 0$ if $d(x,y) > \varepsilon$), and proves L^2 -convergence of the Z_ε to some non-trivial limit as $\varepsilon \downarrow 0$.

This will be the approach used here, but, since we are concerned *only* with an answer to (Q), and not with the (more refined) L^2 -convergence of the Z_ε , we can weaken the assumptions somewhat. In particular, we give sufficient conditions to ensure the existence of points of intersection for general (i.e. *non-symmetric*) Markov processes.

If we could prove that

- (3.i) for some $\eta > 0$, $\{Z_\varepsilon : 0 < \varepsilon < \eta/k\}$ is bounded in L^2 ;
- (3.ii) $\limsup_{\varepsilon \downarrow 0} E Z_\varepsilon > 0$,

then the answer to (Q) is, "Yes". The point is that $(Z_\varepsilon)_{0 < \varepsilon < \eta/k}$ is then uniformly integrable; if there were no common points in the closed ranges $\bar{X}(I_j)$, then the Z_ε would (almost surely) be zero for all small enough $\varepsilon > 0$, and hence the $Z_\varepsilon \rightarrow 0$ in L^1 , contradicting (3.ii).

2. The main result. We suppose that there is a σ -finite measure μ on S such that for all $x \in S$

$$(4) \quad \mu(B_\varepsilon(x)) > 0 \quad \forall \varepsilon > 0.$$

Here, $B_\varepsilon(x) \equiv \{y : d(x,y) \leq \varepsilon\}$. (The assumption (4) is no great restriction, since we could always confine ourselves to the closed set of x for which it is true.)

We shall suppose that the Green's functions of X have densities with respect to μ : for $0 \leq a < b < \infty$, there exists $g_{a,b}(\cdot, \cdot)$ such that

$$(5) \quad G_{a,b}(x,A) \equiv E^x \left[\int_a^b I_A(X_s) ds \right] = \int_A g_{a,b}(x,y) \mu(dy) \quad (\forall x \in S, A \in \mathcal{S}).$$

We suppose also that there are open $U \subset V \subset S$ such that for some $\eta > 0$ the η -neighbourhood of U is contained in V , and that there are positive finite K, T such that

$$(A) \quad \mu(B_{2\varepsilon}(x)) \leq K \mu(B_\varepsilon(x)) \quad \forall \varepsilon \in (0, \eta], \forall x \in V;$$

$$(B) \quad \int_{V \times V} g_{0,T}(x,y)^k \mu(dx) \mu(dy) < \infty;$$

$$(C) \quad \text{for each } \delta \in (0, 2T),$$

$$\sup_{x,y \in V} g_{\delta,2T}(x,y) < \infty;$$

$$(D) \quad \text{for each } 0 < a < b < \infty, g_{a,b}(\cdot, \cdot) \text{ is lower semicontinuous on } V \times V;$$

$$(E) \quad \text{for some } \xi \in U \text{ and } \tau \in (0, T),$$

$$g_{0,\tau}(\xi, \xi) > 0.$$

Remarks on conditions (A)-(E). Condition (A) seems fairly mild; it is trivially satisfied for Lebesgue measure on Euclidean space. The purpose of (A) is to let us take

$$(6) \quad f_\varepsilon(x,y) \equiv \mu(B_\varepsilon(x))^{-1} I_{\{d(x,y) \leq \varepsilon\}}$$

and estimate

$$(7) \quad \begin{aligned} f_\varepsilon(x,y) &\leq K \mu(B_{2\varepsilon}(x))^{-1} I_{\{d(x,y) \leq \varepsilon\}} \\ &\leq K \mu(B_\varepsilon(y))^{-1} I_{\{d(x,y) \leq \varepsilon\}} \\ &= K f_\varepsilon(y,x). \end{aligned}$$

Condition (B) is the ‘folklore’ condition for k -multiple points. Condition (C) may appear severe, but is frequently satisfied. Conditions (A)-(C) will give us (3.i), and conditions (D) and (E) will give us (3.ii). We may (and shall) suppose that the τ appearing in (E) is a point of increase of $g_{0,\cdot}(\xi, \xi)$.

THEOREM 1. *Assuming conditions (A), (B), and (C), the family $\{Z_\varepsilon : 0 < \varepsilon < \eta/k\}$ is bounded in L^2 . Assuming also conditions (D) and (E), there exist initial distributions such that for some disjoint compact intervals I_1, \dots, I_k*

$$P(\bar{X}(I_1) \cap \dots \cap \bar{X}(I_k) \neq \emptyset) > 0.$$

Proof. (i) Let m be the law of X_0 . For ease of exposition, we shall suppose that X has a transition density $p_t(\cdot, \cdot)$ with respect to μ ; the result remains true without this assumption though.

The time-parameter set $C = I_1 \times \cdots \times I_k$ used in the definition of Z is chosen so that $j\tau$ is in the interior of I_j for each j , so that $0 < \delta \leq t - s \leq 2T$ if $t \in I_j, s \in I_{j-1}$ ($j = 2, \dots, k$), and so that $|I_j| < T$ for all j . Then

$$\begin{aligned} E Z_\varepsilon^2 &= E \int_{C \times C} ds dt I_U(X_{s_1}) I_U(X_{t_1}) F_\varepsilon(X_s) F_\varepsilon(X_t) \\ &= \sum_R \int_{C_\varepsilon^2} ds dt \int m(dy_0) I_U(x_1) I_U(y_1) F_\varepsilon(x') F_\varepsilon(y') \prod_{j=1}^k p_{s_j-t_{j-1}}(y_{j-1}, x_j) p_{t_j-s_j}(x_j, y_j) \mu(dx_j) \mu(dy_j), \end{aligned}$$

where $C_\varepsilon^2 = \{(s, t) \in C^2 : s_i \leq t_i \text{ for } i = 1, \dots, k\}$, $t_0 \equiv 0$, the sum is taken over all subsets R of $\{1, \dots, k\}$, and

$$\begin{aligned} x'_i &= x_i, \quad y'_i = y_i && \text{if } i \in R \\ x'_i &= y_i, \quad y'_i = x_i && \text{if } i \notin R. \end{aligned}$$

The typical term in the sum is bounded above by some constant times

$$\int m(dy_0) I_U(x_1) I_U(y_1) F_\varepsilon(x') F_\varepsilon(y') \prod_{j=1}^k q(y_{j-1}, x_j) g(x_j, y_j) \mu(dx_j) \mu(dy_j),$$

where we have made the abbreviations

$$\begin{aligned} q(x, y) &\equiv g_{\delta, 2T}(x, y), \\ g(x, y) &\equiv g_{0, T}(x, y). \end{aligned}$$

By assumption (C), the factors $q(y_{j-1}, x_j)$ are globally bounded, because $x_1, y_1 \in U$, and $d(x'_i, x'_{i+1}) \leq \varepsilon < \eta/k$ for each i , and therefore by assumption $x_i \in V$ for all $i = 1, \dots, k$. Thus we have an upper bound in terms of

$$\begin{aligned} &\int I_U(x_1) I_U(y_1) F_\varepsilon(x') F_\varepsilon(y') \prod_{j=1}^k g(x_j, y_j) \mu(dx_j) \mu(dy_j) \\ &\leq \prod_{j=1}^k \left(\int I_U(x_1) I_U(y_1) F_\varepsilon(x') F_\varepsilon(y') g(x_j, y_j)^k \mu(dx) \mu(dy) \right)^{1/k}, \end{aligned}$$

by Hölder's inequality, where, of course $\mu(dx) \equiv \prod_1^k \mu(dx_j)$. The j^{th} term in this product, raised to the power k , is bounded by

$$\int I_V(x_j) I_V(y_j) g(x_j, y_j)^k \prod_{i=1}^{k-1} f_\varepsilon(x'_i, x'_{i+1}) f_\varepsilon(y'_i, y'_{i+1}) \mu(dx) \mu(dy),$$

which we deal with by integrating out successively $x_k, y_k, x_{k-1}, \dots, x_{j+1}, y_{j+1}$, and then,

exploiting (6), integrating out $x_1, y_1, \dots, x_{j-1}, y_{j-1}$ to leave as an upper bound

$$K^{2j-2} \int I_V(x_j) I_V(y_j) g(x_j, y_j)^k \mu(dx_j) \mu(dy_j)$$

which is finite, by assumption (B). Hence for $0 < \varepsilon < \eta/k$, $E(Z_\varepsilon^2)$ is bounded above by a finite constant independent of ε , which proves the first statement.

(ii) We next exploit (D) and (E) to give us (3.ii). By the choice of the set C , we have that for some small enough $\theta > 0$,

$$C \supseteq C_0 = \{(t_1, \dots, t_k) : |t_i - t_{i-1} - \tau| < \theta \text{ for } i = 1, \dots, k\},$$

where $t_0 = 0$. Hence

$$\begin{aligned} EZ_\varepsilon &\geq E \left[\int_{C_0} dt I_U(X_{t_1}) F_\varepsilon(X_t) \right] \\ &= \int m(dx_0) I_U(x_1) \prod_{i=1}^k g(x_{i-1}, x_i) \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}) \mu(dx), \end{aligned}$$

where we write g as an abbreviation for $g_{\tau-\theta, \tau+\theta}$. Since τ is a point of increase of $g_{0, \cdot}(\xi, \xi)$, we know that $g(\xi, \xi) > 0$. Thus

$$(8) \quad EZ_\varepsilon \geq \int m(dx_0) I_U(x_1) g(x_0, x_1) \underline{g}_\varepsilon(x_1)^{k-1} \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}) \mu(dx),$$

where

$$\underline{g}_\varepsilon(x_1) \equiv \inf\{g(x, y) : d(x, x_1) \leq k\varepsilon, d(y, x_1) \leq k\varepsilon\},$$

which, in view of (D), increases as $\varepsilon \downarrow 0$ to $g(x_1, x_1)$. By integrating out the variables x_k, x_{k-1}, \dots, x_2 in (8), we obtain the lower bound

$$EZ_\varepsilon \geq \int m(dx_0) I_U(x_1) g(x_0, x_1) \underline{g}_\varepsilon(x_1)^{k-1} \mu(dx_1),$$

and hence the estimate

$$\liminf_{\varepsilon \downarrow 0} EZ_\varepsilon \geq \int m(dx_0) I_U(x_1) g(x_0, x_1) g(x_1, x_1)^{k-1} \mu(dx_1).$$

By lower semi-continuity and the fact that $g(\xi, \xi) > 0$, we know that $g(x, y)$ is positive in a neighbourhood of (ξ, ξ) and so taking $m = \delta_\xi$, for example, yields

$$\liminf_{\varepsilon \downarrow 0} EZ_\varepsilon > 0. \quad \diamond$$

We now turn to the more difficult question Q. Let us suppose further that *every singleton is polar*:

$$(F) \quad P^x(X_t = y \text{ for some } t > 0) = 0 \quad \forall x, y \in S,$$

and that

(G) *for each* $\mu \in Pr(S)$, *for each previsible stopping time* $\tau > 0$ *we have*

$$X_\tau = X_{\tau-} \quad P^\mu - \text{ a.s. on } \{\tau < \infty\}.$$

For example, if S is locally compact and separable, and the process X is Feller-Dynkin, then (G) holds; see Rogers and Williams [7], Theorem VI.15.1.

THEOREM 2. *Assuming conditions (A)-(G), there exist initial distributions such that for some disjoint compact intervals I_1, \dots, I_k*

$$P(X(I_1) \cap \dots \cap X(I_k) \neq \emptyset) > 0.$$

Proof. The proof uses Theorem 1, and proceeds by induction on k . For $k = 1$, the result is trivial. We suppose the result is true for $k \leq K$, and, using Theorem 1, take some initial distribution, and disjoint compact intervals I_1, \dots, I_{K+1} such that I_{j+1} is to the right of I_j for each j , and

$$(9) \quad P(\bar{R}_K \cap \bar{X}(I_{K+1}) \neq \emptyset) > 0,$$

where $\bar{R}_K = \bar{X}(I_1) \cap \dots \cap \bar{X}(I_K)$. Let $R_K = X(I_1) \cap \dots \cap X(I_K)$. Then

$$P(\bar{R}_K \cap X(I_{K+1}) \neq \emptyset) > 0,$$

because, if not, from (9), the previsible time set

$$\{t \in I_{K+1} : X_{t-} \in \bar{R}_K\}$$

is non-empty with positive probability and can therefore be sectioned by a previsible time τ ; but, by (G), $X_\tau = X_{\tau-} \in \bar{R}_K$.

Finally we deduce that

$$P(R_K \cap X(I_{K+1}) \neq \emptyset) > 0,$$

for if not, we would have to have

$$(10) \quad P((\bar{R}_K \setminus R_K) \cap X(I_{K+1}) \neq \emptyset) > 0;$$

since $\bar{R}_K \setminus R_K \subset \bigcup_{j=1}^K (\bar{X}(I_j) \setminus X(I_j))$, and $\bar{X}(I_j) \setminus X(I_j)$ is contained in the (countable) set of left endpoints of jumps of X during time interval I_j , it follows from (F) that the set $\bar{R}_K \setminus R_K$ is *polar*, contradicting (10). \diamond

3. Multiple points of Lévy processes. Let X be a Lévy process in \mathbb{R}^n , with resolvent $(U_\lambda)_{\lambda > 0}$. We shall assume that the resolvent is strong Feller (equivalently, that each $U_\lambda(x, \cdot)$ has a density with respect to Lebesgue measure - see Hawkes [5]), in which case there is for each $\lambda > 0$ a λ -excessive lower semi-continuous function u_λ such that

$$U_\lambda f(x) = \int u_\lambda(y) f(y+x) dy .$$

To establish sufficient conditions for k -multiple points, we shall need three lemmas on Lévy processes of interest in their own right.

LEMMA 1. *The resolvent $(U_\lambda)_{\lambda > 0}$ is strong Feller if and only if for every $0 \leq a < b < \infty$ the kernel $G_{a,b}$ has a density $g_{a,b}$.*

If this happens, the densities $g_{a,b}(\cdot)$ may be chosen so that

- (i) $g_{a,b}(\cdot)$ is lower semicontinuous for each $0 \leq a < b < \infty$;
- (ii) $(a,b) \rightarrow g_{a,b}(x)$ is left-continuous increasing in b and right-continuous decreasing in a for each x ;
- (iii) for all $0 \leq a < b < \infty$ and all $x \in \mathbb{R}^n$

$$g_{a,b}(x) = \lim_{\delta \downarrow 0} \delta^{-1} \int g_{0,\delta}(y) g_{a,b-\delta}(x-y) dy .$$

LEMMA 2. *For a Lévy process with a strong Feller resolvent, the following are equivalent:*

- (i) for some $\varepsilon, T > 0$,

$$\int_{\{|x| \leq \varepsilon\}} g_{0,T}(x)^k dx < \infty ;$$

- (ii) for some $T > 0$, $g_{0,T} \in L^k$;
 (iii) for some $\lambda > 0$, $u_\lambda \in L^k$;
 (iv) for some $\varepsilon, \lambda > 0$,

$$\int_{\{|x| \leq \varepsilon\}} u_\lambda(x)^k dx < \infty.$$

LEMMA 3. Let X be a Lévy process with a strong Feller resolvent such that $g_{0,T}(0) > 0$ for some T , and $\{\xi\}$ is non-polar for some $\xi \in \mathbb{R}^n$. Then $\{x\}$ is non-polar for every $x \in \mathbb{R}^n$.

We defer the proofs of these lemmas so as to show how to deduce the following result from them and Theorem 2. Fix some integer $k > 1$.

THEOREM 3 (LeGall-Rosen-Shieh; Evans). Assuming that the Lévy process X has a strong Feller resolvent, the conditions

- (11.i) for some $\varepsilon, T > 0$

$$\int_{\{|x| \leq \varepsilon\}} g_{0,T}(x)^k dx < \infty;$$

- (11.ii) for some $T > 0$, $g_{0,T}(0) > 0$

are sufficient to ensure that the paths of X have points of multiplicity k almost surely.

Proof. In view of Lemma 3, we may assume that every singleton is polar, for, if not, every singleton is non-polar, and the existence of multiple points is trivial! To apply Theorem 2, we must check conditions (A)-(G); (A) is immediate, (B) is guaranteed by (11.i), (D) follows from Lemma 1, (E) comes from (11.ii), (F) is by assumption, and (G) is valid because the Lévy process is a Feller-Dynkin process. Finally, to check (C), (11.i) implies that $g_{0,T}$ is square-integrable in a neighbourhood of 0, so, by Lemma 2, $g_{0,T} \in L^2$. Hence $g_{0,T}^* g_{0,T}$ is bounded and continuous. But for $f \geq 0$ measurable, of compact support, and $0 < \delta < T$

$$\begin{aligned} \int g_{0,T}^* g_{0,T}(x) f(x) dx &= \int_0^T dt \int_0^T ds P_{t+s} f(0) \\ &\geq \delta \sqrt{2} \int_\delta^{2T-\delta} P_t f(0) dt \\ &= \delta \sqrt{2} \int g_{\delta, 2T-\delta}(x) f(x) dx, \end{aligned}$$

whence $g_{\delta,T}(\cdot)$ is bounded globally (exploiting lower semi-continuity).

This completes the proof that (11.i-ii) implies that X has k -multiple points with positive probability, and hence, by Borel-Cantelli, there are almost surely k -multiple points.

Proof of Lemma 1. The arguments used are similar to those of Hawkes [5], so we will just give an outline. The first statement of the lemma is immediate. To get good versions of the densities $g_{a,b}$, firstly take any densities $g'_{p,q}(\cdot)$ for $G_{p,q}$, $0 \leq p < q < \infty$ rational, then define

$$g''_{a,b}(x) \equiv \sup \{g'_{p,q}(x) : a < p < q < b\},$$

which have property (ii) (which remains preserved under the subsequent modifications). Next, for $n > (b-a)^{-1}$ define

$$\tilde{g}_{a,b}^n(x) = n \int g_{0,\delta}(y) g_{a,b-\delta}(x-y) dy, \quad (\delta \equiv n^{-1})$$

which is lower semicontinuous in x (it is the increasing limit as $M \uparrow \infty$ of

$$n \int g_{0,\delta}(y) (M \wedge g_{a,b-\delta}(x-y)) dy,$$

which are continuous by the strong Feller property of $G_{0,\delta}$). Finally, we take

$$g_{a,b}(\cdot) \equiv \sup \{\tilde{g}_{a,b}^n(\cdot) : n > (b-a)^{-1}\}.$$

Since, for fixed $a < b$, $\tilde{g}_{a,b}^n$ is increasing almost everywhere to a version of the density of $G_{a,b}$, this provides a version with the desirable properties (i) - (iii). \diamond

Proof of Lemma 2. The implications (iii) \Rightarrow (iv) \Rightarrow (i) are trivial. The implication (ii) \Rightarrow (iii) follows easily from the estimate

$$\begin{aligned} \int g_{a,a+T}(x)^k dx &= \int \left(\int P_a(dy) g_{0,T}(x-y) \right)^k dx \\ &\leq \int dx \int P_a(dy) g_{0,T}(x-y)^k \\ &= \int g_{0,T}(z)^k dz. \end{aligned}$$

So, finally, we assume (i) and prove (ii). Specifically, let K denote the cube

$$K \equiv \{x \in \mathbb{R}^n : |x_i| \leq \frac{1}{2} \text{ for } i = 1, \dots, n\},$$

and assume without loss of generality that

$$\int_{K+K} g(x)^k dx < \infty,$$

where we have abbreviated $g_{0,T}$ to g . For $j \in \mathbb{Z}^n$, let

$$\tau_j \equiv \inf \{t > 0 : X_t \in j + K\}.$$

Then for $x \in j + K$, we have from the strong Markov property at τ_j that

$$g(x) \leq \int_{j+K} P(\tau_j < T, X(\tau_j) \in dy) g(x-y),$$

from which

$$g(x)^k \leq P(\tau_j < T)^{k-1} \int_{j+K} P(\tau_j < T, X(\tau_j) \in dy) g(x-y)^k,$$

and, integrating,

$$\int_{j+K} g(x)^k dx \leq P(\tau_j < T)^k \int_{K+K} g(z)^k dz.$$

The proof is finished if we can show that

$$\phi(T) \equiv \sum_j P(\tau_j < T) < \infty.$$

Since ϕ is evidently increasing, it is enough to prove that

$$\int_0^\infty \lambda e^{-\lambda T} \phi(T) dT = \sum_j P(\tau_j < \zeta) < \infty,$$

where ζ is an $\exp(\lambda)$ random variable independent of X . But we have the lower bound

$$(12) \quad \int_{j+K+K} u_\lambda(x) dx \geq P(\tau_j < \zeta) \int_K u_\lambda(x) dx.$$

The sum over $j \in \mathbb{Z}^n$ of the left-hand sides of (12) is clearly finite, and $\int_K u_\lambda(x) dx > 0$, so the proof is finished.

Proof of Lemma 3. If $\{\xi\}$ is non-polar, the resolvent density $u_\lambda(\bullet)$ must be bounded, since

$$E^x \exp(-\lambda H_\xi) = c_\lambda \bullet u_\lambda(\xi - x)$$

for some constant c_λ . (Here, $H_\xi = \inf\{t > 0 : X_t = \xi\}$.) By lower semicontinuity, $u_\lambda(0) > 0$ implies that $u_\lambda > 0$ in some neighbourhood of zero and hence, by the resolvent equation, $u_\lambda > 0$ everywhere. Thus $P^x(H_\xi < \infty) > 0$ for every x , and translation invariance implies that every point is non-polar. \diamond

Remarks. (i) It is evident that (11.ii) is equivalent to the condition

(9.ii) for some $\lambda > 0$, $u_\lambda(0) > 0$.

Hence, in view of Lemma 2, the conditions (11) are equivalent to those imposed by Evans [3].

(ii) Similar techniques can be used to study the problem of the existence of common points in the ranges of k independent Markov processes, a technically easier problem.

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Statistical Laboratory
16 Mill Lane
Cambridge CB2 1SB
Great Britain