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Diffusion Semigroups Corresponding to
Uniformly Elliptic Divergence Form Operators

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Chapter I: Aronson's Estimate for Elliptic Operators in Divergence Form

§I.0: INTRODUCTION

The purpose of this chapter is to study the semigroup $\{P_t : t > 0\}$ determined by a second order partial differential operator of the form $L = \nabla \cdot (a\nabla)$ where $a : \mathbb{R}^N \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^N$ is a measurable, symmetric matrix-valued function which satisfies the ellipticity condition

$$\text{(E)} \qquad \lambda I \leq a(\cdot) \leq \frac{1}{\lambda} I$$

(in the sense of non-negative definite matrices) for some $\lambda \in (0, 1]$. Until further notice, we will be assuming that a has bounded derivatives of all orders so that there is no doubt about what we mean by $\{P_t : t > 0\}$: it is then the unique Feller continuous Markov semigroup on $C_b(\mathbb{R}^N)$ with the property that

$$[P_t\phi](x) - \phi(x) = \int_0^t [P_s L\phi](x)\, ds, \quad (t, x) \in [0, \infty) \times \mathbb{R}^N$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$. Moreover, under these assumptions, it is a familiar fact that there is a function $p \in \bigcup_{n=1}^\infty C_b^\infty\left([1/n, n] \times \mathbb{R}^N \times \mathbb{R}^N; (0, \infty)\right)$ with the property that

$$\text{(I.0.1)} \qquad [P_t\phi](x) = \int_{\mathbb{R}^N} \phi(y) p(t, x, y)\, dy;$$

and therefore that

$$\frac{\partial}{\partial t} p(t, x, y) = [Lp(t, x, \cdot)](y), \quad (t, x, y) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N.$$

In addition, it is known that if $\phi \in C_0^\infty(\mathbb{R}^N)$ then $t \in [0, \infty) \longmapsto P_t\phi$ is a smooth mapping into the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ which consists of smooth functions all of whose derivatives are rapidly decreasing. In particular, one has that

$$\frac{\partial}{\partial t}[P_t\phi](x) = [LP_t\phi](x), \quad \phi \in C_0^\infty(\mathbb{R}^N);$$

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and therefore that

$$\frac{\partial}{\partial t}p(t,x,y) = [Lp(t,\cdot,y)](x), \quad (t,x,y) \in (0,\infty)\times \mathbb{R}^N \times \mathbb{R}^N$$

and

$$\text{(I.0.2)} \qquad \frac{d}{dt}\big([P_t\phi],\psi\big) = -\big([\nabla P_t\phi], a\nabla\psi\big), \quad \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \psi \in \mathcal{S}(\mathbb{R}^N),$$

where we have introduced the notation

$$(f,g) = (f,g)_{L^2(\mathbb{R}^N)} \equiv \int_{\mathbb{R}^N} f(x)g(x)\,dx$$

to denote the standard inner-product in $L^2(\mathbb{R}^N)$ (of course, when F and G take values in $\mathbb{R}^N$, (F,G) denotes $\int_{\mathbb{R}^N} F(x)\cdot G(x)\,dx$) and we have integrated $\big([LP_t\phi],\psi\big)$ by parts. As a consequence of (I.0.2), we see that for $T>0$ and $\phi,\psi \in C_0^\infty(\mathbb{R}^N)$

$$\frac{d}{dt}\big([P_{T-t}\phi],[P_t\psi]\big) = 0, \quad 0<t<T$$

and therefore that

$$\text{(I.0.3)} \qquad \big([P_t\phi],\psi\big) = \big(\phi,[P_t\psi]\big), \quad t>0.$$

So far (I.0.3) has only been checked for $\phi,\psi \in C_0^\infty(\mathbb{R}^N)$. However, it is an easy matter to pass from this statement to a more general one. In fact, we have the following.

LEMMA I.0.4. *For each $q \in [1,\infty)$ and all $t>0$,*

$$\|[P_t\phi]\|_q \le \|\phi\|_q, \quad \phi \in C_b(\mathbb{R}^N).$$

(When there is no question about the underlying measure space involved, we will use $\|\cdot\|_q$ to denote the L^q-norm.) In particular, each P_t admits a unique continuous extension $\overline{P}_t$ to $L^2(\mathbb{R}^N)$. Moreover, $\{\overline{P}_t : t>0\}$ is a strongly continuous semigroup of self-adjoint contractions on $L^2(\mathbb{R}^N)$. Finally, let $W_2^{(1)}(\mathbb{R}^N)$ denote the Sobolev space of functions in $L^2(\mathbb{R}^N)$ whose first order (generalized) derivatives are also in $L^2(\mathbb{R}^N)$. That is, $W_2^{(1)}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|\phi\|_2^{(1)} \equiv \big(\|\phi\|_2^2 + \|\nabla\phi\|_2^2\big)^{1/2}.$$

Then, for each $t>0$, $\overline{P}_t$ maps $L^2(\mathbb{R}^N)$ into $W_2^{(1)}(\mathbb{R}^N)$

$$\text{(I.0.5)} \qquad \|[\nabla\overline{P}_t\phi]\|_2 \le \frac{1}{\lambda^{1/2}}\left(\frac{1}{t^{1/2}}\|\phi\|_2\right) \wedge \big(\|\nabla\phi\|_2\big), \quad t>0 \text{ and } \phi \in W_2^{(1)}(\mathbb{R}^N),$$

and

$$\text{(I.0.6)} \qquad \frac{d}{dt}\big([\overline{P}_t\phi],\psi\big) = -\big([\nabla \overline{P}_t\phi], a\nabla\psi\big), \quad t>0 \text{ and } \phi,\psi \in W_2^{(1)}(\mathbb{R}^N).$$

PROOF: To prove the first assertion, let $\phi \in C_0^\infty(\mathbb{R}^N)^+$ and choose a non-decreasing sequence $\{\psi_n\} \subseteq C_0^\infty(\mathbb{R}^N)^+$ which tends point-wise to 1 as $n \longrightarrow \infty$. Then, by (I.0.3) and the monotone convergence theorem, we see that

$$\text{(I.0.7)} \qquad \int_{\mathbb{R}^N} P_t\phi(x)\,dx = \int_{\mathbb{R}^N} \phi(x)\,dx.$$

Given (I.0.7) for all $\phi \in C_0^\infty(\mathbb{R}^N)^+$, it is an easy matter to get the same equality for all non-negative measurable ϕ's. Moreover, since $[P_t\phi](x)$ is computed, for each $x \in \mathbb{R}^N$, by integration with respect to a probability measure, we see that

$$[P_t|\phi|](x)^q \le [P_t|\phi|^q](x), \quad t>0 \text{ and } q\in[1,\infty),$$

for all measurable ϕ's. Hence, the first assertion is now proved.

In view of the preceding and (I.0.3), the existence and self-adjointness of $\overline{P}_t$ are clear; and the continuity of $\{\overline{P}_t : t>0\}$ is an easy consequence of the properties of $\{P_t : t>0\}$ mentioned above. To see that $\overline{P}_t$ maps $L^2(\mathbb{R}^N)$ into $W_2^{(1)}(\mathbb{R}^N)$ and the inequality in (I.0.5), let $\{E_\mu : \mu \in [0,\infty)\}$ be the spectral resolution of the identity in $L^2(\mathbb{R}^N)$ for which

$$\text{(I.0.8)} \qquad \overline{P}_t = \int_{[0,\infty)} e^{-\mu t}\,dE_\mu.$$

It is then immediate from (I.0.2) that

$$\text{(I.0.9)} \qquad \big([\nabla\overline{P}_t\phi], a\nabla\psi\big) = \int_{[0,\infty)} \mu e^{-\mu t}\,d\big(E_\mu\phi,\psi\big) = \big(\nabla[\overline{P}_{t/2}\phi], a[\nabla\overline{P}_{t/2}\psi]\big)$$

for all $\phi,\psi \in C_0^\infty(\mathbb{R}^N)$. Taking $\phi=\psi$ in (I.0.9) and using the ellipticity of a, we conclude that (I.0.5) holds first for elements of $C_0^\infty(\mathbb{R}^N)$ and then for all elements of $W_2^{(1)}(\mathbb{R}^N)$.

Finally, the proof of (I.0.6) is now just an easy matter of passing to limits in (I.0.2). ∎

It should be noted that although we required a to be smooth in order to know that $\{P_t : t>0\}$ exists and maps $C_0^\infty(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^N)$, all the conclusions drawn in Lemma I.0.4 make sense even when a is not smooth. In fact, none of them relies directly on any properties of a other than ellipticity. What we will show in the succeeding sections is that there are several other properties of $\{P_t : t>0\}$ which depend only on the ellipticity of a. In particular, we are going to show that there is an $M = M(\lambda, N) \in [1,\infty)$ such that

$$\text{(I.0.10)} \qquad \frac{1}{Mt^{N/2}}\exp\big(-M|y-x|^2/t\big) \le p(t,x,y) \le \frac{M}{t^{N/2}}\exp\big(-|y-x|^2/Mt\big)$$

for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$.

The estimate in (I.0.10) is *Aronson's estimate* (cf. [**A**]). It constitutes a beautiful summary of the results contained in the sequence of articles starting with those of E. DEGEORGI [**DG**] and J. NASH [**N**] and culminating in the article [**M**] by J. MOSER. Section I.1 contains a derivation of the upper bound in (I.0.10), and the lower bound is derived in Section I.2. In the first section of Chapter II, we will use (I.0.10) to recover the estimates of NASH, DEGIORGI, and MOSER. The proofs given in all of these sections are adopted from the article [**F.-S.**]. In Section I.4, we will discuss some extensions of these results to operators which are lower order perturbations of L. Finally, in Section I.5 we will apply our estimates to the construction of diffusion semigroups corresponding to elliptic operators in which the coefficients are merely bounded and measurable.

§I.1: THE UPPER BOUND

Our derivation of the upper bound turns on the following basic analytic fact about $\mathbb{R}^N$.

LEMMA I.1.1. (NASH'S INEQUALITY) *There is a* $C_N \in (0,\infty)$ *such that*

$$\|\phi\|_2^{2+4/N} \le C_N \|\nabla\phi\|_2^2 \|\phi\|_1^{4/N}, \quad \phi \in L^1(\mathbb{R}^N) \cap W_2^{(1)}(\mathbb{R}^N).$$

PROOF: Clearly it suffices to prove the result for $\phi \in C_0^\infty(\mathbb{R}^N)$. In order to emphasize just how basic it is, we will give three derivations.

i) (FOURIER) For every $r > 0$:

$$(2\pi)^N \|\phi\|_2^2 = \int_{B(0,r)} |\hat{\phi}(\xi)|^2 \, d\xi + \int_{B(0,r)^c} |\hat{\phi}(\xi)|^2 \, d\xi \le \Omega_N r^N \|\phi\|_1^2 + \frac{(2\pi)^N}{r^2} \|\nabla\phi\|_2^2,$$

where $B(x,r) \equiv \{y \in \mathbb{R}^N : |y-x| < r\}$ and $\Omega_N \equiv |B(0,1)|$, and we use $|\Gamma|$ to denote the Lebesgue measure of a measurable set $\Gamma \subseteq \mathbb{R}^N$. The desired inequality results from the preceding by choosing the r which minimizes the right hand side.

ii) (HEAT FLOW) For $(t,x) \in (0,\infty) \times \mathbb{R}^N$ define the *heat kernel*

$$\gamma_t(x) = (4\pi t)^{-N/2} \exp(-|x|^2/4t). \tag{I.1.2}$$

Then

$$\gamma_T * \phi = \phi + \int_0^T \Delta\gamma_t * \phi \, dt, \quad T > 0,$$

where "$*$" is used to denote convolution. Note that

$$|(\gamma_T * \phi, \phi)| \le \|\gamma_T * \phi\|_\infty \|\phi\|_1 \le (4\pi T)^{-N/2} \|\phi\|_1^2,$$

and that, by (I.0.5),

$$|(\phi, \Delta(\gamma_t * \phi))| = |(\nabla\phi, \nabla(\gamma_t * \phi))| \le \|\nabla\phi\|_2^2.$$

Combining these with the preceding, we conclude that

$$\|\phi\|_2^2 \le (4\pi T)^{-N/2} \|\phi\|_1^2 + T\|\nabla\phi\|_2^2, \quad T > 0.$$

Again the desired inequality follows upon optimizing the right hand side.

iii) (SOBOLEV) At least when $N \geq 3$, one has the *critical Sobolev inequality*

(I.1.3) $$\|\phi\|_p \leq c_N \|\nabla\phi\|_2 \text{ with } 1/p = 1/2 - 1/N,$$

for some $c_N < \infty$. To get NASH's inequality from this, define $\theta \in (0,1)$ by $1/2 = \theta/p + (1-\theta)$, use Hölder's inequality:

$$\|\phi\|_2 \leq \|\phi\|_p^\theta \|\phi\|_1^{1-\theta},$$

and use the estimate for $\|\phi\|_p$ which comes from Sobolev's inequality. ∎

As a demonstration of the power and relevance of NASH's inequality to upper bound in (I.0.10), we begin by showing how NASH himself used it. Namely, let $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi\|_1 = 1$ be given, and set $\phi_t = P_t\phi$. Then, by Lemma I.0.4, $\|\phi_t\|_1 \leq 1$; and so

$$\frac{d}{dt}\|\phi_t\|_2^2 = -2(\nabla\phi_t, a\nabla\phi_t) \leq -2\lambda\|\nabla\phi_t\|_2^2 \leq -\frac{2\lambda}{C_N}\|\phi_t\|_2^{2+4/N}$$

for all $t \geq 0$. After integrating this elementary differential inequality, one finds that $\|\phi_t\|_2 \leq \left(\frac{NC_N}{4\lambda t}\right)^{N/4}$. In other words, the norm $\|P_t\|_{1\to 2}$ of P_t as a bounded linear map from $L^1(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ is dominated by a constant $K = K(\lambda, N) \in (0,\infty)$ times $t^{-N/4}$. Since P_t is symmetric in $L^2(\mathbb{R}^N)$, an easy duality argument shows that the norm $\|P_t\|_{2\to\infty}$ is equal to $\|P_t\|_{1\to 2}$. Hence, since $P_{2t} = P_t \circ P_t$, we conclude that

$$\|P_{2t}\|_{1\to\infty} \leq \|P_t\|_{2\to\infty}\|P_t\|_{1\to 2} \leq \frac{K^2}{t^{N/2}};$$

and it is an easy step from this to the estimate

(I.1.4) $$p(t,x,y) \leq \frac{2^{N/2}K^2}{t^{N/2}}, \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N.$$

Actually, (I.1.4) should be viewed as the right hand side of (I.0.10) on the diagonal in $\mathbb{R}^N \times \mathbb{R}^N$. Indeed, the symmetry of P_t tells us that

(I.1.5) $$p(t,x,y) = p(t,y,x), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N,$$

and so, by the Chapman-Kolmogorov equation,

$$\begin{aligned} p(t,x,y) &= \int_{\mathbb{R}^N} p(t/2,x,\xi)p(t/2,\xi,y)\,d\xi \\ &\leq \|p(t/2,x,\cdot)\|_2\|p(t/2,y.\cdot)\|_2 = p(t,x,x)^{1/2}p(t,y,y)^{1/2}. \end{aligned}$$

In order to get the off-diagonal upper bound in (I.0.10), we will use a trick which was introduced in this context by E.B. DAVIES [D]. Namely, let $\psi \in C_b^\infty(\mathbb{R}^N)$ and define

(I.1.6)
$$\begin{aligned} &p^\psi(t,x,y) = \exp(-\psi(x))p(t,x,y)\exp(\psi(y)), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \\ &[P_t^\psi\phi](x) = \int_{\mathbb{R}^N} \phi(y)p^\psi(t,x,y)\,dy, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \text{ and } \phi \in C_b(\mathbb{R}^N), \\ &[L^\psi\phi](x) = \exp(-\psi(x))\left[L(\exp(\psi)\phi)\right](x), \quad x \in \mathbb{R}^N \text{ and } \phi \in C^2(\mathbb{R}^N). \end{aligned}$$

It is easy to check that $\{P_t^\psi : t > 0\}$ is a non-negativity preserving semigroup on $C_b(\mathbb{R}^N)$ and that, for $\phi \in C_0^\infty(\mathbb{R}^N)$, $t \longmapsto [P^\psi \phi]$ is a smooth mapping into $\mathcal{S}(\mathbb{R}^N)$ with

$$\text{(I.1.7)} \qquad \frac{d}{dt}[P_t^\psi \phi](x) = [L^\psi P_t^\psi \phi](x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^N.$$

What we would like to do is apply NASH's idea to get an estimate this time on $p^\psi(t,x,y)$. However, there is now a problem which we did not face before. Namely, it is no longer true that P_t^ψ is a contraction on the Lebesque spaces. Thus we will have to be a little more clever.

LEMMA I.1.8. *For each $q \in [1,\infty)$ and all $\phi \in C_0^\infty(\mathbb{R}^N)^+$,*

$$\text{(I.1.9)} \qquad \frac{d}{dt}\|\phi_t\|_{2q} \le -\frac{\lambda}{q\|\phi_t\|_{2q}^{2q-1}}\|\nabla\phi_t^q\|_2^2 + q\Gamma(\psi)^2\|\phi_t\|_{2q},$$

where $\phi_t = [P_t^\psi \phi]$ and

$$\text{(I.1.10)} \qquad \Gamma(\psi) \equiv \sup_{x\in\mathbb{R}^N}\left(\nabla\psi(x)\cdot a(x)\nabla\psi(x)\right)^{1/2}.$$

In particular,

$$\text{(I.1.11)} \qquad \|P_t^\psi\|_{2\to 2} \le \exp(t\Gamma(\psi)^2).$$

PROOF: From (I.1.7) we see that

$$\frac{d}{dt}\|\phi_t\|_{2q} = \|\phi_t\|_{2q}^{1-2q}(\phi_t^{2q-1}, L^\psi\phi_t).$$

Integrating by parts, we obtain

$$\begin{aligned}(\phi_t^{2q-1}, L^\psi\phi_t) &= -\left(\nabla(e^{-\psi}\phi_t^{2q-1}), a\nabla(e^\psi\phi_t)\right)\\ &= -(2q-1)\left(\phi_t^{2q-2}\nabla\phi_t, a\nabla\phi_t\right) - 2(q-1)\left(\phi_t^{2q-1}\nabla\phi_t, a\nabla\psi\right) + \left(\phi_t^{2q}\nabla\psi, a\nabla\psi\right)\\ &\le -q\left(\phi_t^{2q-2}\nabla\phi_t, a\nabla\phi_t\right) + q\left(\phi_t^{2q}\nabla\psi, a\nabla\psi\right) \le -\frac{\lambda}{q}\|\nabla\phi_t^q\|_2^2 + q\Gamma(\psi)^2\|\phi_t\|_{2q}^{2q}.\end{aligned}$$

Together with the preceding, this now yields (I.1.10). ∎

Continuing with the notation $\phi_t = P_t^\psi\phi$, set

$$u_q(t) = \|\phi_t\|_{2q}, \quad t \ge 0 \text{ and } q \in [1,\infty),$$

and

$$w_q(t) = \sup_{0\le s\le t} s^{N(q-2)/4q}u_{q/2}(s), \quad t \ge 0 \text{ and } q \in [2,\infty).$$

By the preceding,

$$w_2(t) \leq \exp\bigl(t\Gamma(\psi)^2\bigr)\|\phi\|_2, \quad t \geq 0. \tag{I.1.12}$$

Morover, by (I.1.9) and NASH'S inequality, for $q \in [2,\infty)$,

$$\begin{aligned} u_q'(t) &\leq -\frac{\lambda}{q u_q(t)^{2q-1}}\bigl\|\nabla\phi_t^q\bigr\|_2^2 + q\Gamma(\psi)^2 u_q(t) \\ &\leq -\frac{\lambda}{C_N q}\frac{\bigl\|\phi_t^q\bigr\|_2^{2+4/N}}{u_q(t)^{2q-1}\bigl\|\phi_t^q\bigr\|_1^{4/N}} + q\Gamma(\psi)^2 u_q(t) \\ &= -\frac{\lambda}{C_N q}\frac{u_q(t)^{1+4q/N}}{u_{q/2}(t)^{4/N}} + q\Gamma(\psi)^2 u_q(t). \end{aligned}$$

Hence,

$$u_q'(t) \leq -\frac{\lambda}{C_N q}\frac{t^{(q-2)}u_q(t)^{1+4q/N}}{w_q(t)^{4/N}} + q\Gamma(\psi)^2 u_q(t), \quad t \geq 0 \text{ and } q \in [2,\infty). \tag{I.1.13}$$

LEMMA I.1.14. *Let α, β, ϵ be positive numbers and $q \in [2,\infty)$. Let $w : [0,\infty) \longrightarrow (0,\infty)$ be a non-decreasing continuous function. If $u \in C^1([0,\infty);(0,\infty))$ satisfies*

$$u'(t) \leq -\frac{\epsilon}{q}\frac{t^{(q-2)}u(t)^{1+\beta q}}{w(t)^{\beta q}} + q\alpha^2 u(t), \quad t \geq 0,$$

then there exists a $K = K(\epsilon,\beta) \in (0,\infty)$ such that

$$t^{(1-1/q)\beta}u(t) \leq \left(\frac{Kq^2}{\delta}\right)^{1/\beta q}\exp\bigl(\delta\alpha^2 t/q\bigr)w(t), \quad t \geq 0$$

for every $\delta \in (0,1]$.

PROOF: Clearly

$$\left(e^{-q\alpha^2 t}u(t)\right)' \leq -\frac{\epsilon}{q}\frac{t^{(q-2)}e^{\beta q^2\alpha^2 t}}{w(t)^{\beta q}}\left(e^{-q\alpha^2 t}u(t)\right)^{1+\beta q}$$

Thus,

$$\left(\frac{e^{q\alpha^2 t}}{u(t)}\right)^{\beta q} \geq \epsilon\beta\int_0^t \frac{s^{(q-2)}e^{\beta q^2\alpha^2 s}}{w(s)^{\beta q}}\,ds$$

$$\geq \frac{\epsilon\beta}{w(t)^{\beta q}}\int_{(1-\delta/q^2)t}^t s^{(q-2)}e^{\beta q^2\alpha^2 s}\,ds \geq \frac{\epsilon e^{\beta q^2\alpha^2 t}e^{-\delta\beta\alpha^2 t}t^{(q-1)}\bigl(1-(1-\delta/q^2)^{q-1}\bigr)}{(q-1)w(t)^{\beta q}}$$

Noting that $\frac{1-(1-\delta/q^2)^{q-1}}{q-1} \geq \delta/4q^2$ for $q \in [2,\infty)$, one can easily deduce the required estimate from this. ∎

Returning to the notation introduced just prior to Lemma I.1.14 and taking $\alpha = \Gamma(\psi)$, $\beta = 4/N$, and $\epsilon = \frac{\lambda}{C_N q}$ in that lemma, we now see from (I.1.13) that there is a $K = K(\lambda, N) \in (0,\infty)$ such that

$$w_{2q}(t) \leq \left(\frac{Kq^2}{\delta}\right)^{N/4q} e^{\delta\Gamma(\psi)^2 t/q} w_q(t), \quad q \in [2,\infty) \text{ and } t \geq 0.$$

Combining this with (I.1.12), we find that there is a $K' = K'(\lambda, N) \in (0,\infty)$ such that

$$w_{2^n}(t) \leq \frac{K'}{\delta^{N/4}} e^{(1+\delta)\Gamma(\psi)^2 t} \|\phi\|_2;$$

and from this it is clear that

$$\|\phi_t\|_{C_b(\mathbb{R}^N)} \leq \varliminf_{n\to\infty} \|\phi_t\|_{2^n} \leq K'(\delta t)^{-N/4} e^{(1+\delta)\Gamma(\psi)^2 t} \|\phi\|_2.$$

In other words, we now know that

$$\left\|P_t^\psi\right\|_{2\to\infty} \leq \frac{K'}{(\delta t)^{N/4}} e^{(1+\delta)\Gamma(\psi)^2 t}, \quad t > 0.$$

But the formal adjoint of P_t^ψ is just $P_t^{-\psi}$, which clearly satisfies the same estimate. Hence, by duality, we have the same bound on $\left\|P_t^\psi\right\|_{1\to 2}$. Finally, since $P_t^\psi = P_{t/2} \circ P_{t/2}^\psi$, we have now proved that

$$\left\|P_t^\psi\right\|_{1\to\infty} \leq \frac{2^{N/2}K'}{(\delta t)^{N/2}} e^{(1+\delta)\Gamma(\psi)^2 t}, \quad t > 0.$$

THEOREM I.1.15. *There is a $K = K(\lambda, N) \in (0,\infty)$ such that for every $\delta \in (0,1]$ and every $\psi \in C^1(\mathbb{R}^N)$ with $\Gamma(\psi) < \infty$,*

$$p(t,x,y) \leq \frac{K}{(\delta t)^{N/2}} \exp\big(\psi(x) - \psi(y) + (1+\delta)\Gamma(\psi)^2 t\big) \tag{I.1.16}$$

for $(t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$.

PROOF: From the preceding we see that (I.1.16) holds for all $\psi \in C_b^\infty(\mathbb{R}^N)$. It is therefore clear, by elementary approximation, that (I.1.16) holds for all $\psi \in C_b^1(\mathbb{R}^N)$. Finally, if $x, y \in \mathbb{R}^N$ are given, choose $r \geq 1$ so that $|x_i| \vee |y_i| < r$, $1 \leq i \leq N$, choose $\eta \in C_0^\infty((-2r, 2r))$ so that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $[-r, r]$, and define $H : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by

$$H(x)_i = \int_0^{x_i} \eta(t)\, dt, \quad 1 \leq i \leq N \text{ and } x \in \mathbb{R}^N.$$

Then for any $\psi \in C^1(\mathbb{R}^N)$, $\psi \circ H \in C_b^1(\mathbb{R}^N)$, $\Gamma(\psi \circ H) \leq \Gamma(\psi)$, and $\psi \circ H$ equals ψ at x and y. ∎

The upper bound in (I.0.10) is essentially trivial once one has (I.1.16). Indeed, take $\delta = 1$ and consider ψ's of the form $\psi(x) = \theta \cdot x$, where $\theta \in \mathbb{R}^N$. One then finds that

$$p(t,x,y) \leq \frac{K}{t^{N/2}} \exp\big(\theta \cdot (x-y) + 2|\theta|^2 t/\lambda\big).$$

In particular, by taking $\theta = \frac{\lambda(y-x)}{4t}$, one obtains

$$p(t,x,y) \leq \frac{K}{t^{N/2}} \exp(-\lambda|y-x|^2/8t), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N. \tag{I.1.17}$$

However, it is clear that we should be able to get much more out of (I.1.16). To this end, define

$$D_a(x,y) = \inf\{\psi(y) - \psi(x) : \psi \in C^1(\mathbb{R}^N) \text{ and } \Gamma(\psi) \leq 1\}. \tag{I.1.18}$$

It is then an easy matter to see that

$$\sup\{\psi(y) - \psi(x) - (1+\delta)t\Gamma(\psi)^2 : \psi \in C^1(\mathbb{R}^N)\} = \frac{D_a(x,y)^2}{4(1+\delta)t}.$$

Therefore, the sharp conclusion which can be dran from (I.1.16) is that

$$p(t,x,y) \leq \frac{K}{(\delta t)^{N/2}} \exp\left(\frac{D_a(x,y)^2}{4(1+\delta)t}\right). \tag{I.1.19}$$

Because of (I.1.19), it becomes interesting to identify the metric $D_a(x,y)$ with some more familiar metric. In particular, one suspects that $D_a(x,y)$ is related to the *Riemannian distance* $d_a(x,y)$ between x and y computed relative to the Riemannian metric a^{-1}. To confirm this suspicion, we describe $d_a(x,y)$ as follows. Let

$$\mathbf{H} = \{h \in C([0,\infty);\mathbb{R}^N) : h(0) = 0 \text{ and } \dot{h} \in L^2([0,\infty);\mathbb{R}^N)\}, \tag{I.1.20}$$

where $\dot{h}$ denotes the (generalized) derivative of the function h. It is then clear that $\mathbf{H}$ becomes a separable Hilbert space when we take

$$\|h\|_{\mathbf{H}} \equiv \|\dot{h}\|_{L^2([0,\infty);\mathbb{R}^N)}. \tag{I.1.21}$$

For $h \in \mathbf{H}$, define $(t,x) \in [0,\infty) \times \mathbb{R}^N \longmapsto \Phi(t,x;h) \in \mathbb{R}^N$ so that

$$\dot{\Phi}(t,x;h) \equiv \frac{d}{dt}\Phi(t,x;h) = a(\Phi(t,x;h))^{1/2}\dot{h}(t) \text{ and } \Phi(0,x;h) = x. \tag{I.1.22}$$

One can then quite easily show that for any $t > 0$

$$d_a(x,y) = t^{1/2} \inf\{\|h\|_{\mathbf{H}} : h \in \mathbf{H} \text{ and } \Phi(t,x;h) = y\}. \tag{I.1.23}$$

LEMMA I.1.24. *For all* $x, y \in \mathbb{R}^N$, $d_a(x,y) = D_a(x,y)$.

PROOF: Suppose that $h \in \mathbf{H}$ and that $\Phi(1,x;h) = y$. If $\psi \in C^1(\mathbb{R}^N)$ satisfies $\Gamma(\psi) \leq 1$, then

$$\psi(y) - \psi(x) = \psi(\Phi(1,x;h)) - \psi(\Phi(0,x;h)) = \int_0^1 \nabla\psi(\Phi(t,x;h)) \cdot \dot{\Phi}(t,x;h)\,dt$$
$$= \int_0^1 (a^{1/2}\nabla\psi) \circ \Phi(t,x;h) \cdot \dot{h}(t)\,dt \leq \Gamma(\psi)\|h\|_{\mathbf{H}};$$

and therefore $D_a(x,y) \leq d_a(x,y)$.

To prove the opposite inequality, it is convenient to first extend the class of ψ's entering the definition of $D_a(x,y)$. Namely, for $\theta \in S^{N-1}$, set $h_\theta(t) = t\theta$, $t \geq 0$. Next, define Ψ to be the class of $\psi \in C(\mathbb{R}^N)$ with the property that $|\psi(\Phi(t,x;h_\theta)) - \psi(x)| \leq t$ for all $(t,x,\theta) \in (0,1] \times \mathbb{R}^N \times S^{N-1}$. It is easy to check that if $\psi \in C^1(\mathbb{R}^N)$ then $\psi \in \Psi$ if and only if $\Gamma(\psi) \leq 1$. In addition, by a standard approximation procedure, one can show that if $\psi \in \Psi$ then there exist $\{\psi_n\} \subseteq C^1(\mathbb{R}^N)$ such that $\psi_n \longrightarrow \psi$ uniformly on compacts and $\overline{\lim}_{n\to\infty} \Gamma(\psi_n) \leq 1$. With these remarks, it becomes clear that

$$D_a(x,y) = \sup\{\psi(y) - \psi(x) : \psi \in \Psi\}.$$

In particular, since, by I.1.23, for each $x \in \mathbb{R}^N$, $d_a(x,\cdot) \in \Psi$, we now see that $d_a(x,y) \leq D_a(x,y)$. ∎

In view of the preceding lemma and (I.1.19), we have now proved that

$$\text{(I.1.25)} \qquad p(t,x,y) \leq \frac{K}{(\delta t)^{N/2}} \exp\left(-\frac{d_a(x,y)^2}{4(1+\delta)t}\right), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N,$$

for every $\delta \in (0,1]$.

§I.2: THE LOWER BOUND

In this section we will prove the lower bound in (I.0.10). Once again, the proof turns on an idea in [N]. The main step is to show that there is an $A = A(\lambda, N) \in (0,\infty)$ such that

$$\text{(I.2.1)} \qquad \int_{\mathbb{R}^N} e^{-\pi|y|^2} \log\big(p(1,x,y)\big)\,dy \geq -A, \quad x \in B(0,2).$$

In order to show that such an A exists, we will require the following elementary fact, which will certainly be familiar to anyone who has studied the Hermite operator.

LEMMA I.2.2. *For any* $\phi \in C_b^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} e^{-\pi|y|^2} \big(\phi(y) - \langle \phi \rangle\big)^2\,dy \leq 2\pi \int_{\mathbb{R}^N} e^{-\pi|y|^2} |\nabla\phi(y)|^2\,dy,$$

where $<\phi>\equiv\int_{\mathbb{R}^N} e^{-\pi|y|^2}\phi(y)\,dy$.

PROOF: After a change of variable, it becomes clear that the asserted inequality is equivalent to the statement that

$$\text{(I.2.3)}\qquad \int_{\mathbb{R}^N}\phi(y)^2\,\gamma(dy)\le\int_{\mathbb{R}^N}|\nabla\phi(y)|^2\,\gamma(dy)$$

for all $\phi\in C_b^1(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N}\phi(y)\,\gamma(dy)=0$, where $\gamma(dy)=\gamma_{1/2}(y)\,dy$ and $\gamma_t(\cdot)$ is the heat kernel given in (I.1.2). Further, by an approximation argument, it is obviously enough to prove this when $\phi\in C_0^\infty(\mathbb{R}^N)$.

Let $\phi\in C_0^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\phi(y)\,\gamma(dy)=0$ be given, and set

$$\phi_t(x)=[\Gamma_t\phi](x)\equiv\int_{\mathbb{R}^N}\phi(y)\gamma_{\frac{1-e^{-2t}}{2}}(y-e^{-t}x)\,dy.$$

Clearly $\phi_t\longrightarrow 0$ ($\phi_t\longrightarrow\phi$) boundedly and point-wise as $t\longrightarrow\infty$ ($t\longrightarrow 0$). Moreover,

$$\frac{\partial}{\partial t}\phi_t(x)=(\Delta-x\cdot\nabla)\phi_t(x);$$

and so, after integration by parts, we see that

$$\frac{d}{dt}\int_{\mathbb{R}^N}\phi_t(y)^2\,\gamma(dy)=-2\int_{\mathbb{R}^N}|\nabla\phi_t|^2\,\gamma(dy).$$

In particular, this shows that $\|\Gamma_t\psi\|_{L^2(\gamma)}$ is a non-increasing function of $t\ge 0$ for any $\psi\in C_b(\mathbb{R}^N)$. Next, note that $\nabla\phi_t=e^{-t}\big[\Gamma_t(\nabla\phi)\big]$. Hence, we now see that

$$\int_{\mathbb{R}^N}\phi(y)^2\,\gamma(dy)\le 2\int_0^\infty e^{-2t}\left(\int_{\mathbb{R}^N}[\Gamma_t(|\nabla\phi|^2)](y)\,\gamma(dy)\right)dt\le\int_{\mathbb{R}^N}|\nabla\phi(y)|^2\,\gamma(dy),$$

which is precisely (I.2.3). ∎

We now turn to the proof of (I.2.1).

LEMMA I.2.4. *There is an* $A=A(\lambda,N)\in(0,\infty)$ *such that* (I.2.1) *holds for all* $x\in B(0,2)$.

PROOF: Let $x\in B(0,2)$ and $\theta\in[1/2,1)$ be given, and set $u(t,y)=\theta p(t,x,y)+(1-\theta)$ and

$$G(t)=\int_{\mathbb{R}^N}e^{-\pi|y|^2}\log(u(t,y))\,dy.$$

In order to prove the required estemate, it suffices to estimate $G(1)$ from below by a quantity which is independent of x and θ. (It may be helpful to keep in mind that, by Jensen's inequality, $G(1)\le 0$.)

Now, using integration by parts and Lemma I.2.2, we see that:

$$
\begin{aligned}
G'(t) &= -\int_{\mathbb{R}^N} \nabla\left(\frac{e^{-\pi|y|^2}}{u(t,y)}\right)\cdot a(y)\nabla u(t,y)\,dy \\
&= 2\pi\int_{\mathbb{R}^N} e^{-\pi|y|^2} y\cdot a(y)\nabla\big(\log(u(t,y))\big)\,dy \\
&\qquad + \int_{\mathbb{R}^N} e^{-\pi|y|^2}\nabla\big(\log(u(t,y))\big)\cdot a(y)\nabla\big(\log(u(t,y))\big)\,dy \\
&\geq -2\pi^2\int_{\mathbb{R}^N} e^{-\pi|y|^2} y\cdot a(y)y\,dy \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^N} e^{-\pi|y|^2}\nabla\big(\log(u(t,y))\big)\cdot a(y)\nabla\big(\log(u(t,y))\big)\,dy \\
&\geq -\frac{\pi}{\lambda} + \lambda\pi\int_{\mathbb{R}^N} e^{-\pi|y|^2}\big(\log(u(t,y) - G(t)\big)^2\,dy.
\end{aligned}
$$

Also, for any $r > 0$ and $K > 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} e^{-\pi|y|^2}\big(\log(u(t,y)) - G(t)\big)^2\,dy \\
&\qquad\qquad \geq e^{-\pi r^2}|\{y\in B(0,r) : u(t,y)\geq e^{-K}\}|\big(|G(t)| - K\big)^2.
\end{aligned}
$$

Choose $r > 0$ so that $\int_{B(0,r)^c} p(t,x,y)\,dy \leq \frac{1}{4}$ for $t\in(0,1]$. By our upper bound on $p(t,x,y)$, this can not only be done, but can even be done with an r which is independent of $x\in B(0,2)$. Next, again using the upper bound, choose $M\in[1,\infty)$ so that $p(t,x,y)\leq M$ for all $(t,x,y)\in[1/2,1]\times\mathbb{R}^N\times\mathbb{R}^N$. One then has that for $t\in[1/2,1]$ (and any $x\in B(0,2)$ and $\theta\in[1/2,1)$):

$$
\frac{3}{4}\leq\int_{B(0,r)} u(t,y)\,dy \leq \Omega_N r^N e^{-K} + M|\{y\in B(0,r) : u(t,y)\geq e^{-K}\}|.
$$

Hence we can choose $K\in(0,\infty)$ so that

$$
|\{y\in B(0,r) : u(t,y)\geq e^{-K}\}| \geq \frac{1}{8M}
$$

for all $t\in[1/2,1]$ (and any $\theta\in[1/2,1)$ and $x\in B(0,2)$.) Together with the preceding, this implies that

$$
G'(t)\geq -B^2 + \epsilon^2 G(t)^2, \quad t\in[1/2,1] \tag{I.2.5}
$$

for some $B\in(0,\infty)$ and $\epsilon\in(0,1]$.

Now suppose that $G(1)\leq -Q$ where $Q\equiv -B^2 - 2B/\epsilon$. Then, since, by (I.2.5), $G(1) - G(t)\geq -B^2/2$ for $t\in[1/2,1]$, we would that $G(t)\leq -2B/\epsilon$, $t\in[1/2,1]$; and therefore, again by (I.2.5), that

$$
G'(t)\geq\frac{3\epsilon^2}{4}G(t)^2, \quad t\in[1/2,1].
$$

But this means that $\frac{1}{G(1)} \leq -\frac{3\epsilon^2}{8}$; or, equivalently, that $G(1) \geq -\frac{2}{3\epsilon^2}$. Hence, either $G(1) \geq -Q$ or $G(1) \geq -\frac{8}{3\epsilon^2}$; and so we can take $A = Q \vee \frac{8}{3\epsilon^2}$. ∎

Although, at first sight, (I.2.1) appears to be only a small step toward our goal, it turns out to be the crucial one. Indeed, by the Chapman-Kolmogorov equation, the symmetry of $p(t,x,y)$, and Jensen's inequality, we see that it leads immediatley to

$$\begin{aligned}\log(p(2,x,y)) &= \log\left(\int_{\mathbb{R}^N} p(1,x,\xi)p(1,y,\xi)\,d\xi\right)\\ &\geq \log\left(\int_{\mathbb{R}^N} e^{-\pi|\xi|^2}p(1,x,\xi)p(1,y,\xi)\,d\xi\right)\\ \geq \int_{\mathbb{R}^N} e^{-\pi|\xi|^2}\log(p(1,x,\xi))\,d\xi &+ \int_{\mathbb{R}^N} e^{-\pi|\xi|^2}\log(p(1,y,\xi))\,d\xi \geq -2A\end{aligned}$$

for all $x, y \in B(0,2)$. In other words, we now know that

$$p(2,x,y) \geq e^{-2A}, \quad x,y \in B(0,2). \tag{I.2.6}$$

The next step is to take advantage of the scaling and translation invariance of the hypotheses under which (I.2.6) has been proved.

LEMMA I.2.7. *Using the notation $p^a(t,x,y)$ to emphasize the coefficeint matrix, one has that for all $r > 0$ and $\xi \in \mathbb{R}^N$:*

$$p^a(r^2t, rx+\xi, ry+\xi) = r^{-N}p^{a_{r,\xi}}(t,x,y),$$

where $a_{r,\xi}(\cdot) \equiv a(r\cdot+\xi)$.

PROOF: Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be given and set

$$u(t,x) = \int_{\mathbb{R}^N} \phi((y-\xi)/r)p(t,x,y)\,dy \text{ and } w(t,x) = u(r^2t, rx+\xi).$$

Then one can easily check that

$$\frac{\partial}{\partial t}w(t,x) = \left(\nabla\cdot\left(a_{r,\xi}\right)\nabla w(t,\cdot)\right))(x) \text{ and } \lim_{t\to 0} w(t,\cdot) = \phi.$$

Hence, by uniqueness,

$$r^N\int_{\mathbb{R}^N}\phi(y)p(r^2t, rx+\xi, ry+\xi)\,dy = w(t,x) = \int_{\mathbb{R}^N}\phi(y)p^{a_{r,\xi}}(t,x,y)\,dy.$$

Since this is true for every $\phi \in C_0^\infty(\mathbb{R}^N)$, the result follows immediately. ∎

Because, for any choice of r and ξ, $a_{r,\xi}$ satisfies the same hypotheses as a, (I.2.6) continues to hold when $p(2,x,y)$ is replaced by $p^{a_{r,\xi}}(2,x,y)$. Combined with Lemma I.2.7, this allows us to conclude that

$$p(2t,x,y) \geq \frac{e^{-2A}}{t^{N/2}}, \quad (t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \text{ with } |y-x| < 4t^{1/2}. \tag{I.2.8}$$

Clearly (I.2.8) is the lower bound "near the diagonal." To get away from the diagonal, we use a *chaining procedure*. Namely, suppose that $n \le |y-x|^2 < n+1$ for some $n \in Z^+$. Set $x_m = x + \frac{m}{n+1}(y-x)$ and $B_m = B(x_m, 1/n^{1/2})$ for $0 \le m \le n+1$. Notice that if $\xi_m \in B_m$, $0 \le m \le n+1$, then $|\xi_m - \xi_{m-1}| < 3/n^{1/2}$ and therefore that $p(2/(n+1), \xi_{m-1}, \xi_m) \ge n^{N/2}e^{-2A}$. Hence, by the Chapman-Kolmogorov equation,

$$\begin{aligned} &p(2,x,y) \\ \ge \int_{B_1} \cdots \int_{B_n} p\left(\frac{2}{n+1}, x, \xi_1\right) p\left(\frac{2}{n+1}, \xi_1, \xi_2\right) \cdots p\left(\frac{2}{n+1}, \xi_n, y\right) d\xi_1 \cdots d\xi_n \\ &\ge \left(n^{N/2}e^{-2A}\right)^{n+1} \left(\Omega_N n^{-N/2}\right)^n . \end{aligned}$$

Thus, if we choose $\beta > 0$ so that $e^{-\beta} \le \Omega_N e^{-2A}$, then we have that

$$p(2,x,y) \ge e^{-2A} e^{-\beta|y-x|^2}.$$

After combining this with (I.2.8) and then using Lemma I.1.7 to re-scale, we arrive at the conclusion that there is an $M = M(\lambda, N) \in [1,\infty)$ for which the left hand side of (I.0.10) holds.

REMARK.

Unlike the estimate (I.1.25), the estimate just derived is not sharp. Indeed, it is known that

$$\text{(I.2.9)} \qquad \lim_{t\to 0} -4t \log(p(t,x,y)) = d_a(x,y)^2.$$

The hard part of (I.2.9) is the domination of the "$\overline{\lim}$," and this follows easily from (I.1.25). Once one knows the crude lower bound contained in (I.0.10), the "$\underline{\lim}$" part of (I.2.9) is an easy application of elementary ideas from the theory of large deviations.

Chapter II: Some Applications and Extensions

§II.1: APPLICATIONS TO HARMONIC ANALYSIS

In this section we will see that the existence of both upper and lower bounds in (I.0.10) leads to far more powerful applications than the existence of just one or the other.

In discussing these applications, it will be convenient to use the language provided by probability theory. Thus, let $\Omega = C([0,\infty); \mathbf{R}^N)$, endow Ω with the topology of uniform convergence on compact intervals, let $\mathcal{M}$ denote the Borel field over Ω, for $\omega \in \Omega$ and $t \in [0,\infty)$ denote by $x(t,\omega)$ the position of ω at time t, and set $\mathcal{M}_t = \sigma(\{x(s) : s \in [0,t]\})$ (the σ-algebra over Ω generated by the maps $x(s) : \Omega \longrightarrow$

$\mathbb{R}^N$, $s \in [0,t]$). For each $x \in \mathbb{R}^N$ let P_x denote the unique probability measure on $(\Omega, \mathcal{M})$ which satisfies

$$P_x(x(0) = x) = 1$$
$$(\text{II.1.1}) \qquad P_x(x(s+t) \in \Gamma) = \left[P_t\chi_\Gamma\right](x(s))\,(a.s., P_x), \quad \Gamma \in \mathcal{B}_{\mathbb{R}^N}.$$

The existence of such P_x's presents no challenge, since the upper bound in (I.0.10) is more than enough to guarantee that Kolmogorov's criterion is satisfied. In addition, it is clear that the family $\{P_x : x \in \mathbb{R}^N\}$ is strongly Feller continuous and Markov. In particular, it is certainly strongly Markovian.

As a preliminary application of (I.0.10), we will prove the following.

LEMMA II.1.2. *There is an $M = M(\lambda, N) \in [1,\infty)$ such that*

$$P_x\left(\sup_{s\in[0,t]} |x(s) - x| \geq r\right) \leq M \exp(-r^2/Mt)$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^N$ and $r > 0$.

PROOF: We use a crude version of the reflection principle. Namely, let $\zeta_r(\omega) = \inf\{t \geq 0 : |x(t,\omega) - x| \geq r\}$. Then, by the strong Markov property,

$$P\big(t, x, \overline{B(x,r)}^c\big) = E^{P_x}\left[P\big(t - \zeta_r, x(\zeta_r), \overline{B(x,r)}^c\big), \zeta_r < t\right].$$

Note that for $\xi \in \partial B(x,r)$ and $s > 0$, the lower bound in (I.0.10) shows that $P\big(s, \xi, \overline{B(x,r)}^c\big) \geq \epsilon$, where ϵ depends only on λ and N. Thus, from the above, we conclude that

$$P_x(\zeta_r < t) \leq \frac{1}{\epsilon} P\big(t, x, \overline{B(x,r)}^c\big).$$

At the same time, it is clear from the upper bound in (I.0.10) that

$$P\big(t, x, \overline{B(x,r)}^c\big) \leq A \exp(-r^2/Bt)$$

for some pair $A, B \in (0,\infty)$ which depend only on λ and N. Hence, the required estimate has been proved. ∎

As is well-known, the estimate in Lemma II.1.2 is extremely useful when one wants to *localize*. For example, consider any diffusion associated with an operator L' such that $[L'\phi] = [L\phi]$ whenever $\phi \in C_0^\infty(B(\xi,r))$; and denote by $P'(t,x,\cdot)$ the corresponding transition probability function. It is then a relatively easy matter to show that $P'(t,x,dy) = p'(t,x,y)\,dy$ on $(0,\infty) \times \mathbb{R}^N \times B(\xi,r)$ and that for any $\delta \in (0,1)$ the function $p'(t,x,y)$ will satisfy an upper bound of the sort in (I.0.10) so long as $(t,x,y) \in (0,r^2] \times B(\xi,\delta r) \times B(\xi,\delta r)$.

The next result shows that we can also localize the lower bound.

THEOREM II.1.3. *Given $r > 0$ and $\xi \in \mathbb{R}^N$, define*

$$P^{\xi,r}(t,x,\Gamma) = P_x\big(x(t) \in \Gamma, \zeta_r(\xi) > t\big),$$

where $\zeta_r(\xi,\omega) = \inf\{t \geq 0 : x(t,\omega) \notin B(\xi,r)\}$. Then $P^{\xi,r}(t,x,dy) = p^{\xi,r}(t,x,y)\,dy$, where $p^{\xi,r}(t,x,\cdot) \in C_b(B(\xi,r))$ for each $(t,x) \in (0,\infty) \times B(\xi,r)$. Moreover, for each $\delta \in (0,1)$ there exists an $M = M(\delta,\lambda,N) \in [1,\infty)$ such that for every $(r,\xi) \in (0,\infty) \times \mathbb{R}^N$,

(II.1.4)

$$p^{\xi,r}(t,x,y) \geq \frac{1}{Mt^{N/2}} \exp\left(-M|y-x|^2/t\right), \quad (t,x,y) \in (0,r^2] \times B(\xi,\delta r) \times B(\xi,\delta r)$$

PROOF: The proof of this (and related results) is based on the formula

$$P^{\xi,r}(t,x,\cdot) = P(t,x,\cdot) - E^{P_x}\left[P(t-\zeta_r(\xi), x(\zeta_r(\xi)),\cdot), \zeta_r(\xi) \leq t\right],$$

which is a standard application of the strong Markov property. In particular, the existence and continuity of $p^{\xi,r}(t,x,\cdot)$ is immediate when one uses the expression

(II.1.5) $$p^{\xi,r}(t,x,\cdot) = p(t,x,\cdot) - E^{P_x}\left[p(t-\zeta_r(\xi), x(\zeta_r(\xi)),\cdot), \zeta_r(\xi) \leq t\right]$$

as a definition.

By translation and re-scaling, it suffices to prove (II.1.4) in the case when $\xi = 0$ and $r = 1$; thus we will restrict our attention to this case. Further, for notational convenience, we will use $\hat{p}(t,x,\cdot)$ instead of $p^{0,1}(t,x,\cdot)$. From (II.1.5) and (I.0.10), it is clear that

$$\hat{p}(t,x,y) \geq \frac{1}{Mt^{N/2}} e^{-M|y-x|^2/t} - \sup_{s\in(0,t]} \frac{M}{s^{N/2}} e^{(1-\delta)^2/Ms}$$

so long as $x, y \in B(0,\delta)$. Thus there is a $\rho = \rho(\delta,\lambda,N) \in (0, 1-\delta)$ such that

$$\hat{p}(t,x,y) \geq \frac{1}{2Mt^{N/2}} e^{-M|y-x|^2/t}$$

for all $x, y \in B(0,\delta)$ and $t > 0$ satisfying $t^{1/2} \vee |y-x| \leq \rho$.

To complete the argument, we use a chaining procedure again. (Notice that $\hat{p}(t,x,y)$ satisfies the Chapman-Kolmogorov equation.) Namely, if $t \in (0,\rho^2]$ and $x, y \in B(0,\delta)$ with $|y-x| \geq \rho$, set $x_m = x + \frac{m}{n+1}(y-x)$, where $n \in Z^+$ is chosen so that $n \geq 6/\rho$. Next, set $\Gamma_m = B(x_m, 1/n) \cap B(0,\delta)$, and note that $|\xi_m - \xi_{m-1}| \leq 3/n < \rho$ if $\xi_m \in \Gamma_m$. Also, observe that there is a $\gamma > 0$, depending only on n, N, and δ, such that $|\Gamma_m| \geq \gamma$. Hence, by the preceding and the Chapman-Kolmogorov equation,

$$\hat{p}(t,x,y) \geq \gamma^n \left(\frac{(n+1)^{N/2}}{2Mt^{N/2}} e^{-M(n+1)\rho^2/t}\right)^{n+1} \geq \frac{\epsilon}{t^{N/2}} e^{-\rho^2/\epsilon t} \geq \frac{\epsilon}{t^{N/2}} e^{-|y-x|^2/\epsilon t}$$

for some $\epsilon = \epsilon(n, N, M) \in (0, 1/M]$. Finally, suppose that $t \in [\rho^2, 1]$ and that $x, y \in B(0, \delta)$, and this time choose $n \in Z^+$ so that $n \geq 1/\rho^2$. Then, by the Chapman-Kolmogorov equation and what we have just proved,

$$\hat{p}(t, x, y) \geq |B(0, \delta)|^n \left(\frac{(n+1)^{N/2}\epsilon}{t^{N/2}} e^{-4(n+1)/\epsilon\rho^2} \right)^{n+1};$$

which is enough to complete the proof. ∎

The estemate (II.1.4) becomes an extremely powerful tool when it is applied to the harmonic analysis for the operator L. Indeed, we will see below that it leads quite quickly to the famous continuity theorem of J. NASH as well as the Harnack principles proved by DE GIORGI and J. MOSER. The particular route which we will take in passing from (II.1.4) to these results is based on ideas introduced by N. KRYLOV. But, whatever route one adopts, the key to everything is contained in the following sort of *super-mean-value property.*

THEOREM II.1.6. *Let $\alpha, \beta \in (0, 1)$ be given. Then there is an $\epsilon = \epsilon(\alpha, \beta, \lambda, N) \in (0, 1)$ such that for all $(\sigma, \xi) \in \mathbf{R} \times \mathbf{R}^N$, $r > 0$, and $u \in C^{1,2}([\sigma, \sigma + r^2] \times \overline{B(\xi, r)})^+$ satisfying $\frac{\partial}{\partial t}u(t, x) + [Lu](t, x) \leq 0$:*

$$u(s, x) \geq \frac{\epsilon}{|B(\xi, \beta r)|} \int_{B(\xi, \beta r)} u(\sigma + r^2, y)\, dy, \quad (s, x) \in [\sigma, \sigma + \alpha r^2] \times \overline{B(\xi, \beta r)}.$$

PROOF: Set $t = \sigma + r^2 - s$. Then $t \in [(1 - \alpha)r^2, r^2]$ and

$$E^{P_x}\left[u\big(s + t \wedge \zeta_r(\xi), x(t \wedge \zeta_r(\xi))\big)\right] \leq u(s, x).$$

At the same time, because $u \geq 0$,

$$\begin{aligned}
E^{P_x}\left[u\big(s + t \wedge \zeta_r(\xi), x(t \wedge \zeta_r(\xi))\big)\right] &\geq \int_{B(\xi, \beta r)} u(\sigma + r^2, y) p^{\xi, r}(t, x, y)\, dy \\
&\geq \frac{1}{M t^{N/2}} \int_{B(\xi, \beta r)} u(\sigma + r^2, y) e^{-M|y-x|^2/t}\, dy \\
&\geq \frac{1}{M r^N} e^{-4M\beta^2 r^2/(1-\alpha)r^2} \int_{B(\xi, \beta r)} u(\sigma + r^2, y)\, dy,
\end{aligned}$$

where we have used here the $M = M(\beta, \lambda, N)$ coming from Lemma II.1.3. After combining this with the above, one now sees how to choose ϵ. ∎

Given $(\sigma, \xi) \in \mathbf{R} \times \mathbf{R}^N$ and $r > 0$, set $Q\big((\sigma, \xi), r\big) = [\sigma, \sigma + r^2] \times \overline{B(\xi, r)}$. For $u \in C\big(Q\big((\sigma, \xi), r\big)\big)$, define the *oscillation* of u on $Q\big((\sigma, \xi), r\big)$ by

$$\operatorname{Osc}(u; (\sigma, \xi), r) = \max\{u(s', x') - u(s, x) : (s, x), (s', x') \in Q\big((\sigma, \xi), r\big)\}.$$

LEMMA II.1.7. *For each $\delta \in (0, 1)$ there is a $\rho = \rho(\delta, \lambda, N) \in (0, 1)$ such that for all $(\sigma, \xi) \in \mathbf{R} \times \mathbf{R}^N$, $r > 0$, and $u \in C^{1,2}\big(Q\big((\sigma, \xi), r\big)\big)$ satisfying $\frac{\partial}{\partial t}u(t, x) + [Lu](t, x) = 0$,*

$$Osc(u; (\sigma, \xi), \delta r) \leq Osc(u; (\sigma, \xi), r).$$

PROOF: For $r' \in (0,r]$, set

$$M(r') = \max\{u(s,x) : (s,x) \in Q\big((\sigma,\xi),r'\big)\}$$
$$\text{and } m(r') = \min\{u(s,x) : (s,x) \in Q\big((\sigma,\xi),r'\big)\}.$$

Let $\Gamma = \{y \in B(\xi,\delta r) : u(\sigma+r^2,y) \geq \frac{M(r)+m(r)}{2}\}$, and suppose that $|\Gamma| \geq \frac{1}{2}|B(\xi,\delta r)|$. Then, for all $(s,x) \in Q\big((\sigma,\xi),\delta r\big)$:

$$u(s,x) - m(r) \geq \frac{\epsilon}{|B(\xi,\delta r)|}\int_{B(\xi,\delta r)} \big(u(\sigma + R^2,y) - m(r)\big)\,dy \geq \frac{\epsilon}{4}\big((M(r) - m(r)\big),$$

where we have taken ϵ to be the $\epsilon(\delta^2,\delta,\lambda,N)$ from Lemma II.1.6. Hence, $m(\delta r) - m(r) \geq \frac{\epsilon}{4}\big(M(r) - m(r)\big)$; and so

$$M(\delta r) - m(\delta r) \leq (1 - \epsilon/4)\big(M(r) - m(r)\big).$$

When we assume that $|\Gamma| \leq \frac{1}{2}|B(\xi,\delta r)|$, we can draw the same conclusion by considering $M(r) - u$ and $B(\xi,\delta r) \setminus \Gamma$ in place of $u - m(r)$ and Γ, respectively. Thus, in either case, we can take $\rho = 1 - \frac{\epsilon}{4}$. ∎

THEOREM II.1.8. *(J. NASH) Let $\delta \in (0,1)$ be given. Then there exist an $\alpha = \alpha(\delta,\lambda,N) \in (0,1]$ and a $C = C(\delta,\lambda,N) \in (0,\infty)$ such that for all $(\sigma,\xi) \in \mathbb{R}\times\mathbb{R}^N$, $r > 0$, and $u \in C^{1,2}\big(Q\big((\sigma,\xi),r\big)\big)$ satisfying $\frac{\partial}{\partial t}u(t,x) + [Lu](t,x) = 0$,*

$$|u(s',x') - u(s,x)| \leq C\left(\frac{|s'-s|^{1/2} \vee |x'-x|}{r}\right)^{\alpha} \|u\|_{C(Q((\sigma,\xi),r))}$$

for $(s,x),(s',x') \in Q\big((\sigma,\xi),\delta r)\big)$. In particular, if $u \in C^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ is a bounded solution to $\frac{\partial}{\partial t}u(t,x) + [Lu](t,x) = 0$, then u is constant.

PROOF: Let ρ be the $\rho(1-\delta,\lambda,N)$ from Lemma II.1.7. Assume that $s \leq s'$ and set $\ell = (s'-s)^{1/2} \vee |x'-x|$. If $\frac{\ell}{r} \geq 1-\delta$, there is nothing to be done. If $\frac{\ell}{r} < 1-\delta$, choose $k \in Z^+$ so that $(1-\delta)^{k+1} \leq \frac{\ell}{r} < (1-\delta)^k$. Then

$$\begin{aligned}|u(s',x') - u(s,x)| &\leq \mathrm{Osc}(u;(s,x),(1-\delta)^k r) \leq \rho^{k-1}\mathrm{Osc}(u;(s,x),(1-\delta)r)\\ &\leq \rho^{k-1}\mathrm{Osc}(u;(\sigma,\xi),r) \leq \frac{2}{\rho^2}\|u\|_{C(Q((\sigma,\xi),r)))}\rho^{k+1}.\end{aligned}$$

Finally, define α so that $\rho = \big((1-\delta)\wedge\rho\big)^{\alpha}$. Then,

$$\rho^{k+1} \leq \left(\delta^{k+1}\right)^{\alpha} \leq \left(\frac{\ell}{r}\right)^{\alpha}. \quad ∎$$

As a consequence of this result applied to $p(t,x,y)$ itself, we have the following.

COROLLARY II.1.9. *There exist* $C = C(\lambda, N) \in (0,\infty)$ *and* $\alpha = \alpha(\lambda, N) \in (0,1)$ *such that for every* $\delta > 0$

$$|p(t',x',y') - p(t,x,y)| \le \frac{C}{\delta^N}\left(\frac{|t'-t|^{1/2} \vee |x'-x| \vee |y'-y|}{\delta}\right)^{\alpha} \tag{II.1.10}$$

for all $(t',x',y'), (t,x,y) \in [\delta^2,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $|y'-y| \vee |x'-x| \le \delta$.

THEOREM II.1.11. (MOSER) *Let* α, β, *and* $\gamma \in (0,1)$ *with* $\alpha < \beta$ *be given. Then there exists an* $M = M(\alpha,\beta,\gamma,\lambda,N) \in [1,\infty)$ *such that for all* $(\sigma,\xi) \in \mathbb{R}\times\mathbb{R}^N$, $r > 0$, *and* $u \in C^{1,2}\big(Q((\sigma,\xi),r)\big)^+$ *satisfying* $\frac{\partial}{\partial t}u(t,x) + [Lu](t,x) = 0$,

$$u(s,x) \le Mu(\sigma,\xi), \quad (s,x) \in [\sigma + \alpha r^2, \sigma + \beta r^2] \times \overline{B(\xi,\gamma r)}.$$

In particular, if $u \in C^2(\mathbb{R}^N)$ *satisfies* $[Lu] = 0$ *and* u *is bounded above or below, then* u *is constant.*

PROOF: We may and will assume that $\sigma = 0$, $\xi = 0$, and that $r = 1$. Set $\epsilon = \epsilon(1-\alpha,(1+\gamma)/2,\lambda,N)$ and $\rho = \rho(1/2,\lambda,N)$ as in Theorem II.1.6 and Lemma II.1.7, respectively; and put $\mu = (1-\rho)/2$ and $\kappa \equiv (1-\mu)/\rho = \frac{1}{2\rho}(1+\rho) > 1$.

Note that

$$u(0,0) \ge \epsilon \int_{B(0,(1+\gamma)/2)} u(t,y)\,dy, \quad t \in [\alpha,1].$$

Thus, there is nothing to prove if $u(0,0) = 0$; and so, without loss in generality, we assume that $u(0,0) = 1$ and attempt to prove that there is an $M \in [1,\infty)$ such that $u(s,x) \le M$ for $(s,x) \in [\alpha,\beta] \times \overline{B(0,\gamma)}$. Next, set $\Sigma(s,M) = \{y \in B(0,(1+\gamma)/2) : u(s,y) \ge \mu M\}$, and observe that

$$|\Sigma(s,M)| \le \frac{1}{\epsilon M}, \quad M > 0 \text{ and } s \in [\alpha,1].$$

Now suppose that $Q\big((s,x),2r\big) \subseteq [\alpha,1] \times \overline{B(0,(1+\gamma)/2)}$ and that $u(s,x) \ge M$ where $\Omega_N r^N > \frac{1}{\epsilon\mu M}$. Then $|\Sigma(s,M)| < |B(x,r)|$; and so there must be a $y \in B(x,r)$ for which $u(s,y) \le \mu M$. But, by Lemma II.1.7, this means that there is an $(s',x') \in Q\big((s,x),2r\big)$ such that

$$u(s',x') \ge \operatorname{Osc}(u;(s,x),2r) \ge \frac{1}{\rho}\operatorname{Osc}(u;(s,x),r) \ge \frac{1}{\rho}|u(s,y) - u(s,x)| \ge \kappa M.$$

Thus if we define $r(M)$ for $M > 0$ by $\Omega_N r(M)^N = 2/\epsilon\mu M$, and if $Q\big((s,x),2r(M)\big) \subseteq [\alpha,1]\times\overline{B(\xi,\frac{1+\gamma}{2})}$, then $u(s,x) \ge M$ implies that there is an $(s',x') \in Q\big((s,x),2r(M)\big)$ for which $u(s',x') \ge \kappa M$.

Finally, choose $M \in [1,\infty)$ so that both

$$\beta + \sum_{n=0}^{\infty}\big(2r(\kappa^n M)\big)^2 \le \frac{1+\beta}{2} \text{ and } \gamma + \sum_{n=0}^{\infty} 2r(\kappa^n M) \le \frac{1+\gamma}{2}.$$

If $u(s,x) \geq M$ for some $(s,x) \in [\alpha,\beta] \times \overline{B(0,\gamma)}$, then one can use the preceding paragraph to produce a sequence $\{(s_n,x_n)\}_0^\infty \subseteq [\alpha,(1+\beta)/2] \times \overline{B(0,(1+\gamma)/2)}$ such that $u(s_n,x_n) \geq \kappa^n M$. But clearly this is impossible, since u must be bounded on $[\alpha,(1+\beta)/2] \times \overline{B(0,(1+\gamma)/2)}$.

To prove the final statement of the theorem, assume that $u \geq 0$ and conclude that u must be bounded and therefore, by Theorem II.1.8, must be constant. ∎

§II.2: Extension to Perturbations of Divergence Form Operators

Again let $a : \mathbb{R}^N \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^N$ be a symmetric matrix-valued function having bounded derivatives of all orders and satisfying **(E)**. Next, let $b : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $\hat{b} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, and $c : \mathbb{R}^N \longrightarrow \mathbb{R}$ be smooth functions with bounded derivatives of all orders, and suppose that

$$\text{(II.2.1)} \qquad ||c||_{C_b(\mathbb{R}^N)} + ||b \cdot ab||_{C_b(\mathbb{R}^N)} + ||\hat{b} \cdot a\hat{b}||_{C_b(\mathbb{R}^N)} \leq \Lambda$$

for some $\Lambda \in (0,\infty)$. Finally, define the operator $\mathcal{L}$ on $C^2(\mathbb{R}^N)$ by

$$\text{(II.2.2)} \qquad [\mathcal{L}\phi](x) = [\nabla \cdot (a\nabla\phi)](x) + [b \cdot (a\nabla\phi)](x) - [\nabla \cdot (\phi a\hat{b})](x) + [c\phi](x).$$

Just as before, one knows that there exists a unique semigroup $\{Q_t : t > 0\}$ of non-negativity preserving bounded operators on $C_b(\mathbb{R}^N)$ such that

$$[Q_t\phi](x) - \phi(x) = \int_0^t [Q_s\mathcal{L}\phi](x)\,ds, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \text{ and } \phi \in C_0^\infty(\mathbb{R}^N);$$

and once again there is a $q \in \bigcup_{n=1}^\infty C_b\infty([1/n,n] \times \mathbb{R}^N \times \mathbb{R}^N;(0,\infty))$ such that

$$[Q_t\phi](x) = \int_{\mathbb{R}^N} \phi(y)q(t,x,y)\,dy, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \text{ and } \phi \in C_b(\mathbb{R}^N).$$

In particular, if $\hat{\mathcal{L}}$ denotes the operator obtained by reversing the roles of b and $\hat{b}$ in the (II.2.2), then

$$\frac{\partial}{\partial t}q(t,x,y) = [\hat{\mathcal{L}}q(t,x,\cdot)](y), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N.$$

Also, for each $\phi \in C_0^\infty(\mathbb{R}^N)$ the map $t \in (0,\infty) \longmapsto [Q_t\phi] \in \mathcal{S}(\mathbb{R}^N)$ is smooth and

$$\frac{\partial}{\partial t}[Q_t\phi] = [\mathcal{L}Q_t\phi], \quad t \in (0,\infty).$$

In particular,

$$\frac{\partial}{\partial t}q(t,x,y) = [\mathcal{L}q(t,\cdot,y)](x), \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N.$$

What, in general, no longer holds is either $[Q_t 1] = 1$ or $q(t,x,y) = q(t,y,x)$. Although their loss makes life somewhat more difficult, one can often circumvent these difficulties by taking advantage of the observation that, for all $\phi, \psi \in C_0^\infty(\mathbf{R}^N)$,

$$(Q_t\phi, \psi) = (\phi, \hat{Q}_t\psi), \quad t \in (0,\infty), \tag{II.2.3}$$

where $\{\hat{Q}_t : t > 0\}$ denotes the semigroup corresponding to the operator $\hat{\mathcal{L}}$. The proof of this equation is essentially the same as that of (I.0.3). As a consequence of (II.2.3), we see that

$$\hat{q}(t,x,y) = q(t,y,x), \quad (t,x,y) \in (0,\infty) \times \mathbf{R}^N \times \mathbf{R}^N;$$

and this fact can be used to replace symmetry. In addition, from (II.2.3) one sees that $\|Q_t\phi\|_1 \le \|\hat{Q}_t 1\|_\infty \|\phi\|_1$; and therefore that

$$\|Q_t\phi\|_q \le \|Q_t 1\|_\infty^{1/q'} \|Q_t|\phi|^q\|_1^{1/q} \le \|\hat{Q}_t 1\|_\infty^{1/q} \|Q_t 1\|_\infty^{1/q'} \|\phi\|_q,$$

which often serves as a substitute for Lemma I.0.4.

Of course, if we rely on standard probabilistic machinery (e.g. the Feynman-Kac formula) to estimate quantities like $\|Q_t 1\|_\infty$ or $\|\hat{Q}_t 1\|_\infty$, we get bounds which depend badly on b and $\hat{b}$; that is, on their derivatives. What we again want are bounds which depend only on λ, Λ, and N. In fact, what we are going to show is that there is an $M = M(\lambda, \Lambda, N) \in [1,\infty)$ such that

$$\frac{1}{Mt^{N/2}} \exp\bigl(-M(t + |y-x|^2/t)\bigr) \le q(t,x,y) \le \frac{M}{t^{N/2}} \exp\bigl(Mt - |y-x|^2/Mt\bigr) \tag{II.2.4}$$

for all $(t,x,y) \in (0,1] \times \mathbf{R}^N \times \mathbf{R}^N$.

The estimate (II.2.4) is again due to D. Aronson [**A**]. We will model our proof on the method used to prove (I.0.10). Because much of the argument is the same as that given in Chapter I, we will merely outline the proof, giving details only at those points where new difficulties arrise.

Step 1).

Set $\phi_t = [Q_t\phi]$, where $\phi \in C_0^\infty(\mathbf{R}^N)^+$. Just as in the proof of Lemma I.1.8, one finds that for any $q \in [1,\infty)$:

$$\|\phi_t\|_{2q}^{2q-1} \frac{d}{dt}\|\phi_t\|_{2q} \le -\frac{\lambda}{q}\|\nabla\phi_t^q\|_2^2 + q\bigl(c_q, \phi_t^{2q}\bigr),$$

where

$$c_q \equiv \frac{c}{q} + \frac{1}{2}\left(\frac{b}{q} + (2-\frac{1}{q})\hat{b}\right) \cdot a \left(\frac{b}{q} + (2-\frac{1}{q})\hat{b}\right).$$

In particular,

$$\|\phi_t\|_2 \le e^{\kappa t}\|\phi\|_2, \quad t \in [0,\infty),$$

where

$$\kappa = \bigl\|c + \frac{1}{2}\bigl[(b+\hat{b}) \cdot a(b+\hat{b})\bigr]\bigr\|_{C_b(\mathbf{R}^N)}.$$

Thus, an application of Nash's inequality yields

$$\frac{d}{dt}\|\phi_t\|_{2q} \le -\frac{\lambda}{C_N q}\frac{\|\phi_t\|_{2q}^{1+4q/N}}{\|\phi_t\|_q^{4q/N}} + q\Lambda\|\phi_t\|_{2q}, \quad t\in(0,\infty) \text{ and } q\in[2,\infty).$$

If we now use Lemma I.1.14 in exactly the same way as we did in Section I.1, we arrive at the existence of a $K = K(\lambda, N) \in (0,\infty)$ such that

$$\|Q_t\|_{2\to\infty} \le \frac{K}{(\delta t)^{N/4}} e^{t(\kappa+\delta\Lambda)}, \quad t\in(0,\infty) \text{ and } \delta\in(0,1].$$

Since the same estimate holds for $\|Q_t\|_{1\to 2} = \|\hat{Q}_t\|_{2\to\infty}$, we can conclude that

$$\text{(II.2.5)}\quad q(t,x,y) \le \frac{K}{(\delta t)^{N/2}} e^{t(\kappa+\delta\Lambda)}, \quad (t,x,y)\in(0,\infty)\times\mathbf{R}^N\times\mathbf{R}^N \text{ and } \delta\in(0,1]$$

for some other K having the same dependence properties.

STEP 2).

For $\psi \in C_b^\infty(\mathbf{R}^N)$, set $q^\psi(t,x,y) = e^{-\psi(x)}q(t,x,y)e^{\psi(y)}$. One can easily compute that $q^\psi(t,x,y)$ corresponds to the operator $\mathcal{L}^\psi$ which is obtained from $\mathcal{L}$ by replacing b and $\hat{b}$, repectively, with $b^\psi \equiv b + \nabla\psi$ and $\hat{b}^\psi \equiv \hat{b} - \nabla\psi$ and replacing c with $c^\psi \equiv c + (b-\hat{b})\cdot a\nabla\psi + \nabla\psi\cdot a\nabla\psi$. Hence, by (I.2.5) applied to $q^\psi(t,x,y)$, we obtain

$$q(t,x,y) \le \frac{K}{(\delta t)^{N/2}} \exp\left(\psi(x) - \psi(y) + t\left(\frac{\alpha}{\delta}\Lambda + (1+\delta)\Gamma(\psi)^2\right)\right),$$

where K, κ and Λ are the same as they were in Step 1), $\alpha\in(0,\infty)$ is independent of a, b, $\hat{b}$, and c, and $\Gamma(\psi)$ is defined as in (I.1.10). Proceeding from here in exactly the same way as we passed from (I.1.16) to (I.1.25), we see that there is a $K = K(\lambda,\Lambda,N)\in(0,\infty)$ such that

(II.2.6)

$$q(t,x,y) \le \frac{Ke^{\alpha\Lambda t/\delta}}{(\delta t)^{N/2}} \exp\left(-\frac{d_a(x,y)^2}{4(1+\delta)t}\right), \quad (t,x,y)\in(0,\infty)\times\mathbf{R}^N\times\mathbf{R}^N \text{ and } \delta\in(0,1],$$

where $d_a(x,y)$ is the Riemannian distance determined by the Riemann metric a^{-1}. Certainly (II.2.6) is more than sufficient to prove the upper bound in (II.2.4).

STEP 3).

Before attempting to repeat the argument given in Lemma I.2.4 to prove (I.2.1), we need here to handle a problem which did not arise there. Namely, we need to get a lower bound on $\int_{\mathbf{R}^N} q(t,x,y)\,dy$. To this end, let $x\in B(0,2)$ and $\theta\in[1/2,1)$ be given, and set $u(t,y) = \theta q(t,x,y) + (1-\theta)$. Next, set

$$H_\beta(t) = \int_{\mathbf{R}^N} e^{-\eta(y)}u(t,y)^\beta\,dy, \quad \beta\in(0,1],$$

where $\eta(y) = c_N(1+|y|^2)^{1/2}$ and c_N is chosen so that $\int_{\mathbb{R}^N} e^{-\eta(y)}\,dy = 1$. One can then show that for each $\beta \in (0,1)$ there exist $\mu_\beta = \mu_\beta(\lambda,\Lambda,N) \in (0,\infty)$ and $\nu_\beta = \nu_\beta(\lambda,\Lambda,N) \in (0,\infty)$ such that

$$H'_\beta(t) \geq \frac{\beta(1-\beta)}{2}\int_{\mathbb{R}^N} e^{-\eta(y)}u(t,y)^{\beta-2}\nabla u(t,y)\cdot a\nabla u(t,y)\,dy - \mu_\beta H_\beta(t)$$

and

$$H'_1(t) \geq -\frac{\beta(1-\beta)}{2}\int_{\mathbb{R}^N} e^{-\eta(y)}u(t,y)^{\beta-2}\nabla u(t,y)\cdot a\nabla u(t,y)\,dy \\ -\nu_\beta\int_{\mathbb{R}^N} e^{-\eta(y)}u(t,y)^{2-\beta}\,dy.$$

Combining these with the upper bound in (II.2.4), we see that

$$(H_\beta + H_1)'(t) \geq -\left(\mu_\beta + \frac{\nu_\beta M^{1-\beta}}{t^{(1-\beta)N/2}}\right)(H_\beta + H_1)(t), \quad t \in (0,1].$$

Hence, if we take $\beta = 1 - 1/N$ and note that $H_1(0) \geq e^{-4c_N}$, then we conclude that there is an $\alpha = \alpha(\lambda,\Lambda,N) \in (0,1)$ such that

$$H_\beta(t) + H_1(t) \geq \alpha, \quad t \in [0,1].$$

Finally, since $H_\beta(t) \leq \left(H_1(t)\right)^\beta$, it follows that for some other $\alpha \in (0,1)$, with the same dependence,

$$\text{(II.2.7)} \qquad \int_{\mathbb{R}^N} q(t,x,y)\,dy \geq \alpha, \quad (t,x) \in [0,1]\times B(0,2).$$

STEP 4.

We can now prove the analogue of Lemma I.2.4. Namely, define $u(t,y)$ as in Step 3) and set $G(t) = \int_{\mathbb{R}^N} e^{-\pi|y|^2}\log(u(t,y))\,dy$. One then finds that there is a $B = B(\lambda,\Lambda,N) \in (0,\infty)$ such that

$$G'(t) \geq -B + \frac{1}{B}e^{-\pi r^2}\big|\{y \in B(0,r) : u(t,y) \geq e^{-K}\}\big|\big(|G(t)| - K\big)^2, \quad t \in (0,\infty),$$

for every choice of $r > 0$ and $K > 0$. At this point one needs (II.2.7) in order to show that there is a choice of r and K, depending only on λ,Λ, and N, such that

$$\big|\{y \in B(0,r) : u(t,y) \geq e^{-K}\}\big| \geq \delta$$

for some $\delta = \delta(\lambda,\Lambda,N) \in (0,\infty)$ and all $(t,x) \in [1/2]\times B(0,2)$ and $\theta \in [1/2,1]$. But once one has (II.2.7), the proof is exactly like the one given to prove the analoguous result in Lemma I.2.4; and from here the rest of the proof contains no changes. In other words, we now know that there is an $A = A(\lambda, \Lambda, N) \in (0, \infty)$ such that

$$\text{(II.2.8)} \qquad \int_{\mathbb{R}^N} e^{-\pi|y|^2} \log(q(1,x,y))\, dy \geq -A, \quad x \in B(0,2).$$

STEP 5.

Because of the symmetry of our hypotheses, (II.2.8) holds for $\hat{q}(t,x,y)$ as well. Hence, just as in Section I.2 (cf. (I.2.6)) we find that

$$\text{(II.2.9)} \qquad q(2,x,y) \geq e^{-2A}, \quad x, y \in B(0,2).$$

We next want to introduce translation and scaling. Although translation poses no problems, scaling now has a flaw. Indeed, by the reasoning used to prove Lemma I.2.7, one finds that $r^N q(r^2 t, rx + \xi, ry + \xi) = q_{\xi,r}(t,x,y)$ where $q_{\xi,r}(t,x,y)$ corresponds to the coefficients $a(r\cdot +\xi), rb(r\cdot +\xi)$, and $r^2 c(r\cdot +\xi)$. Thus, if we want to stay within the class of coefficients which satisfy our hypotheses, then we have got to restrict our scaling factor r to $(0,1]$. In particular, we can get from (II.2.10) only that

$$\text{(II.2.11)} \qquad q(2t,x,y) \geq \frac{e^{-2A}}{t^{N/2}}, \quad t \in (0,1] \text{ and } |y-x| \leq 4t^{1/2}.$$

Finally, once one has (II.2.11), one completes the derivation of the lower bound in (II.2.4) by the chaining procedure used at the end of Section 2.2. More precisely, for $t \in [1,\infty)$, choose $n \in Z^+$ so that $n \leq t \leq (n+1)$ and note that, by the Chapman-Kolmogorov equation,

$$q(t,x,y) \geq \int_{B(x,1)} \cdots \int_{B(x,1)} q(t/(n+1), x, \xi_1) \cdots q(t/(n+1), \xi_n, y)\, d\xi_1 \cdots d\xi_n,$$

from which one sees that there exist $\mu = \mu(\lambda, \Lambda, N) \in (0,\infty)$ and $\epsilon = \epsilon(\lambda, \Lambda, N) \in (0,\infty)$ for which

$$q(t,x,y) \geq \frac{\epsilon e^{-\mu t}}{t^{N/2}}, \quad t > 0 \text{ and } |y-x| \leq t^{1/2}.$$

Starting with the preceding and repeating the chaining argument used at the end of Section 2.2, one quickly arrives at the required lower bound.

Specializing to the case in which $\hat{b}$ and c vanish identically, and therefore

$$\int_{\mathbb{R}^N} q(t,x,y) \equiv 1,$$

one can now re-run the arguments given in Section II.1 to prove the following variation on the theorems of Nash and Moser.

THEOREM II.2.12. *Let $\mathcal{L} = \nabla \cdot a\nabla + b \cdot \nabla$ where a satisfies* (**E**) *and* $\|b\|_{C_b(\mathbb{R}^N)} \le \Lambda$. *Then for each $\delta \in (0,1)$ there exist $C = C(\delta, \lambda, \Lambda, N) \in (0,\infty)$ and $\alpha = \alpha(\delta, \lambda, \Lambda, N) \in (0,1]$ such that for all $(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $r \in (0, 1/\delta)$*

$$|u(s',x') - u(s,x)| \le C \left(\frac{|s'-s|^{1/2} \vee |x'-x|}{r} \right)^{\alpha} \|u\|_{C(Q((\sigma,\xi),r))}$$

for $(s,x),(s',x') \in Q\big((\sigma,\xi),\delta r\big)$ whenever $u \in C^{1,2}\big(Q\big((\sigma,\xi),r\big)\big)$ satisfies $\frac{\partial}{\partial t}u(t,x) + [\mathcal{L}u](t,x) = 0$. In particular,

$$|q(t',x',y) - q(t,x,y)| \le \frac{C}{\delta^N} \left(\frac{|t'-t|^{1/2} \vee |x'-x|}{\delta} \right)^{\alpha} \tag{II.2.13}$$

for all $(t',x',y),(t,x,y) \in [\delta^2, 1/\delta^2] \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x'-x| \le \delta$. Finally, for each $0 < \alpha < \beta < 1$ and $\gamma \in (0,1)$, there exists an $M = M(\alpha,\beta,\gamma,\lambda,\Lambda,N) \in [1,\infty)$ such that for every $(\sigma,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $r \in (0,1]$, and $u \in C^{1,2}\big(Q\big((\sigma,\xi),r\big)\big)^+$ satisfying $\frac{\partial}{\partial t}u(t,x) + [\mathcal{L}u](t,x) = 0$,

$$u(s,x) \le M u(\sigma,\xi), \quad (s,x) \in [\sigma + \beta r^2, \sigma + \beta r^2] \times B(\xi, \gamma r).$$

There are two more directions in which it is easy to extend the results discussed thus far. In the first place, one can consider operators $\mathcal{L}^{\omega} = \frac{1}{\omega}\mathcal{L}$, where $\mathcal{L}$ is of the form we have been discussing thus far and $\omega \in C_b^{\infty}(\mathbb{R}^N)$ is a function with values in an interval $[\mu, 1/\mu]$ with $\mu \in (0,1]$. When considering such an operator, it is natural to work in the L^p-spaces $L^p(\omega)$ corresponding to the weight ω:

$$\|\phi\|_{L^p(\omega)} \equiv \left(\int_{\mathbb{R}^N} |\phi(y)|^p \omega(y)\, dy \right)^{1/p}.$$

With this in mind, if one simply repeats the arguments already given, one concludes that the semigroup $\{Q_t^{\omega} : t > 0\}$ corresponding to $\mathcal{L}^{\omega}$ admits the representation

$$[Q_t^{\omega}\phi](x) = \int_{\mathbb{R}^N} \phi(y) q^{\omega}(t,x,y) \omega(y)\, dy,$$

where $q^{\omega}(t,x,y)$ satisfies (II.2.4) and (II.2.13) with M, C, and α now depending on μ as well as λ, Λ, and N. Of course, when b and $\hat{b}$ vanish identically, $q^{\omega}(t,x,y)$ is symmetric in x and y; and, when c vanishes as well, one gets the results in Chapter I and Section II.1 (with constants depending also on μ.)

The second direction in which it is a relatively easy matter to extend our results is to the case when the coefficients depend on time as well as space. Although there are several places in which the argument has to be adjusted in order to accomodate time-dependence, the basic ideas apply with no changes. (See [**F.-S.**] for more details.)

§II.3: Diffusion Semigroups Corresponding to Measurable Coefficients

One of the advantages to our having estimates which do not depend on smoothness properties of the coefficients is that it opens up the possibility of constructing nice semigroups corresponding to (possibly) discontinuous coefficients. We devote this section to carrying out such a construction and discussing a few of the properties of the resulting semigroups.

Given numbers $\lambda \in (0,1]$, let $\mathcal{A}(\lambda)$ denote the class of all measurable, symmetric matrix-valued functions $a : \mathbf{R}^N \longrightarrow \mathbf{R}^N \otimes \mathbf{R}^N$ which satisfy (**E**). We begin by discussing semigroups determined, in a certain generalized sense, by the operator $L = \nabla \cdot a\nabla$.

Theorem II.3.1. *For each $a \in \mathcal{A}(\lambda)$ there is a unique Feller continuous, Markov semigroup $\{P_t : t > 0\}$ with the properties that*

i) The map $t \in [0,\infty) \longmapsto [P_t\phi] \in W_2^{(1)}(\mathbf{R}^N)$ is weakly continuous map for each $\phi \in C_0^\infty(\mathbf{R}^N)$.

ii) For all $\phi, \psi \in C_0^\infty(\mathbf{R}^N)$

$$(P_t\phi, \psi) = (\phi, \psi) + \int_0^t (\nabla P_t\phi, a\nabla\psi)\, ds, \quad t \in (0,\infty). \tag{II.3.2}$$

In fact, $\{P_t : t > 0\}$ determines a unique strongly continuous semigroup $\{\overline{P}_t : t > 0\}$ of selfadjoint contractions on $L^2(\mathbf{R}^N)$, $\{\overline{P}_t : t > 0\}$ is strongly continuous on $W_2^{(1)}(\mathbf{R}^N)$, and (I.0.5) holds. Moreover, there is a $p \in C((0,\infty) \times \mathbf{R}^N \times \mathbf{R}^N)$ satisfying both (I.0.10) and (II.1.10) such that

$$[P_t\phi](x) = \int_{\mathbf{R}^N} \phi(y)p(t,x,y)\, dy, \quad \phi \in C_0^\infty(\mathbf{R}^N).$$

In particular, $\{P_t : t > 0\}$ is strongly Feller continuous. Finally, if $\{a^n\}_1^\infty \subseteq \mathcal{A}(\lambda)$ and $a^n \longrightarrow a$ almost everywhere, then $p^n(t,x,y) \longrightarrow p(t,x,y)$ uniformly on compacts (in $(0,\infty) \times \mathbf{R}^N \times \mathbf{R}^N$) and, for each $t \in [0,\infty)$ and $\phi \in C_0^\infty(\mathbf{R}^N)$, $[P_t^n\phi] \longrightarrow [P_t\phi]$ in $W_2^{(1)}(\mathbf{R}^N)$.

Proof: Choose $\{a^n\}_1^\infty \subseteq \mathcal{A}(\lambda) \cap C_b^\infty(\mathbf{R}^N; \mathbf{R}^N \otimes \mathbf{R}^N)$ so that $a^n \longrightarrow a$ almost everywhere. Because of (II.1.10) and (I.0.10), we may and will assume that there is a $p \in C((0,\infty) \times \mathbf{R}^N \times \mathbf{R}^N)$, which again satisfies (II.1.10) and (I.0.10), such that $p^n(t,x,y) \longrightarrow p(t,x,y)$ boundedly and uniformly on compacts. In particular, we have that $p(t,x,y) = p(t,y,x)$ and the Chapman-Kolmogorov equation

$$\begin{aligned} p(s+t,x,y) &= \lim_{n\to\infty} p^n(s+t,x,y) \\ &= \lim_{n\to\infty} \int_{\mathbf{R}^N} p^n(s,x,\xi)p^n(t,\xi,y)\, d\xi = \int_{\mathbf{R}^N} p(s,x,\xi)p(t,\xi,y)\, d\xi. \end{aligned}$$

Next, define $[P_t\phi](x) = \int_{\mathbf{R}^N} \phi(y)p(t,x,y)\, dy$ for $\phi \in C_b(\mathbf{R}^N)$. Clearly $\{P_t : t > 0\}$ is a strongly Feller continuous, Markov semigroup. Moreover, by the preceding, we see that each P_t is a symmetric contraction on $L^2(\mathbb{R}^N)$ and that $[P_t^n\phi] \longrightarrow [P_t\phi]$ in $L^2(\mathbb{R}^N)$ for each $\phi \in C_0^\infty(\mathbb{R}^N)$. In particular (cf. Lemma I.0.4), $\{P_t : t > 0\}$ determines a unique strongly continuous semigroup $\{\overline{P}_t : t > 0\}$ of selfadjoint contractions on $L^2(\mathbb{R}^N)$; and, by (I.0.5) applied to the P_t^n's, each P_t maps $C_0^\infty(\mathbb{R}^N)$ into $W_2^{(1)}(\mathbb{R}^N)$ and satisfies (I.0.5). (Recall that if $\{\psi_n\}_1^\infty$ is a bounded sequence in $W_2^{(1)}(\mathbb{R}^N)$ and if $\psi_n \longrightarrow \psi$ in $L^2(\mathbb{R}^N)$, then $\psi \in W_2^{(1)}(\mathbb{R}^N)$, $\psi_n \longrightarrow \psi$ weakly in $W_2^{(1)}(\mathbb{R}^N)$, and therefore $\|\psi\|_2^{(1)} \le \underline{\lim}_{n\to\infty} \|\psi_n\|_2^{(1)}$.) From these considerations, we see that $\{\overline{P}_t : t > 0\}$ is weakly continuous on $W_2^{(1)}(\mathbb{R}^N)$; and therefore, by general semigroup theory, it must also be strongly continuous there.

To see that (II.3.2) holds, note that it holds for each of the semigroups $\{P_t^n : t > 0\}$, and use the facts that $[P_t^n\phi] \longrightarrow [P_t\phi]$ (strongly) in $L^2(\mathbb{R}^N)$ and weakly in $W_2^{(1)}(\mathbb{R}^N)$ together with (I.0.5) in order to justify the passage to the limit.

To prove the uniqueness assertion, suppose that $\{P_t' : t > 0\}$ is a second Feller continuous Markov semigroup which satisfies i) and ii). Then, for $T > 0$ and $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$, $t \in [0,T] \longmapsto (P_t'\phi, P_{T-t}\psi)$ is once continuously differentiable and

$$\frac{\partial}{\partial t}(P_t'\phi, P_{T-t}\psi) = 0, \quad t \in (0,T).$$

Hence, $([P_T'\phi],\psi) = (\phi,[P_T\psi]) = ([P_T\phi],\psi)$; and so $P_t' = P_t$, $t \in (0,\infty)$.

Finally, suppose that $\{a^n\}_1^\infty \subseteq \mathcal{A}(\lambda)$ and that $a^n \longrightarrow a$ almost everywhere. Arguing as in the preceding proof of existence, one sees that $\{p^n(t,x,y)\}_1^\infty$ is relatively compact with respect to local uniform convergence and that every convergent subsequence gives rise to a Feller continuous, Markov semigroup which satisfies i) and ii). Hence, by uniqueness, we conclude that $p^n(t,x,y) \longrightarrow p(t,x,y)$ uniformly on compacts. Certainly this means that, for each $\phi \in C_0^\infty(\mathbb{R}^N)$, $[P_t^n\phi] \longrightarrow [P_t\phi]$ in $L^2(\mathbb{R}^N)$ uniformly on compacts. To see that $[\nabla P_t^n\phi] \longrightarrow [\nabla P_t\phi]$ in $L^2(\mathbb{R}^N;\mathbb{R}^N)$, set $u_n(t) = ([\nabla P_t^n\phi], a^n[\nabla P_t^n\phi])$ and $u(t) = ([\nabla P_t\phi], a[\nabla P_t\phi])$. Because $[\nabla P_t^n\phi]$ tends weakly to $[\nabla P_t\phi]$ in $L^2(\mathbb{R}^N;\mathbb{R}^N)$, we know that $\overline{\lim}_{n\to\infty} u_n(t) \le u(t)$ for each $t \in [0,\infty)$. On the other hand, using the spectral theorem to represent $u_n(t)$ (cf. (I.0.9)), one sees that $\{u_n\}_1^\infty$ is equicontinuous on $[0,\infty)$. Thus, if $\underline{\lim}_{n\to\infty} u_n(T) < u(T)$ for some $T \ge 0$, then we would have $\underline{\lim}_{n\to\infty} \int_0^{T+1} u_n(t)\,dt < \int_0^{T+1} u(t)\,dt$. But

$$2\int_0^{T+1} u_n(t)\,dt = \|\phi\|_2^2 - \|[P_{T+1}^n\phi]\|_2^2 \longrightarrow \|\phi\|_2^2 - \|[P_{T+1}\phi]\|_2^2 = 2\int_0^{T+1} u(t)\,dt;$$

and so it must be that $u_n(T) \longrightarrow u(T)$ for each $T \ge 0$. From this it is clear that $\|[\nabla P_t^n\phi]\|_2^2 \longrightarrow \|[\nabla P_t\phi]\|_2^2$; which, together with the corresponding weak convergence, implies that $[\nabla P_t^n\phi] \longrightarrow [\nabla P_t\phi]$ in $L^2(\mathbb{R}^N;\mathbb{R}^N)$. ∎

We next turn to the non-symmetric case. Given $\Lambda \in (0,\infty)$, use $\mathcal{B}(\Lambda)$ to denote the class of measurable $b : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ such that $|b(\cdot)| \le \Lambda$. We want to study the semigroup corresponding to "$\mathcal{L} = \nabla \cdot a\nabla + b \cdot a\nabla$," where $a \in \mathcal{A}(\lambda)$ and $b \in \mathcal{B}(\Lambda)$.

Lemma II.3.3. *Suppose that $a \in \mathcal{A}(\lambda)$ and $b \in \mathcal{B}(\Lambda)$, and let $\{P_t : t > 0\}$ denote the semigroup in* Theorem *II.3.1 corresponding to a. Then for each $\phi \in W_2^{(1)}(\mathbb{R}^N)$ there* *is at most one weakly continuous map* $t \in [0,\infty) \longmapsto \phi_t \in W_2^{(1)}(\mathbf{R}^N)$ *which satisfies* (II.3.4)

$$(\phi_t,\psi) = ([P_t\phi],\psi) + \int_0^t (\nabla\phi_t, a[P_{t-s}\psi]b)\,ds, \quad t \in [0,\infty) \text{ and } \psi \in C_0^\infty(\mathbf{R}^N).$$

In fact, if $t \longmapsto \phi_t$ *is such a map, then*

$$\|\phi_t\|_2^{(1)} \le \mu e^{\mu t}\|\phi\|_2^{(1)}, \quad t \in [0,\infty), \tag{II.3.5}$$

where $\mu = \mu(\lambda,\Lambda) \in (0,\infty)$. *Finally, if* $t \in [0,\infty) \longmapsto \phi_t \in W_2^{(1)}(\mathbf{R}^N)$ *is a weakly continuous map and*

$$(\phi_t,\psi) = (\phi,\psi) + \int_0^t (\nabla\phi_s, a(\psi b - \nabla\psi))\,ds, \quad t \in [0,\infty) \text{ and } \psi \in C_0^\infty(\mathbf{R}^N) \tag{II.3.6}$$

for some $\phi \in W_2^{(1)}(\mathbf{R}^N)$, *then* $t \longmapsto \phi_t$ *is the unique solution to* (II.3.4).

PROOF: If we prove (II.3.5), then the uniqueness statement will follow trivially.

Before proving (II.3.5), first recall the space $W_2^{(-1)}(\mathbf{R}^N)$ of (slightly) generalized functions which is the dual space of $W_2^{(1)}(\mathbf{R}^N)$ when one uses the $L^2(\mathbf{R}^N)$-inner product to define the duality relation. Thus, the norm $\|\cdot\|_2^{(-1)}$ on $W_2^{(-1)}(\mathbf{R}^N)$ is defined by

$$\|\psi\|_2^{(-1)} = \sup\{\,(\phi,\psi) : \phi \in C_0^\infty(\mathbf{R}^N) \text{ with } \|\phi\|_2^{(1)} \le 1\}$$

for $\psi \in C_0^\infty(\mathbf{R}^N)$; and the space $W_2^{(-1)}(\mathbf{R}^N)$ is obtained by completing $C_0^\infty(\mathbf{R}^N)$ with respect to this norm. In particular, we have that

$$\|\phi\|_2^{(1)} = \sup\{\,(\phi,\psi) : \psi \in C_0^\infty(\mathbf{R}^N) \text{ with } \|\psi\|_2^{(-1)} \le 1\};$$

and so, by (I.0.5) and symmetry,

$$\|[P_t\psi]\|_2 \le \frac{1}{(\lambda t)^{1/2}}\|\psi\|_2^{(-1)}, \quad t \in (0,\infty) \text{ and } \psi \in C_0^\infty(\mathbf{R}^N). \tag{II.3.7}$$

Returning to the proof of (II.3.5), note that (II.3.4) together with (I.0.5) and (II.3.7) yield

$$|(\phi_t,\psi)| \le \frac{1}{\lambda^{1/2}}\|\phi\|_2^{(1)}\|\psi\|_2^{(-1)} + \frac{2\Lambda t^{1/2}}{\lambda^{3/2}} \sup_{s\in[0,t]} \|\phi_s\|_2^{(1)}\|\psi\|_2^{(-1)}$$

for $t \in [0,\infty)$ and $\psi \in C_0^\infty(\mathbf{R}^N)$; and therefore we can choose $T = T(\lambda,\Lambda) \in (0,\infty)$ so that $\|\phi_t\|_2^{(1)} \le \frac{2}{\lambda^{1/2}}\|\phi\|_2^{(1)}$ for all $t \in [0,T]$. Working by induction, one concludes that $\|\phi_t\|_2^{(1)} \le \left(\frac{2}{\lambda^{1/2}}\right)^n \|\phi\|_2^{(1)}$ for $nT \le t \le (n+1)T$; and so the required $\mu(\lambda,\Lambda) \in (0,\infty)$ is seen to exist.

Finally, suppose that $t \in [0,\infty) \longmapsto W_2^{(1)}(\mathbf{R}^N)$ is a weakly continuous solution to (II.3.6). Then, one can easily check that $(s,t) \in [0,\infty)^2 \longmapsto (\phi_s,[P_t\psi])$ is a once continuously differentiable function for each $\psi \in C_0^\infty(\mathbf{R}^N)$ and that

$$\frac{\partial}{\partial s}(\phi_s,[P_{t-s}\psi]) = (\nabla\phi_s, a[P_{t-s}\psi]b), \quad s \in [0,t].$$

Hence, (II.3.4) follows by integration. ∎

THEOREM II.3.8. *For each $a \in (\lambda)$ and $b \in \mathcal{B}(\Lambda)$ there is a unique Feller continuous, Markov semigroup $\{Q_t : t > 0\}$ such that*

i) The map $t \in [0,\infty) \longmapsto [Q_t\phi] \in W_2^{(1)}(\mathbb{R}^N)$ is weakly continuous for each $\phi \in C_0^\infty(\mathbb{R}^N)$.

ii) For all $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$,

$$([Q_t\phi], \psi) = (\phi, \psi) + \int_0^t ([\nabla Q_s\phi], a(\psi b - \nabla\psi))\, ds, \quad t \in [0,\infty). \tag{II.3.9}$$

In fact, $\{Q_t : t > 0\}$ determines a unique strongly continuous semigroup $\{\overline{Q}_t : t > 0\}$ on $L^2(\mathbb{R}^N)$, $\{\overline{Q}_t : t > 0\}$ is strongly continuous on $W_2^{(1)}(\mathbb{R}^N)$, and there is a $\mu = \mu(\lambda, \Lambda N) \in (0,\infty)$ such that

$$\|\overline{Q}_t\phi\|_2 \le \mu e^{\mu t}\|\phi\|_2 \text{ and } \|\overline{Q}_t\phi\|_2^{(1)} \le \mu e^{\mu t}\|\phi\|_2^{(1)} \tag{II.3.10}$$

for ϕ in $L^2(\mathbb{R}^N)$ and $W_2^{(1)}(\mathbb{R}^N)$, respectively. Moreover, there exists a measurable $q : (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,\infty)$ satisfying (II.2.4) and (II.2.13) such that

$$[Q_t\phi](x) = \int_{\mathbb{R}^N} \phi(y) q(t,x,y)\, dy, \quad t \in (0,\infty) \text{ and } \phi \in C_b(\mathbb{R}^N).$$

In particular, $\{Q_t : t > 0\}$ is strongly Feller continuous. Finally, if $\{a^n\}_1^\infty \subseteq \mathcal{A}(\lambda)$, $\{b^n\}_1^\infty \subseteq \mathcal{B}(\Lambda)$ and if $a^n \longrightarrow a$ and $b^n \longrightarrow b$ almost everywhere, then, for all bounded measurable $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$, $[Q_t^n\phi](x) \longrightarrow [Q_t\phi](x)$ uniformly on compact subsets of $(0,\infty) \times \mathbb{R}^N$.

PROOF: We first observe that uniqueness is an immediate consequence of Lemma II.3.3. We next prove existence. For this purpose, choose

$$\{a^n\}_1^\infty \subseteq \mathcal{A}(\lambda) \cap C_b^\infty(\mathbb{R}^N; \mathbb{R}^N \otimes \mathbb{R}^N) \text{ and } \{b^n\}_1^\infty \subseteq \mathcal{B}(\Lambda) \cap C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$$

so that $a^n \longrightarrow a$ and $b^n \longrightarrow b$ almost everywhere. By (II.2.4) and (II.2.13), we may and will assume that there is a measurable $q : (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,\infty)$, which again satisfies (II.2.4) and (II.2.13), such that for every bounded measurable $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$

$$[Q_t^n\phi](x) \longrightarrow [Q_t\phi](x) \equiv \int_{\mathbb{R}^N} \phi(y) q(t,x,y)\, dy$$

uniformly with respect to (t,x) in compacts. In particular, one sees that $\{Q_t : t > 0\}$ is a strongly Feller continuous semigroup and that, for $\phi \in C_0^\infty(\mathbb{R}^N)$, $[Q_t^n\phi] \longrightarrow [Q_t\phi]$ in $L^2(\mathbb{R}^N)$ uniformly with respect to t in compacts. Also, note that, by (II.2.4), $\|[Q_t\phi]\|_2 \le \mu e^{\mu t}\|\phi\|_2$ for some $\mu = \mu(\lambda, \Lambda, N) \in (0,\infty)$ and that, by Lemma II.3.3, for each $\phi \in W_2^{(1)}(\mathbb{R}^N)$, $\sup_{n\ge 0} \|[\overline{Q}_t^n\phi]\|_2^{(1)}$ is uniformly bounded, where $\{\overline{Q}_t : t > 0\}$ is the strongly continuous semigroup which $\{Q_t : t > 0\}$ determines on $L^2(\mathbb{R}^N)$. In particular, we have that $[Q_t^n\phi] \longrightarrow [Q_t\phi]$ weakly in $W_2^{(1)}(\mathbb{R}^N)$ uniformly with respect to t in compact subsets of $[0,\infty)$ for each $\phi \in C_0^\infty(\mathbf{R}^N)$. Hence, since, for each $\phi \in C_0^\infty(\mathbf{R}^N)$, $t \longmapsto [Q_t^n \phi]$ satisfies (II.3.8) with a^n and b^n, we can conclude that $t \longmapsto [Q_t\phi] \in W_2^{(1)}(\mathbf{R}^N)$ is a weakly continuous solution to (II.3.9); and another application of Lemma II.3.3 shows that $\{\overline{Q}_t : t > 0\}$ is weakly, and therefore strongly, continuous on $W_2^{(1)}(\mathbf{R}^N)$ and satisfies (II.3.10) for some μ with the required dependence.

Finally, the asserted convergence result is an easy consequence of the compactness provided by (II.2.4) and (II.2.13) plus uniqueness. ∎

It is possible to make an improvement in the convergence part of Theorem II.3.7. Namely, the conclusion continues to hold even when one assumes that the b^n 's tend to b only in the sense of weak convergence (i.e. in the sense of distributions.) In order to prove this improved statement, we will require the following preliminary result.

LEMMA II.3.11. *For each $\delta \in (0,1)$, $\lambda \in (0,1)$, and $\Lambda \in (0,\infty)$ there is a non-decreasing function $\alpha : (0,\infty) \longrightarrow (0,\infty)$ with the properties that $\lim_{\rho\to 0}\alpha(\rho) = 0$ and that for all $a \in \mathcal{A}(\lambda)$ and $b \in \mathcal{B}(\Lambda)$*

(II.3.12)

$$\sup_{x\in\mathbf{R}^N}\int_\delta^{1/\delta}\int_{\mathbf{R}^N} |q(s+\sigma,x,y+\xi) - q(s,x,y)|\,dyds \le \alpha(\sigma \vee |\xi|), \quad \sigma \ge 0 \textit{ and } \xi \in \mathbf{R}^N.$$

PROOF: By (II.2.4) and (II.2.13), it suffices for us to show that for each $T > 0$

$$\sup_{x\in\mathbf{R}^N}\int_0^T |q(s+\sigma,x,y+\xi) - q(s,x,y)|\,dy \longrightarrow 0 \text{ as } |\xi| \to 0$$

at a rate which is independent of $a \in \mathcal{A}(\lambda)$ and $b \in \mathcal{B}(\Lambda)$. Moreover, we may and will restrict our attention to a 's and b 's which have bounded continuous derivatives of all orders. But, for such coefficents, one knows that the measures Q_x corresponding to a and b are absolutely continuous on $\mathcal{M}_T$ to the measures P_x associated with a and that the Radon-Nikodym derivative $R(T,x)$ is given by the expression

$$R(T,x) = \exp\left[\int_0^T b(x(t))\cdot d\beta(t,x) - \frac{1}{2}\int_0^T |b(x(t))|^2\,dt\right]$$

where

$$(\beta(t,x), \mathcal{M}_t, P_x)$$

is an $N-$dimensional Brownian Motion. In particular, $\|R(T,x)\|_{L^2(P_x)} \le \exp[\Lambda T/\lambda]$. Hence,

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbf{R}^N}\big(q(t,x,y+\xi) - q(t,x,y)\big)\phi(y)\,dydt\right| \\
&= \left|E^{P_x}\left[R(T,x)\int_0^T \big(\phi(x(t)-\xi) - \phi(x(t))\big)\,dt\right]\right| \\
&\le \exp[\Lambda T/\lambda]E^{P_x}\left[\left(\int_0^T \big(\phi(x(t)-\xi) - \phi(x(t))\big)\,dt\right)^2\right]^{1/2}
\end{aligned}$$

Finally,

$$E^{P_x}\left[\left(\int_0^T (\phi(x(t)-\xi)-\phi(x(t)))\,dt\right)^2\right]$$

$$= 2\int_0^T E^{P_x}\left[(\phi(x(s)-\xi)-\phi(x(s)))E^{P_{x(s)}}\left[\int_s^T (\phi(x(t)-\xi)-\phi(x(t)))\,dt\right]\right] ds$$

$$\leq 4T\|\phi\|^2_{C_b(\mathbb{R}^N)} \sup_{\eta\in\mathbb{R}^N}\int_0^T |p(t,\eta,y+\xi)-p(t,\eta,y)|\,dy;$$

and, by (I.0.10) and Corollary II.1.9, it is clear that

$$\sup_{\eta\in\mathbb{R}^N}\int_0^T |p(t,\eta,y+\xi)-p(t,\eta,y)|\,dy \longrightarrow 0 \text{ as } |\xi|\to 0$$

at a rate which is independent of $a\in\mathcal{A}(\lambda)$. ∎

THEOREM II.3.13. *Let $\{a^n\}_1^\infty \subseteq \mathcal{A}(\lambda)$ and $\{b^n\}_1^\infty \subseteq \mathcal{B}(\Lambda)$ be sequences. Let $a\in\mathcal{A}(\lambda)$ and $b\in\mathcal{B}(\Lambda)$, and assume that $a^n \longrightarrow a$ almost everywhere and that $b^n\longrightarrow b$ in the sense that*

$$\text{(II.3.14)}\qquad \int_{\mathbb{R}^N} b^n(y)\phi(y)\,dy \longrightarrow \int_{\mathbb{R}^N} b(y)\phi(y)\,y,\quad \phi\in C_0^\infty(\mathbb{R}^N).$$

Then, for all bounded measurable $\phi:\mathbb{R}^N\longrightarrow\mathbb{R}$, $[Q_t^n\phi](x)\longrightarrow[Q_t\phi](x)$ uniformly for (t,x) in compact subsets of $(0,\infty)\times\mathbb{R}^N$.

PROOF: We first note that for any $a\in\mathcal{A}(\lambda)$, $b\in\mathcal{B}(\Lambda)$, and $\psi\in L^2(\mathbb{R}^N)$, $t\in[0,\infty)\longmapsto[\hat{Q}_t\psi]\in L^2(\mathbb{R}^N)$ is the unique weakly continuous map $t\in[0,\infty)\longmapsto\psi_t\in L^2(\mathbb{R}^N)$ such that

$$\text{(II.3.15)}\qquad (\phi,\psi_t)=(\phi,\psi)+\int_0^t([\nabla P_{t-s}\phi],a\psi_s b)\,ds,\quad t>0 \text{ and } \phi\in C_0^\infty(\mathbb{R}^N),$$

where $\{\hat{Q}_t : t>0\}$ denotes the semigroup which is adjoint (in $L^2(\mathbb{R}^N)$) to $\{\overline{Q}_t : t>0\}$. The proof of uniqueness is very much like the proof of the uniqueness statement in Lemma II.3.3, and the fact that $t\longmapsto[\hat{Q}_t\psi]$ satisfies (II.3.15) is easy when a and b have bounded continuous derivatives of all orders and follows in the general case after taking limits. The details are left as an exercise.

Now let $\psi\in C_0^\infty(\mathbb{R}^N)$ be given, and set $\psi_t^n=[\hat{Q}_t^n\psi]$. If we can show that $\psi_t^n\longrightarrow[\hat{Q}_t\psi]$ in $L^2(\mathbb{R}^N)$ for each $t>0$, then we will be done. Because of (II.2.4) and (II.2.13), we know that there is a weakly continuous $t\in[0,\infty)\longmapsto\psi_t\in L^2(\mathbb{R}^N)$ to which a subsequence of $\{\psi_t^n\}_1^\infty$ converges weakly in $L^2(\mathbb{R}^N)$, uniformly for t's in compacts. Thus, we need only show that such a $t\longmapsto\psi_t$ satisfies (II.3.15); and, in doing so, we will simplify the notation by supposing that the original sequence converges weakly in $L^2(\mathbb{R}^N)$ to ψ_t, uniformly for t's in compacts. But, for each $n \geq 1$, ψ_t^n satisfies (II.3.15) when $[P_{t-s}\phi]$ and a are replaced by $[P_{t-s}^n\phi]$ and a^n, respectively. Moreover, we know that $[P_{t-s}^n\phi] \longrightarrow [P_{t-s}\phi]$ in $W_2^{(1)}(\mathbb{R}^N)$, boundedly for $s \in [0,t]$. In particular, $a^n[\nabla P_{t-s}^n\phi] \longrightarrow [\nabla P_{t-s}\phi]$ in $L^2(\mathbb{R}^N)$, boundedly for $s \in [0,t]$. At the same time, $\sup_{s\in[0,t]} \|\psi_s^n\|_2$ is bounded indepentent of $n \geq 1$, and, by Lemma II.2.13, $\int_\delta^t \|\psi_s^n - \psi_s\|_{L^2(\mathbb{R}^N)}^2 \, ds \longrightarrow 0$ for every $\delta \in (0,t)$. After combining these remarks, one can easily deduce that $t \longmapsto \psi_t$ does indeed satisfy (II.3.15). ∎

CONCLUDING REMARK.

It should be obvious that the results obtained in this section can be used to construct a diffusion process on Ω (cf. the beginning of Section II.1) corresponding to any one of the semigroups discussed herein. In addition, the convergence results for the semigroups give rise to weak convergence of the corresponding measures on Ω. It remains an open and challenging problem to provide a better probabilistic interpretation of these essentially analytic facts.

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