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THE STATISTICAL EQUILIBRIUM OF AN ISOTROPIC STOCHASTIC FLOW WITH NEGATIVE LYAPOUNOV EXPONENTS IS TRIVIAL

R.W.R.Darling and Yves Le Jan

Summary. It is well known that every stochastic flow on $\mathbb{R}_d$, whose one-point motion has an invariant measure m, gives rise to a measure-valued process $\{\nu_t, t\geq 0\}$ with $\nu_0 = m$, which converges almost surely to a random measure ν_∞, called the statistical equilibrium. We prove here that if the flow is spatially homogeneous and isotropic, and if either the covariance is smooth and the top Lyapounov exponent is strictly negative, or if the flow is "of coalescing type" (these phenomena can only occur when $d \leq 3$), then $\nu_\infty = 0$ a.s.

1. Isotropic stochastic flows on $\mathbb{R}^d$ with C^4 covariances.

A rather complete description of isotropic stochastic flows on $\mathbb{R}^d$, $d \geq 2$, with a smooth covariance structure, may be found in the papers of Baxendale and Harris [1], and of Le Jan [4]. We shall reproduce here only a brief definition, and a few formulas that we need.

A stochastic flow on $\mathbb{R}^d$ is a family of random mappings (actually diffeomorphisms in the cases studied in [1] and [4]) from $\mathbb{R}^d$ to itself, denoted $\{X_{st}, 0 \leq s \leq t < \infty\}$, such that $X_{tu} \circ X_{st} = X_{su}$ if $s \leq t \leq u$, X_{ss} is the identity map, and X_{st}, $X_{s't'}$, ... are independent if $s \leq t \leq s' \leq t' \leq \ldots$. In certain contexts it is helpful to consider "backwards time" flows, i.e. $\{X_{st}, -\infty < s \leq t \leq 0\}$. We treat the case where:

(a) For each x in $\mathbb{R}^d$, the one-point motion $\{X_{0t}(x), t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}^d$ with $X_{00}(x) = x$.

(b) The covariance structure for the motions of k points, $k \geq 2$, is spatially homogeneous, and isotropic in the vector sense (see Baxendale and Harris [1]). Such a structure is obtained as follows:

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Take any pair of positive, finite measures F_L and F_N on $(0,\infty)$, with finite fourth moments, suitably normalized. Let

(1.1) $$A = d^{-1}\int \rho^2 F_L(d\rho), \quad B = d^{-1}\int \rho^2 F_N(d\rho).$$

From F_L and F_N we calculate the "longitudinal and transverse correlation functions" B_L and B_N, which are scalar functions on $(0,\infty)$ tending to zero at infinity, satisfying

(1.2) $$B_N(r) = 1 - \beta r^2 + O(r^4),$$
$$B_L(r) - B_N(r) = -\gamma r^2 + O(r^4), \text{ as } r \downarrow 0,$$

where β and γ are defined in terms of (1.1) by

$$A = 2\beta + (d+1)\gamma, \quad B = 2\beta - \gamma.$$

From B_L and B_N we obtain a C^4 $d\times d$ matrix-valued function b on $\mathbb{R}^d$, called the covariance matrix, by the formula

$$b^{pq}(z) = (B_L(|z|) - B_N(|z|))z^p z^q/|z|^2 + B_N(|z|)\delta^{pq}, \ z \neq 0.$$

The law of an isotropic stochastic flow is uniquely specified by this covariance matrix, because the latter determines, for each $k \geq 2$, the generator of the k-point motion, namely

$$A^{(k)}f = \tfrac{1}{2} \Sigma_{1\leq i,j\leq k} \Sigma_{1\leq p,q\leq d}\, b^{pq}(z_i - z_j)\, \partial^2 f/\partial z_i^p \partial z_j^q, \ f \in C_0^\infty((\mathbb{R}^d)^k).$$

Given an isotropic stochastic flow of this type, the "distance process" $d_t = |X_{0,t}(x) - X_{0,t}(y)|$, for given x and y in $\mathbb{R}^d$, is a one-dimensional diffusion on $(0,\infty)$ with generator

(1.3) $$Rg(r) = (1 - B_L(r))g''(r) + (d-1)r^{-1}(1 - B_N(r)g'(r).$$

The almost sure limiting behaviour of the distance process may be described in terms of the scale function $\int_1^r s'(u)du$ and the top Lyapounov exponent λ_1 (see Le Jan [4,p.618]); namely

for $d = 2$: $\lambda_1 < 0 \Rightarrow$

(1.4) $$|X_{0,t}(x) - X_{0,t}(y)| \to 0 \text{ a.s for all } x,y \in \mathbb{R}^d.$$

for $d = 3$: $\lambda_1 < 0 \Rightarrow \int_0^\infty s'(u)du < \infty$, and

$$P(|X_{0,t}(x) - X_{0,t}(y)| \to \infty) = 1 - P(|X_{0,t}(x) - X_{0,t}(y)| \to 0)$$

$$\equiv \eta(|x-y|) = \frac{\int_0^{|x-y|} s'(u)du}{\int_0^{\infty} s'(u)du}$$

(1.5) $$\sim c|x-y|^{-\mu-1} \text{ as } |x-y| \downarrow 0,$$

where

(1.6) $$\mu = -2 + \frac{(d-1)[A + (d+1)B]}{3A + (d-1)B} = -2 + \frac{(d-1)\beta}{\beta + \gamma}.$$

Recall that

(1.7) $$\lambda_1 = \frac{1}{2(d+2)}[(d-4)A + d(d-1)B].$$

(= (B-A)/4 if d = 2, and = (6B-A)/10 if d = 3). Thus λ_1 can be strictly negative only when $d \le 3$, and (1.6) shows that in that case

(1.8) $$\mu \in \begin{cases} (-5/3, -1) \text{ if } d = 2, \\ (-4/3, -1) \text{ if } d = 3, \end{cases}$$

which verifies that $\eta(|x-y|) \to 0$ as $|x-y| \to 0$.

To calculate the density of an invariant measure for the distance process $(d_t, t \ge 0)$, we solve the adjoint equation $R^*u = 0$. One solution h is the invariant measure with respect to which the transition semigroup is self-adjoint, namely:

(1.9) $$\ln h(r) = -\ln(1-B_L(r)) + (d-1)\ln r - (d-1)\int_r^{\infty} \frac{B_L(s) - B_N(s)}{s(1 - B_L(s))} ds.$$

The asymptotics of h are:

(1.10) $$h(r) \sim \begin{cases} r^{d-1} \text{ as } r \to \infty, \\ c_0 r^{\mu} \text{ as } r \to 0, \text{ where } c_0 > 0. \end{cases}$$

To find a solution g to "$R^*u = 0$" which is independent of h, we write g(r) = h(r).k(r) and solve for k; we find that

(1.11) $$\frac{g(r)}{h(r)} = c\int_1^r \left(\frac{1}{h(t)(1-B_L(t))}\right)^2 \left(\exp\int_1^t \frac{(d-1)(1-B_N(s))}{s(1-B_L(s))} ds\right)dt + c'.$$

When d=3, we verify below that a suitable choice of constants in (1.11) gives a solution g with the following asymptotics:

(1.12) $g(r) \sim \begin{cases} c_1 r^{-1} \text{ as } r \downarrow 0, \text{ for some } c_1 > 0; \\ -c_2 r \text{ as } r \uparrow \infty, \text{ for some } c_2 > 0; \end{cases}$

In particular, since $-4/3 < \mu < -1$ when $d=3$, it follows from (1.10) that for all $a \in \mathbb{R}$,

(1.13) $a g(r) + h(r) \sim h(r)$ as $r \downarrow 0$.

The verification of (1.12) goes as follows: for simplicity, suppose $c > 0$ in (1.11). Then according to (1.2), (1.6), (1.10) and (1.11), for some $c_1 > 0$,

$$\frac{g(r)}{h(r)} \sim c_3 \int_1^r t^{-2\mu-4} \exp\left(\frac{2\beta}{\beta+\gamma} \ln t\right) dt + c_4 \quad \text{as } r \downarrow 0,$$

$$= c_3 \int_1^r t^{-4-2\mu+\mu+2} dt + c_4 = \frac{c_3}{-\mu-1} r^{-1-\mu} + c_4, \text{ as } r \downarrow 0.$$

and

$$\frac{g(r)}{h(r)} \sim c_5 \int_1^r t^{-4} \exp(2 \ln t)\, dt + c_6$$

$$= c_5 \int_1^r t^{-2} dt + c_6 = c_6 - c_7 r^{-1}, \text{ as } r \uparrow \infty.$$

Setting $c_4 = c_6 = 0$ (to remove the multiples of h), we see that g(r) has the desired asymptotic behaviour.

Statistical equilibrium.

Since the one-point motion is Brownian, it has the Lebesgue measure on $\mathbb{R}^d$, denoted dx, as its invariant measure. As in Le Jan [4], [5], we study here the processes $\{\nu_{s,t},\ s \le t\}$, for $t \in (-\infty, \infty)$, taking values in the space of (unnormalized) mass distributions on $\mathbb{R}^d$, defined by

(1.14) $\langle \varphi, \nu_{s,t}(\omega) \rangle = \int \varphi(X_{st}(\omega)(x))\, dx, \quad \varphi \in C_K(\mathbb{R}^d)$.

(Here $C_K(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$ with compact support.) For a discussion of the "backwards-time" stochastic process $\{X_{st},\ -\infty < s \le t \le 0\}$, see Le Jan [5].

Le Jan [4] shows that, for each fixed t, $\{\langle\varphi,\nu_{s,t}\rangle,\ s\le t\}$ is a real-valued backwards martingale in s, and in fact the difference of two positive backwards martingales, and therefore converges almost surely as $s \downarrow -\infty$. Since the convergence occurs for all φ in $C_K(\mathbb{R}^d)$, this proves that for each fixed t, $\{\nu_{s,t},\ s\le t\}$ converges almost surely in the vague topology to a random measure $\nu_{-\infty,t}$ as $s \downarrow -\infty$. Since the flow is time-homogeneous, $\nu_{-\infty,t}$ has the same law for every t. We abbreviate $\nu_{-\infty,0}$ to $\nu_{-\infty}$.

Le Jan [4] has shown that, for an isotropic stochastic flow in $\mathbb{R}^d$ with $\lambda_1 > 0$, the martingales above are square integrable, and a nontrivial statistical equilibrium $\nu_{-\infty}$ exists which is a.s. diffuse when $A \neq 0$ (see (1.1) above), and which is simply Lebesgue measure when $A = 0$ (the volume-preserving case).

The purpose of this paper is to show that when $\lambda_1 < 0$, $\nu_{-\infty} = 0$ a.s., at least for the isotropic case. Our methods are analytic rather than geometric, and depend heavily on the fact that the distance process is a one-dimensional diffusion in the isotropic case. It would be interesting to have an alternative proof which would apply to the non-isotropic case, when $\lambda_1 < 0$. No result is known for the case $\lambda_1 = 0$, even for isotropic flows.

2. Isotropic stochastic flows on $\mathbb{R}^d$ of coalescent type.

On the basis of a suggestion of Harris, Darling has given a construction of isotropic stochastic flows in $\mathbb{R}^d$, for d = 2 [2] and d = 3 [3], for which the covariance function b(z) is not differentiable at z = 0, and instead of (1.2) we have

$$(2.1)\qquad \begin{aligned} B_N(r) &= 1 - \beta r^{\delta-1} + O(r^2),\\ B_L(r) - B_N(r) &= -\gamma r^{\delta-1} + O(r^2), \text{ as } r \downarrow 0, \end{aligned}$$

for some constant δ with $d-1<\delta<3$.

These flows are called coalescing because the distance process $\{d_t, t\ge 0\}$ may actually reach zero in finite time, whereupon it is absorbed at zero.

As in the smooth case, the distance process has an invariant measure h(r)dr, with respect to which the transition semigroup is self-adjoint, given by formula (1.9), i.e.

$$(2.2)\qquad \ln h(r) = -\ln(1-B_L(r)) + (d-1)\ln r - (d-1)\int_r^\infty \frac{B_L(s) - B_N(s)}{s(1 - B_L(s))}ds.$$

To see that the final integral is finite, note that by Darling [2,(16.21)] and [3, Appendix], $B_L(r)$ and $B_N(r)$ are $O(r^{-(d+1)/2})$ as $r \uparrow \infty$, and therefore the integrand in (2.2) is $O(s^{-(d+3)/2})$ as $s \uparrow \infty$, and so the integral is finite.

We need to know the asymptotics of h(r) for the coalescing case. As $r \downarrow 0$, (2.2) gives

$$\ln h(r) \sim \{-(\delta-1) + (d-1) - \gamma(d-1)/(\beta+\gamma)\}\ln r.$$

In the "pure potential" case studied in Darling [2], [3], $\gamma = (\delta-1)\beta$ (both for d=2 and d=3), and so

$$\ln h(r) \sim \{1 - \delta + (d-1)/\delta\}\ln r.$$

Referring to (2.2) again, we see that

$$(2.3) \qquad h(r) \sim \begin{cases} cr^{\theta} \text{ as } r \downarrow 0, \\ r^{d-1} \text{ as } r \uparrow \infty, \end{cases}$$

where

$$(2.4) \qquad \theta = 1 - \delta + (d-1)/\delta.$$

(Notice that the formula for θ is consistent with the formula (1.6) for μ, taking $\delta = 3$ in (2.4) and $B = 0$ in (1.6).)

The behaviour of the distance process is as follows: for $x,y \in \mathbb{R}^d$,

$$(2.5) \quad d = 2: \qquad |X_{0,t}(x) - X_{0,t}(y)| \to 0 \text{ a.s.}.$$

$$d = 3: \qquad P(|X_{0,t}(x) - X_{0,t}(y)| \to \infty) = 1 - P(|X_{0,t}(x) - X_{0,t}(y)| \to 0)$$

$$\equiv \eta(|x-y|) = \frac{\int_0^{|x-y|} s'(u)du}{\int_0^{\infty} s'(u)du}$$

$$(2.6) \qquad \sim c|x-y|^{1-2/\delta} \text{ as } |x-y| \downarrow 0,$$

where s(.) is the scale function, which satisfies $\int_0^\infty s'(u)du < \infty$: see Darling [2, Proposition 16.1] and [3, Appendix]. Observe that $1 - 2/\delta > 0$, by choice of δ.

As in the smooth case, there is another solution g to $R^*u = 0$, independent of h as in (2.2), such that $g(r)/h(r)$ satisfies (1.11). We shall demonstrate that when $d = 3$ the asymptotics of g(r) are as follows:

$$(2.7) \qquad g(r) \sim \begin{cases} c_1 r^{2-\delta} \text{ as } r \downarrow 0, \text{ for some } c_1 > 0 \\ -c_2 r \text{ as } r \uparrow \infty, \text{ for some } c_2 > 0. \end{cases}$$

To verify this, suppose $c > 0$ in (1.11); according to (1.11), (2.1) and (2.3),

$$\frac{g(r)}{h(r)} \sim c_3\int_1^r t^{-2\theta-2\delta+2}\exp\left(\frac{2\beta}{\beta+\gamma}\ln t\right)dt + c_4 \quad \text{as } r \downarrow 0,$$

Now $2\beta/(\beta+\gamma) = 2/\delta$, since $\gamma = (\delta-1)\beta$. Hence the last expression is

$$= c_3\int_1^r t^{-2\theta-2\delta+2+2/\delta}dt + c_4 = \frac{c_3}{1-2/\delta}r^{1-2/\delta} + c_4\ , \text{ as } r \downarrow 0,$$

since by (2.4), $-2\theta-2\delta+2 = -4/\delta$. Observe that (2.3) and (2.4) imply

$$h(r)r^{1-2/\delta} \sim r^{1-2/\delta+1-\delta+2/\delta} = r^{2-\delta} \text{ as } r \downarrow 0,$$

which verifies the first part of (2.7) on taking $c_4 = 0$. The proof of the second part is similar to that of (1.12).

3. The statistical equilibrium is trivial when $\lambda_1 < 0$.

THEOREM A

For an isotropic stochastic flow in $\mathbb{R}^d$ with C^4 covariance matrix, if the highest Lyapounov exponent is < 0 (which can only occur when d=2 or 3), then the statistical equilibrium $\nu_{-\infty}$ is zero almost surely.

Proof. As for any random measure on $\mathbb{R}^d$, there are just three alternatives for $\nu_{-\infty}$:

Case I: For almost all ω, $\nu_{-\infty}(\omega)$ is a Dirac measure, concentrated at some $Y(\omega)$ in $\mathbb{R}^d$.
Case II: There exists $\varepsilon > 0$, and non-negative φ_1, $\varphi_2 \in C_K(\mathbb{R}^d)$ such that $\varphi_1(x)\varphi_2(y) = 0$ whenever $|x-y| < \varepsilon$, such that $\mathbb{E}[\langle\varphi_1,\nu_{-\infty}\rangle\langle\varphi_2,\nu_{-\infty}\rangle] > 0$.
Case III: $\nu_{-\infty} = 0$ a.s.

It is easy to eliminate Case I, because if this were true, then $Y(\omega)$ would have to be uniformly distributed on $\mathbb{R}^d$ (by spatial homogeneity of the flow), and this is impossible when Y is a random variable. It remains to prove that Case II cannot occur when $\lambda_1 < 0$.

Suppose that the situation described in Case II is the correct one. Then by Fatou's Lemma and the almost sure convergence of $\langle\varphi_1,\nu_{-t,0}\rangle$ to $\langle\varphi_1,\nu_{-\infty}\rangle$ as $t \to \infty$,

$$0 < \mathbb{E}[\langle\varphi_1,\nu_{-\infty}\rangle\langle\varphi_2,\nu_{-\infty}\rangle] = \mathbb{E}[\lim_{t\to\infty} \langle\varphi_1,\nu_{-t,0}\rangle\langle\varphi_2,\nu_{-t,0}\rangle]$$
$$\le \liminf_{t\to\infty} \mathbb{E}[\langle\varphi_1,\nu_{-t,0}\rangle\langle\varphi_2,\nu_{-t,0}\rangle],$$
$$= \liminf_{t\to\infty} \iint \mathbb{E}[\varphi_1(X_{-t,0}(x))\,\varphi_2(X_{-t,0}(y))]\,dx\,dy,$$
$$(3.1)\qquad = \liminf_{t\to\infty} \iint P_t^{(2)}u(x,y)\,dx\,dy,$$

where $u(x,y) = \varphi_1(x)\varphi_2(y)$, and $\{P_t^{(2)},\ t\ge 0\}$ is the transition semigroup for the two-point motion.

As mentioned in Section 1, the interpoint distance process $\{|X_{0,t}(x) - X_{0,t}(y)|,\ t\ge 0\}$ is a one-dimensional diffusion with an invariant measure $h(r)dr$ on $[0,\infty)$. Therefore the two-point motion $\{(X_{0,t}(x),X_{0,t}(y)), t\ge 0\}$ has (by a change of variables) an invariant measure $H(x,y)\,dx\,dy$ on $\mathbb{R}^{2d}$, where $H(x,y) = h(|x-y|)|x-y|^{1-d}$. The self-adjointness of $P_t^{(2)}$ with respect to this invariant measure gives

$$\iint P_t^{(2)}u(x,y)\,dx\,dy = \iint P_t^{(2)}u(x,y)H(x,y)^{-1}H(x,y)\,dx\,dy$$

$$(3.2)\qquad = \iint \varphi_1(x)\varphi_2(y)P_t^{(2)}(1/H)(x,y)H(x,y)\,dx\,dy.$$

As described in Section 1, (1.6) and the asymptotics of $h(r)$ given in (1.10) show that

$$(3.3)\qquad 1/H(x,y) \sim \begin{cases} c|x-y|^{d-1-\mu} \text{ as } |x-y|\downarrow 0 \\ 1 \text{ as } |x-y|\uparrow \infty. \end{cases}$$

Since $\lambda_1 < 0$ by assumption, it follows from (1.8) that $\mu < -1$, and so $d-1-\mu > d$. Therefore there exists $M_1 > 0$ such that

$$(3.4)\qquad \sup_{x\ne y} 1/H(x,y) \le M_1.$$

According to the description of Case II, there exist $k > \varepsilon > 0$ such that $\varphi_1(x)\varphi_2(y) = 0$ unless $\varepsilon \le |x-y| \le k$. Define

$$(3.5)\qquad M_2 = \max_{\varepsilon\le r\le k}\{h(r)r^{1-d}\} < \infty.$$

By (3.4) and (3.5), the integrand in (3.2) satisfies:

$$(3.6)\qquad \varphi_1(x)\varphi_2(y)P_t^{(2)}(1/H)(x,y)H(x,y) \begin{cases} = 0 \text{ if } |x-y| > k \\ \le M_1M_2\langle\varphi_1,m\rangle\langle\varphi_2,m\rangle \text{ if } |x-y|\le k \end{cases}$$

(Here m denotes Lebesgue measure on $\mathbb{R}^d$.) Observe that the bound is uniform in t. We now consider the cases $d=2$ and $d=3$ separately.

The Case d = 2.

By (1.4), (3.3), (3.4) and Lebesgue's Bounded Convergence Theorem (LBCT),

(3.7) $$P_t^{(2)}(1/H)(x,y) = \mathbb{E}[1/H(X_{0,t}(x),X_{0,t}(y))] \to 0 \text{ as } t \to \infty,$$

for each x,y in $\mathbb{R}^d$. Now (3.2), (3.6), (3.7), and another application of the LBCT show that

$$\lim_{t\to\infty} \int\int P_t^{(2)}u(x,y)\,dx\,dy = 0,$$

which contradicts (3.1). Hence Case II cannot occur, which completes the proof when d = 2.

The Case d = 3.

It follows from (3.1), (3.2), and (3.4) that

(3.8) $$\mathbb{E}[\langle\varphi_1,\nu_{-\infty}\rangle\langle\varphi_2,\nu_{-\infty}\rangle] \le M_1\int\int\varphi_1(x)\varphi_2(y)H(x,y)\,dx\,dy,$$

where the constant M_1 is independent of φ_1 and φ_2. By the Riesz Representation Theorem and the Radon-Nikodym Theorem, there exists a non-negative measurable function G on $\mathbb{R}^d\times\mathbb{R}^d$ such that

(3.9) $$\mathbb{E}[\langle\varphi_1,\nu_{-\infty}\rangle\langle\varphi_2,\nu_{-\infty}\rangle] = \int\int\varphi_1(x)\varphi_2(y)G(x,y)\,dx\,dy.$$

Moreover the measure G(x,y)dxdy is invariant for the two-point motion, in the sense that (for $u(x,y) = \varphi_1(x)\varphi_2(y)$),

(3.10) $$\int\int P_t^{(2)}u(x,y)G(x,y)\,dx\,dy = \int\int u(x,y)G(x,y)\,dx\,dy, \text{ all } t\ge 0;$$

to verify (3.10), observe that by time-homogeneity of the flow, for any $t>0$, the right side may be written as:

$$\int\int\varphi_1(x)\varphi_2(y)G(x,y)\,dx\,dy = \mathbb{E}[\langle\varphi_1,\nu_{-\infty,t}\rangle\langle\varphi_2,\nu_{-\infty,t}\rangle],$$

$$= \mathbb{E}[\lim_{r\to\infty}\int\varphi_1(X_{-r,t}(x))\,dx\int\varphi_2(X_{-r,t}(y))\,dy], \text{ by (1.14) et sequ.,}$$

$$= \mathbb{E}[\lim_{r\to\infty}\int\varphi_1\circ X_{0,t}(X_{-r,0}(x))\,dx\int\varphi_2\circ X_{0,t}(X_{-r,0}(y))\,dy]$$

$$= \mathbb{E}[\langle\varphi_1\circ X_{0,t},\nu_{-\infty}\rangle\langle\varphi_2\circ X_{0,t},\nu_{-\infty}\rangle]$$

$$= \mathbb{E}[\ \int\int \varphi_1(X_{0,t}(x))\varphi_2(X_{0,t}(y))\, G(x,y)\, dx\, dy],$$

by the independence of $X_{0,t}$ and $\nu_{-\infty}$. By Fubini, this equals

$$= \int\int \mathbb{E}[\varphi_1(X_{0,t}(x))\varphi_2(X_{0,t}(y))]\, G(x,y)\, dx\, dy$$

$$= \int\int P_t^{(2)} u(x,y)\, G(x,y)\, dx\, dy,$$

which verifies (3.10).

On the other hand, (3.1) shows that for all φ_1, φ_2, as in Case II,

$$\int\int \varphi_1(x)\varphi_2(y) G(x,y)\, dx\, dy \le \liminf_{t\to\infty} \int\int P_t^{(2)} u(x,y)\, dx\, dy$$

$$\le \int\int \varphi_1(x)\varphi_2(y) M_1\, \eta(|x-y|)\, H(x,y)\, dx\, dy \tag{3.11}$$

by (3.2) and (3.4), where

$$\eta(|x-y|) = P(|X_{0,t}(x) - X_{0,t}(y)| \to \infty).$$

Since this inequality holds for all φ_1, φ_2 as in Case II, it follows that

$$G(x,y) \le M_1\, \eta(|x-y|)\, H(x,y). \tag{3.12}$$

From the translation-invariance and vector isotropy of the law of the flow, we obtain the fact that $G(x,y) = v(|x-y|)|x-y|^{-2}$ for some function $v:(0,\infty) \to \mathbb{R}$, such that $R^*v = 0$, where R is the generator of the distance process, as in (1.3). Let $g:(0,\infty) \to \mathbb{R}$ be the solution of $R^*u = 0$ which was described in (1.12). It is impossible that v is a multiple of g, because $g(|x-y|)dxdy$ is not a positive measure, by (1.12). Hence $v = a_1 g + a_2 h$ for some $a_1 \in \mathbb{R}$ and some $a_2 > 0$, such that $a_1 g(r) + a_2 h(r) \ge 0$ for all r. By (1.13), $v(r)/h(r) \sim a_2$ as $r \downarrow 0$. This contradicts (3.12), which says that $v(r)/h(r) = O(\eta(r)) = O(r^{-\mu-1})$ by (1.5); recall that $-\mu-1 > 0$ by (1.8).

Hence Case II cannot occur, and this completes the proof. □

THEOREM B

For an isotropic stochastic flow in $\mathbb{R}^2$ or $\mathbb{R}^3$ of "coalescent type" (as in Darling [2],[3]), the statistical equilibrium $\nu_{-\infty}$ is zero almost surely.

Proof. The proof is essentially the same as for Theorem A, except that:
(i) μ is replaced by θ in (3.3); according to (2.4),

$$d-1-\theta \in \begin{cases} (0,8/3) \text{ when } d=2, \\ (7/3,10/3) \text{ when } d=3. \end{cases}$$

Thus (3.4) still holds.

(ii) The asymptotics of $\eta(|x-y|)$ in (3.12) are given by (2.6) instead of (1.5); the same conclusions hold however. □

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