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Brownian Excursions From Extremes

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Let $B = (B_t, \mathcal{F}_t, P; t \geq 0)$ be standard Brownian motion starting at zero and define its extreme processes as

$$M_t = \max_{0 \leq s \leq t} B_s \quad \text{and} \quad m_t = \min_{0 \leq s \leq t} B_s.$$

The point of this note is to observe a mapping property of Brownian motion and use it to derive some results about excursions of B from its extremes which are related to the work of Groeneboom [4], Bass[1] and Pitman[9] and of Imhof[7]. It must be pointed out that these results are consequences of general excursion theory as expounded by Gettoor [2],[3] and Jacobs[8], for example. However this mapping property is new and its application to excursions is direct.

Let $r_t = M_t - m_t$ be the range process and for each $\epsilon > 0$ define the increasing processes

$$a(t, \epsilon) = \int_{\epsilon}^t 4r_s^{-2} ds$$

and

$$\tau(t, \epsilon) = \inf\{s : a(s, \epsilon) > t\}.$$

Let

$$(1) \quad X_t = \frac{2B_t - M_t - m_t}{M_t - m_t}$$

and define $X_t^{\epsilon} = X_{\tau(t, \epsilon)}$.

Proposition 1. *The process $X^{\epsilon} = (X_t^{\epsilon}, \mathcal{F}_{\tau(t, \epsilon)}, P; t \geq 0)$ is a reflecting Brownian motion on $[-1, 1]$. Its local times at ± 1 are*

$$\phi_t^{\epsilon, +} = \int_{\epsilon}^{\tau(t, \epsilon)} 4r_s^{-1} dM_s$$

and

$$\phi_t^{\epsilon, -} = \int_{\epsilon}^{\tau(t, \epsilon)} 4r_s^{-1} d(-m_s) \quad \text{respectively}$$

Proof. We may write equation (1) as $X_t = F(B_t, M_t, m_t)$ where

$$F(x, y, z) = (2x - y - z)/(y - z).$$

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Since F is smooth on $\{y \neq z\}$, we may apply Itô's formula there to obtain

$$(2) \quad dX_s = \frac{2dB_s}{M_s - m_s} + \frac{2(M_s - B_s)d(-m_s)}{(M_s - m_s)^2} - \frac{2(B_s - m_s)dM_s}{(M_s - m_s)^2}.$$

Because each $\tau(t, \epsilon)$ is an $\mathcal{F}_t$ -stopping time we may write (2) in the integrated form

$$(3) \quad X_{\tau(t, \epsilon)} = X_\epsilon + W_t^\epsilon + \frac{1}{2}\phi_t^{\epsilon, -} - \frac{1}{2}\phi_t^{\epsilon, +}$$

where

$$(4) \quad \begin{cases} W_t^\epsilon = \int_\epsilon^{\tau(t, \epsilon)} 2r_s^{-1} dB_s \\ \phi_t^{\epsilon, -} = \int_\epsilon^{\tau(t, \epsilon)} 4r_s^{-1} d(-m_s) \\ \phi_t^{\epsilon, +} = \int_\epsilon^{\tau(t, \epsilon)} 4r_s^{-1} dM_s \end{cases}$$

To finish the proof we check that W^ϵ is a standard Brownian motion and that (3) is its Skorohod equation (Tanaka[11]). Clearly W^ϵ is an $\mathcal{F}_{\tau(t, \epsilon)}$ -martingale and

$$[W^\epsilon]_t = \int_\epsilon^{\tau(t, \epsilon)} 4r_s^{-2} ds = a(\tau(t, \epsilon), \epsilon) = t.$$

By Lévy's criterion, W^ϵ is a Brownian motion, independent of X_ϵ . Now $\phi_t^{\epsilon, \pm}$ are continuous, increasing $\mathcal{F}_{\tau(t, \epsilon)}$ -adapted processes which increase only when B attains a new extremum, that is only when $X^\epsilon = \pm 1$. $\diamond$

As the next proposition shows, we may write the extreme processes in terms of the local times $\phi^{\epsilon, \pm}$. Set $\phi_t^\epsilon = \phi_t^{\epsilon, +} + \phi_t^{\epsilon, -}$.

Proposition 2. For $t \geq \epsilon$,

$$(i) \quad M_t = M_\epsilon + r(\epsilon) \int_0^{a(t, \epsilon)} \exp\{\phi_s^\epsilon/4\} d\phi_s^{\epsilon, +}$$

$$(ii) \quad m_t = m_\epsilon + r(\epsilon) \int_0^{a(t, \epsilon)} \exp\{\phi_s^\epsilon/4\} d\phi_s^{\epsilon, -}$$

where $a(t, \epsilon) = \inf\{s : \tau(s, \epsilon) > t\}$ and

$$\tau(t, \epsilon) = \epsilon + \frac{1}{4}r(\epsilon)^2 \int_0^t \exp\{\phi_s^\epsilon/2\} ds.$$

Proof. By (4) we have

$$\phi_t^\epsilon = \int_\epsilon^{\tau(t, \epsilon)} 4r_s^{-1} dr_s = 4 \log \frac{r(\tau(t, \epsilon))}{r(\epsilon)},$$

hence

$$(5) \quad r(\tau(t, \epsilon)) = r(\epsilon) \exp \{ \phi_t^\epsilon / 4 \}$$

Since $a(\tau(t, \epsilon), \epsilon) = t$ it follows that $d\tau(t, \epsilon) = r(\tau(t, \epsilon))^2 dt / 4$. Thus by (5),

$$\begin{aligned} \tau(t, \epsilon) &= \tau(0, \epsilon) + \frac{1}{4} \int_0^t r(\tau(s, \epsilon))^2 ds \\ &= \epsilon + \frac{1}{4} r(\epsilon)^2 \int_0^t \exp \{ \phi_s^\epsilon / 2 \} ds. \end{aligned}$$

It follows that τ and hence a are defined solely in terms of M_ϵ , m_ϵ and $\phi^{\epsilon, \pm}$.

Next, by (4)

$$(6) \quad \phi_t^{\epsilon, +} = \int_\epsilon^{\tau(t, \epsilon)} 4r_s^{-1} dM_s = \int_0^t 4r(\tau(s, \epsilon))^{-1} dM_{\tau(s, \epsilon)},$$

and so

$$(7a) \quad M_{\tau(t, \epsilon)} = M_\epsilon + \frac{1}{4} r_\epsilon \int_0^t \exp \{ \phi_s^\epsilon / 4 \} d\phi_s^{\epsilon, +},$$

Similarly, we have

$$(7b) \quad m_{\tau(t, \epsilon)} = m_\epsilon + \frac{1}{4} r_\epsilon \int_0^t \exp \{ \phi_s^\epsilon / 4 \} d\phi_s^{\epsilon, -},$$

and the proposition follows from a time change in (7a) and (7b). $\diamond$

These propositions allow us to compare excursions of B from its extremes with excursions of reflecting Brownian motion in $[-1, 1]$. To be precise, let

$$(8) \quad f(t, \epsilon) = \inf \{ s : \phi_s^\epsilon > t \}$$

be the inverse of boundary local time and let q^ϵ be the point process of excursions of X^ϵ . That is, let

$$D_{q^\epsilon} = \{ s : f(s, \epsilon) > f(s-, \epsilon) \}$$

and for each $s \in D_{q^\epsilon}$ let

$$(9) \quad q_s^\epsilon(u) = X^\epsilon(f(s-, \epsilon) + u \wedge l_s^\epsilon), \quad u \geq 0$$

where $l_s^\epsilon = f(s, \epsilon) - f(s-, \epsilon)$ is the duration of the excursion. Similarly, consider the point process p of excursions of B from its extremes. Let

$$(10) \quad \mu(t) = \inf \{ s : r(s) > t \},$$

let the domain of p be $D = \{ t : \mu(t) > \mu(t-) \}$ and for each $t \in D$ let

$$(11) \quad p_t(u) = B(\mu(t-) + u \wedge \lambda(t)), \quad u \geq 0$$

where $\lambda(t) = \mu(t) - \mu(t-)$. Proposition 4 provides a formula for p in terms of q^ϵ . To ensure the formula is well defined we need the

Lemma 3. Let $D^\epsilon = \{t : \mu(t) > \mu(t-) \text{ and } \mu(t) > \epsilon\}$. Then

- (i) $f(t, \epsilon) = a(\mu(r(\epsilon)e^{t/4}), \epsilon)$
- (ii) $D_{q^*} = \{s(u, \epsilon) : u \in D^\epsilon\}$ where $s(t, \epsilon) = 4 \log(tr(\epsilon)^{-1})$.

Proof. The lemma follows easily from the equality $r(f(t, \epsilon), \epsilon) = \mu(r(\epsilon)e^{t/4})$, which we now show. Let $\alpha(t)$ and $\beta(t)$ denote the left and right side of this equality, respectively. On the one hand, by (7a) and (7b), we have

$$r(\alpha(t)) = r(\epsilon) \exp\{\phi^\epsilon(f(t, \epsilon))\} = r(\epsilon)e^{t/4} \stackrel{\text{def.}}{=} g(t).$$

On the other hand, by definition $r(\beta(t)) = g(t)$. Thus $r(\alpha(t)) = r(\beta(t))$. Since $g(t)$ is strictly increasing, for any $\delta > 0$

$$\begin{aligned} \alpha(t + \delta) &\geq \mu(r(\alpha(t + \delta)) -) = \mu(g(t + \delta) -) \\ &\geq \mu(g(t)) = \mu(r(\beta(t))) = \beta(t). \end{aligned}$$

Letting $\delta \downarrow 0$ we get $\alpha(t) \geq \beta(t)$. Since the reverse inequality is similar, the lemma is proved. $\diamond$

Proposition 4. Let $\{p_t; t \in D\}$ be the point process of excursions of B from its extremes. For each $t \in D^\epsilon$

$$p_t(u) = \frac{t}{2} q_{s(t, \epsilon)}^\epsilon \left(\frac{4u}{t^2} \right) + \frac{1}{2} (M_{\mu(t)} + m_{\mu(t)})$$

Proof. First note that the statement makes sense, by Lemma 3. Let $s \in D_{q^*}$ where $s = s(t, \epsilon)$ and $t \in D^\epsilon$. The durations $l^\epsilon(s)$ of q_s^ϵ and $\lambda(t)$ of p_t are related, according to Lemma 3, by

$$\begin{aligned} (12) \quad l^\epsilon(s) &= f(s(t, \epsilon), \epsilon) - f(s(t, \epsilon) -, \epsilon) \\ &= a(\mu(t), \epsilon) - a(\mu(t) -, \epsilon) \\ &= 4 \int_{\mu(t-)}^{\mu(t)} r_u^{-2} du \\ &= \frac{\mu(t) - \mu(t-)}{r(\mu(t))^2} = 4 \frac{\lambda(t)}{t^2}. \end{aligned}$$

Thus by the formulas

$$X_u = \frac{2B_u - M_u - m_u}{M_u - m_u}, \quad X_v^\epsilon = X_{r(v, \epsilon)}$$

and the definition of q^ϵ and p we get

$$q_{s(t, \epsilon)}^\epsilon(u) = \frac{1}{t} \left(2p_t \left(\frac{t^2 u}{4} \right) - M_{\mu(t)} - m_{\mu(t)} \right),$$

from which the proposition follows. $\diamond$

An immediate corollary is the identification of the conditional law of excursions of B from its extremes. Indeed, let $-\infty < c < d < \infty$ and introduce the transition density of Brownian motion in $[c, d]$ with absorption at the endpoints (Port-Stone[10]):

$$(14) \quad p_0^{c,d}(t, x, y) = \frac{2}{d-c} \sum_{n=0}^{\infty} \sin \left(n\pi \frac{x-c}{d-c} \right) \sin \left(n\pi \frac{y-c}{d-c} \right) \exp \left\{ -\frac{n^2 \pi^2}{(d-c)^2} \frac{t}{2} \right\}$$

as well as the functions

$$(15) \quad \begin{cases} g^{c,d}(t, y; a) = \frac{1}{2} \frac{\partial}{\partial n_a} p_0^{c,d}(t, a, y), & a = c, d \\ \theta^{c,d}(t, a, b) = \frac{1}{4} \frac{\partial^2}{\partial n_a \partial n_b} p_0^{c,d}(t, a, b), & a, b = c, d. \end{cases}$$

There exist unique probability laws $P_{c,d}^{a,b;l}$ on $C([0, \infty), [c, d])$ with absolute distribution:

$$(16) \quad P_{c,d}^{a,b;l}(e(u) \in dy) = \frac{g^{c,d}(u, y; a) g^{c,d}(l-u, y; b)}{\theta^{c,d}(l, a, b)} dy, \quad 0 \leq u \leq l$$

and transition density

$$(17) \quad P_{c,d}^{a,b;l}(e(v) \in dy | e(u) = x) = p_0^{c,d}(v-u, x, dy) \frac{g^{c,d}(l-v, y; b)}{g^{c,d}(l-u, x; b)} \quad 0 \leq u < v \leq l.$$

Indeed, if $X^{c,d}$ is reflecting Brownian motion in $[c, d]$ then $P_{c,d}^{a,b;l}$ is just the law of the excursion process of $X^{c,d}$ conditioned to begin at a , end at b and have duration l . This is a simple extension of the well-known case of one reflecting barrier (e.g. Ikeda-Watanabe[6]) and also can be proved by imitating the calculations of Hsu[5]. Finally let us note a scaling property of the laws $P_{c,d}^{a,b;l}$ which follows from the invariance of the family $\{p_0^{c,d}, -\infty < c < d < \infty\}$ under affine changes of variable:

$$(18) \quad \text{If } Z = \{Z(t); 0 \leq t \leq l\} \text{ has the law } P_{c,d}^{a,b;l} \text{ then } \{\alpha Z(\alpha^{-2}t) + \beta; 0 \leq t \leq \alpha^2 l\} \text{ has the law } P_{\alpha c + \beta, \alpha d + \beta}^{\alpha a + \beta, \alpha b + \beta; \alpha^2 l}.$$

Theorem 5. Let $t \in D$. Let $-\infty < c < d < \infty$ and let $l > 0$. Then conditional on the event $\xi = [m_{\mu(t)} = c, M_{\mu(t)} = d, p_t(0) = a, p_t(\lambda(t)) = b, \lambda(t) = l]$, the law of the excursion process $p_t(\cdot)$ is $P_{c,d}^{a,b;l}$.

Proof. Fix some ϵ with $t \in D^\epsilon$ and let $s = s(t, \epsilon)$. By (9) and (12), we have

$$\xi = [q_s^\epsilon(0) = \text{sgn}(a), q_s^\epsilon(l^\epsilon(s)) = \text{sgn}(b), l^\epsilon(s) = |d - c|^2 l / 4]$$

But then conditional on ξ , the process $q_s^\epsilon(\cdot)$ has law $P_{-1,1}^{e,f;m}$ with $e = \text{sgn}(a)$, $f = \text{sgn}(b)$ and $m = |d - c|^2 l / 4$. So by Proposition 4 and the invariance property (18) we find that conditional on ξ , $p_t(\cdot)$ has the law $P_{c,d}^{a,b;l}$. $\diamond$

It is known that if X is reflecting Brownian motion in an interval then conditional on the σ -field generated by the boundary local time of X , the various excursions of X from the boundary are mutually independent. This is evident from the construction of the excursions law characterizing the excursion point process in the one reflecting barrier case (Ikeda-Watanabe[6]). Or again, one can either imitate the argument of Hsu[5] or simply quote the results in Jacobs[8]. Let us show that this conditional independence property is shared by excursions of Brownian motion B from its extremes, conditional on $\sigma\{M_s, m_s; s \geq 0\}$.

Lemma 6. Let $\mathcal{B}_\epsilon = \sigma\{\phi_s^{\epsilon,+}, \phi_s^{\epsilon,-}; s \geq 0\}$ and $\mathcal{B} = \sigma\{M_s, m_s; s \geq 0\}$. Then $\mathcal{B}_\epsilon \subset \mathcal{B}$ and $\lim_{\epsilon \rightarrow 0} \mathcal{B}_\epsilon = \mathcal{B}$.

Proof. Since Proposition 2 exhibits M and m as explicit functions of $\phi^{\epsilon,\pm}$, we have the inclusions

$$\sigma\{M_s - M_\epsilon, m_s - m_\epsilon; s \geq \epsilon\} \subset \sigma\{M_\epsilon, m_\epsilon, \phi_s^{\epsilon, \pm}; s \geq 0\} \subset \sigma\{M_s, m_s; s \geq \epsilon\}$$

and the lemma follows from this. $\diamond$

Theorem 7. *Conditional on $\mathcal{B} = \sigma\{M_s, m_s; s \geq 0\}$, the excursions $[p_t(\cdot); t \in D]$ are mutually independent.*

Proof. For $n \geq 1$ consider functionals $F : C([0, \infty), R)^n \rightarrow R$ of the form

$$F(\omega_1, \omega_2, \dots, \omega_n) = \prod_{j=1}^n f_j(\omega_j(s_{j,1}), \dots, \omega_j(s_{j,m(j)}))$$

for bounded continuous functions f_j . Let $t_1, \dots, t_n \in D$. Using Proposition 1, for all sufficiently small ϵ ,

$$E \left[F(p_{t_1}, \dots, p_{t_n}) \middle| \mathcal{B}_\epsilon \right] = \prod_{j=1}^n E \left[f_j(p_{t_j}(s_{j,1}), \dots, p_{t_j}(s_{j,m(j)})) \middle| \mathcal{B}_\epsilon \right]$$

by the conditional independence property of q^ϵ . Thus by the martingale convergence theorems and Lemma 6; taking the limit as $\epsilon \downarrow 0$ yields

$$E \left[F(p_{t_1}, \dots, p_{t_n}) \middle| \mathcal{B} \right] = \prod_{j=1}^n E \left[f_j(p_{t_j}(s_{j,1}), \dots, p_{t_j}(s_{j,m(j)})) \middle| \mathcal{B} \right]. \quad \diamond$$

We close by remarking that Theorem 5 and 7 show that Brownian motion consists of conditionally independent Brownian excursions properly interpolated between endpoints of flat stretches of the extreme process M and m .

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