

SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 22 (1988), p. 89-91

[<http://www.numdam.org/item?id=SPS_1988__22__89_0>](http://www.numdam.org/item?id=SPS_1988__22__89_0)

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A PERTURBATION THEOREM FOR SEMIGROUPS OF LINEAR OPERATORS

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Let $\mathfrak{X}$ be a Banach space. Let $T(t)$ and $S(t)$ be strongly continuous semigroups of bounded linear operators on $\mathfrak{X}$ with infinitesimal generators A and B respectively. The main purpose of this note is to prove the following theorem:

THEOREM 1. Let $x \in \mathcal{D}(A)$. If $T(s)x \in \mathcal{D}(B)$ for $0 < s < t$, and $\int_0^t \|BT(s)x\| ds < \infty$ (or equivalently $\int_0^t \|(B-A)T(s)x\| ds < \infty$), then we have

$$(1) \quad S(t)x - T(t)x = \int_0^t S(t-s)(B-A)T(s)x \, ds$$

REMARK. If $B-A$ is a bounded operator, the conditions are satisfied, and in this case formula (1) is due to R.S. Phillips [1] (see also [2], p. 77).

Before proving theorem 1 we give a useful corollary :

COROLLARY. Assume $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B-A$ is A -bounded in the sense that

$$\|(B-A)y\| \leq a\|y\| + b\|Ay\| \text{ for } y \in \mathcal{D}(A),$$

where $a \geq 0$, $b \geq 0$ are constants. Then (1) holds for $x \in \mathcal{D}(A)$.

Proof. One has $T(s)x \in \mathcal{D}(A) \subset \mathcal{D}(B)$, and

$$\|BT(s)x\| \leq \|(B-A)T(s)x\| + \|AT(s)x\| \leq a\|T(s)x\| + (b+1)\|T(s)Ax\|.$$

Since $\|T(s)\| \leq Me^{\omega s}$ for some $M > 0$ and $\omega \geq 0$, we have $\int_0^t \|BT(s)x\| ds < \infty$, and therefore the conditions of theorem 1 are satisfied.

From now on we concentrate on the proof of theorem 1. Let

$$(2) \quad G(t) = \int_0^t S(t-s)T(s) \, ds.$$

It is easy to see that each $G(t)$ is a bounded linear operator and $G(\cdot)$ is strongly continuous on $\mathfrak{X}$.

LEMMA 1. We have

$$(3) \quad \lim_{h \downarrow 0} G(h)/h = I \text{ strongly on } \mathfrak{X}$$

$$(4) \quad G(s+t) = S(s)G(t) + G(s)T(t)$$

Proof. Left to the reader.

1. This work was supported by the Foundation of Academia Sinica for Scientific Research.

PROPOSITION 1. Let $x \in \mathcal{B}(A)$. Then $G(t)x \in \mathcal{B}(B)$, and we have

$$(5) \quad S(t)x - T(t)x = BG(t)x - G(t)Ax \quad .$$

Proof. By (4) we have, for $h > 0$

$$(6) \quad \frac{1}{h}(G(t+h) - G(t))x = S(t)\frac{G(h)x}{h} + G(t)\frac{T(h) - I}{h}$$

Letting $h \downarrow 0$ we see that the right derivative exists and that

$$(7) \quad G'(t)x = S(t)x + G(t)Ax \quad .$$

On the other hand, the left side of (6) is also equal to

$$\frac{S(h) - I}{h}G(t)x + \frac{G(h)}{h}T(t)x$$

Thus $G(t)x \in \mathcal{B}(B)$, and we have

$$G'(t)x = BG(t)x + T(t)x \quad .$$

Combining this with (7) gives (5).

LEMMA 2. Let $x \in \mathcal{B}(A)$. Assume that $T(s)x \in \mathcal{B}(B)$ for $0 < s < t$ and

$$(8) \quad \int_0^t \|BS(t-s)T(s)x\| ds < \infty$$

Then $s \mapsto BS(t-s)T(s)x$ is Bochner integrable on $[0, t]$ and we have

$$(9) \quad BG(t)x = \int_0^t BS(t-s)T(s)x \, ds \quad .$$

Proof. Let R_λ be the resolvent $\int_0^\infty e^{-\lambda u} S(u) du$, which is defined for λ large enough, and such that $\text{Im}(R_\lambda) = \mathcal{B}(B)$, BR_λ is a bounded operator, $BR_\lambda = R_\lambda B$ on $\mathcal{B}(B)$, $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda = I$ in the strong sense.

Set $f(s) = BS(t-s)T(s)x$, $f_\lambda(s) = \lambda BR_\lambda S(t-s)T(s)x = \lambda R_\lambda f(s)$

We have $\lim_\lambda f_\lambda(s) = f(s)$ for $0 < s < t$, and $f_\lambda(\cdot)$ is continuous since BR_λ is bounded. Hence $f(\cdot)$ is strongly measurable. So (8) is equivalent to the Bochner integrability of f (see [3], p.133). Therefore

$$\begin{aligned} \int_0^t f(s) ds &= \lim_\lambda \lambda R_\lambda \int_0^t f(s) ds = \lim_\lambda \int_0^t \lambda R_\lambda f(s) ds \\ &= \lim_\lambda \int_0^t \lambda R_\lambda BS(t-s)T(s)x ds = \lim_\lambda \lambda R_\lambda BG(t)x = BG(t)x \end{aligned}$$

which proves (9).

Now we are in a position to give the proof of theorem 1.

Proof of theorem 1. Let $x \in \mathcal{B}(A)$ such that $T(s)x \in \mathcal{B}(B)$ for $0 < s < t$, so that $BS(t-s)T(s)x = S(t-s)BT(s)x$, and

$$\begin{aligned} \int_0^t \|BS(t-s)T(s)x\| ds &= \int_0^t \|S(t-s)BT(s)x\| ds \\ &\leq \int_0^t \|S(t-s)\| \|BT(s)x\| ds < \infty \text{ by hypothesis} \quad . \end{aligned}$$

Thus according to (9), we have

$$(10) \quad BG(t)x = \int_0^t BS(t-s)T(s)x \, ds = \int_0^t S(t-s)BT(s)x \, ds \quad .$$

On the other hand, we have

$$(11) \quad G(t)Ax = \int_0^t S(t-s)T(s)Ax \, ds = \int_0^t S(t-s)ATx \, ds$$

Combining (10) and (11) we get

$$BG(t)x - G(t)Ax = \int_0^t S(t-s)(B-A)T(s) \, ds$$

and we conclude using (5). $\square$

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