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Penetration Times and Skorohod Stopping

by

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1. Introduction.

By virtue of a theorem of Kuznetsov [14], given a Borel right semigroup (P_s) on a nice state space $(E, \mathcal{E})$, and a (σ-finite) excessive measure m, one can construct a stationary Markov process $(Y, Q_m) = (\{Y_t: t \in \mathbf{R}\}, Q_m)$ whose transition semigroup is (P_s), and whose one-dimensional distributions are all m. The process Y has random birth and death times, and the measure Q_m is σ-finite.

In a recent paper [4], B. Maisonneuve and the author have used (Y, Q_m) to investigate (among other things) certain "balayage" operations on the convex cone of excessive measures. In particular, a natural extension of Hunt's balayage $L_B m$ was defined in section 5 of [4]. (See also Getoor and Steffens [8,9] and Kaspi [13] for further work on this topic.)

Recall that if the potential kernel $U \equiv \int_0^\infty P_s ds$ is proper then any excessive measure m can be realized as the increasing limit of a sequence $\{\mu_n U\}$ of potentials. Following Hunt [11], one defines for $B \in \mathcal{E}$,

$$L_B m = \uparrow \lim_n \ \mu_n P_B U$$

where P_B is the hitting operator for B. From [11, Prop. 8.3] we know that if B is finely open then

$$L_B m = \wedge \{\xi \quad \text{excessive:} \quad \xi \geq m \quad \text{on} \quad B\}, \tag{1.1}$$

where $\wedge$ denotes infimum in the lattice of excessive measures. R. K. Getoor has asked whether (1.1) remains valid for the extended balayage of [4]. Proposition (2.7), our affirmative answer to this question, while hardly surprising, exploits an interesting connection with the Lebesgue penetration time of B. This result was proved in ignorance of the "semiclassical" potential theory of Kac [12] which concerns itself with such penetration times. Indeed, in the case of Brownian motion, (2.7) follows from work of Ciesielski [2] and Stroock [15,16].

In a third section we apply (2.7) to obtain a "Skorohod stopping" theorem. This result implies that a second excessive measure ξ, "weakly dominated" by m, can be represented as a balayage of m by means of a randomized terminal time.

2. Reduites and Penetration Times.

We recall from [4] the basic facts concerning the stationary process (Y, Q_m). Let $(E, \mathcal{E})$ be a Lusin state space for a Borel right semigroup (P_s). Let $\Delta \notin E$ be the cemetary point; any function f defined on E is extended to $E_\Delta \equiv E \cup \{\Delta\}$ by setting $f(\Delta) = 0$. Let W denote the space of paths $w: \mathbf{R} \to E_\Delta$ which are E-valued and right continuous on some open interval $]\alpha(w),\ \beta(w)[\subset \mathbf{R}$, and which take the value Δ outside $]\alpha(w),\ \beta(w)[$. The case $]\alpha(w),\ \beta(w)[= \phi$ corresponds to the dead path $[\Delta]: t \to \Delta$ for which $\alpha([\Delta]) = +\infty$, $\beta([w]) = -\infty$. Let $\{Y_t: t \in \mathbf{R}\}$ denote the coordinate process on W, and set $\mathcal{G}^0 = \sigma\{Y_t: t \in \mathbf{R}\}$, $\mathcal{G}^0_t = \sigma\{Y_s: s \leq t\}$. Shift operators are defined on W by

$$\begin{aligned} (\tau_t w)(s) &= w(t+s), \quad s > 0,\ t \in \mathbf{R}, \\ &= \Delta, \qquad\quad s \leq 0,\ t \in \mathbf{R}. \end{aligned}$$

Let $\Omega = \{w \in W: \alpha(w) = 0,\ Y_{\alpha+}(w) \text{ exists in } E\} \cup \{[\Delta]\}$, and let X_s, θ_s, $\mathcal{F}^0$, $\mathcal{F}^0_s$ denote the restrictions of Y_{s+}, τ_s, $\mathcal{G}^0$, $\mathcal{G}^0_s$ to Ω, where $s \geq 0$. Since (P_s) is a Borel right semigroup, there is a Borel measurable family $\{P^x: x \in E_\Delta\}$ of measures on $(\Omega, \mathcal{F}^0)$ such that $X = (\Omega, \mathcal{F}^0, \mathcal{F}^0_{t+}, X_t, \theta_t, P^x)$ is a strong Markov realization of (P_s). Note that for $t \in \mathbf{R}$ and $s \geq 0$, $\tau_t: \{\alpha < t\} \to \Omega$ and

$$X_s \circ \tau_t = Y_{t+s} \qquad \text{on} \qquad \{\alpha < t\}.$$

Let Exc denote the class of excessive measures for (P_s): $m \in \text{Exc}$ if and only if m is a σ-finite measure on E with $mP_s \leq m$, $s \geq 0$. Given $m \in \text{Exc}$ there is a unique measure Q_m on $(W, \mathcal{G}^0)$ such that $Q_m(\{[\Delta]\}) = 0$ and

$$Q_m(f \circ Y_t) = m(f), \qquad f \in \mathcal{E}^+, t \in \mathbf{R}; \tag{2.1}$$

$$Q_m(F \circ \tau_t \mid \mathcal{G}^0_{t+}) = P^{Y_t}(F) \qquad \text{a.e. } Q_m \text{ on } \{\alpha < t < \beta\}, \tag{2.2}$$

where $t \in \mathbf{R}$ and $F \in (\mathcal{G}^0)^+$. Note that (2.1) implies that Q_m restricted to $\mathcal{G}^0_{t+} \cap \{\alpha < t < \beta\}$ is σ-finite. (Indeed, (2.1) and (2.2) together imply that Q_m is σ-finite on $\mathcal{G}^0$.) The existence of Q_m follows from our hypotheses on (P_s) and a general theorem of Kuznetsov [14]. See also Getoor and Glover [7] for an excellent account of the construction of such measures. It is evident from (2.1) and (2.2) that (Y, Q_m) is *stationary*: if we define σ_t, $t \in \mathbf{R}$, by

$$(\sigma_t w)(s) = w(t+s), \qquad s, t \in \mathbf{R},$$

then $\sigma_t(Q_m) = Q_m$, $t \in \mathbf{R}$.

A balayage operation was defined in [4] as follows. Let $\mathcal{G}^*_t$ denote the universal completion of $\mathcal{G}^0_t$. Let $T\colon W \to [-\infty, +\infty]$ be a $(\mathcal{G}^*_{t+})$-stopping time such that $\alpha \le T < \beta$ on $\{T < +\infty\}$ and such that

$$t + T(\sigma_t w) = T(w), \qquad \forall\, t \in \mathbf{R}, \forall\, w \in W. \tag{2.3}$$

The *balayage of m via T* is the excessive measure $L_T m$ defined for $m \in$ Exc by

$$L_T m(f) = Q_m(f \circ Y_t; T < t), \quad f \in \mathcal{E}^+, \tag{2.4}$$

where $t \in \mathbf{R}$ is arbitrary. Evidently $L_T m \le m$, and $m \mapsto L_T m$ is an additive, positive homogeneous mapping of Exc into itself. Since $Q_m(T = t) = 0$ for all $t \in \mathbf{R}$, the condition $T < t$ in (2.4) can be replaced by $T \le t$.

A familiar example of a stopping time satisfying (2.3) is the hitting time $T_B \equiv \inf(t > \alpha\colon Y_t \in B)$, where B is Borel measurable. We write $L_B m$ instead of $L_{T_B} m$. It was shown in [4] that if $(\mu_n U)$ is a sequence of potentials increasing to m, then $\mu_n P_B U \uparrow L_B m$.

As a second example consider the Lebesgue penetration time of a set $B \in \mathcal{E}$:

$$\Pi_B \equiv \inf\,\{t > \alpha\colon \int_\alpha^t 1_B(Y_s)ds > 0\}.$$

Clearly $\Pi_B \ge T_B$, and Π_B is a $(\mathcal{G}^*_{t+})$-stopping time satisfying (2.3). Both T_B and Π_B satisfy the "terminal time" property

$$T = t + T \circ \tau_t \qquad \text{on} \qquad \{\alpha < t < T\}. \tag{2.5}$$

Let $B^* = \{x \in E: P^x(\Pi_B = 0) = 1\}$. Then $B^* \in \mathcal{E}$ and from Walsh [17] we know that $B \backslash B^*$ has zero potential, and that $\Pi_B = \Pi_{B^*} = T_{B^*}$ a.s. P^μ for all finite measures μ on E. It follows that for any $m \in \text{Exc}$, $m(B \backslash B^*) = 0$ and $\Pi_B = \Pi_{B^*} = T_{B^*}$ a.s. Q_m.

Finally, consider the réduite of $m \in \text{Exc}$ on $B \in \mathcal{E}$:

$$R_B m \equiv \wedge \{\xi \in \text{Exc}: \xi \geq m \quad \text{on} \quad B\}. \tag{2.6}$$

Here and elsewhere "$\xi \geq m$ on B" means $\xi(A) \geq m(A)$ for all Borel sets $A \subset B$. Note the following facts: $R_B m \in \text{Exc}$, $R_B m \leq m$ with equality on B; if $\xi \geq m$ on B then $R_B \xi \geq R_B m$; if $m(A \Delta B) = 0$ then $R_A m = R_B m$.

Here is our answer to Getoor's question, posed in section 1.

(2.7) Proposition. *For each $m \in \text{Exc}$ and $B \in \mathcal{E}$, $L_B m \geq L_{B^*} m = R_B m$. If $Q_m(T_B \neq \Pi_B) = 0$, then $L_B m = R_B m$. This is the case, for example, if B is finely open.*

Proof. Since $T_B \leq \Pi_B = T_{B^*}$ a.s. Q_m, we have $L_B m \geq L_{\Pi_B} = L_{B^*} m$. It follows easily from $m(B \backslash B^*) = 0$ that $L_{B^*} m = m$ on B; consequently $L_{B^*} m \geq R_B m$. It remains to show that $L_{B^*} m \leq R_B m$; for this we use an old trick, due to Hunt [11]. Given $h \in b\mathcal{E}^+$ note that on $\{\alpha < t < \beta\}$

$$1 - \exp\left(-\int_\alpha^t h(Y_s)ds\right) = \int_\alpha^t \exp\left(-\int_s^t h(Y_u)du\right) h(Y_s)ds. \tag{2.8}$$

Fix $\xi \in \text{Exc}$ with $\xi \geq m$ on B, and choose $h \in b\mathcal{E}^+$ with $\{h > 0\} = B$. By (2.8), (2.1), and (2.2),

$$\begin{aligned}
\xi(f) &\geq Q_\xi\left(f(Y_t)\left(1 - \exp\left(-\int_\alpha^t h(Y_s)ds\right)\right)\right) \\
&= \int_{-\infty}^t ds\, Q_\xi\left(h(Y_s)f(Y_t)\exp\left(-\int_0^{t-s} h(X_u)du\right) \circ \tau_s\right) \\
&= \int_{-\infty}^t ds\, Q_\xi\left(h(Y_s)P^{Y_s}\left(f(Y)_{t-s})\exp\left(-\int_0^{t-s} h(X_u)du\right)\right)\right) \\
&\geq \int_{-\infty}^t ds\, Q_m\left(h(Y_s)P^{Y_s}\left(f(Y_{t-s})\exp\left(-\int_0^{t-s} h(X_u)du\right)\right)\right) \\
&= Q_m\left(f(Y_t)\left(1 - \exp\left(-\int_\alpha^t h(Y_s)ds\right)\right)\right).
\end{aligned} \tag{2.9}$$

Let $(h_n) \subset b\mathcal{E}^+$ be an increasing sequence with $\{h_n > 0\} = B$ and $h_n \uparrow +\infty$ on B. Then

$$1 - \exp\left(- \int_\alpha^t h_n(Y_s)ds \right) \uparrow 1_{\{\Pi_B < t\}}$$

as $n \uparrow \infty$. Taking $h = h_n$ in (2.9) and letting $n \uparrow \infty$ we obtain

$$\xi(f) \geq Q_m(f(Y_t);\ \Pi_B < t) = L_{\Pi_B} m(f) = L_{B^*} m(f).$$

Thus $R_B m \geq L_{B^*} m$, and the proof of (2.7) is complete.

(2.10) Remark. A simple but important consequence of the identification $R_B m = L_{B^*} m$ is the observation that $m \mapsto R_B m$ is additive on Exc.

3. An Integral Representation Theorem.

The main result of this section is the integral representation theorem (3.1), a sort of Lebesgue decomposition for excessive measures. See (3.21) for an interpretation of (3.1) as a Skorohod stopping theorem.

(3.1) Theorem. *Let ξ and m be excessive measures. There is an increasing family $\{T(u): u \geq 0\}$ of $(\mathcal{G}^*_{t+})$-stopping times, each one satisfying (2.3) and (2.5), such that*

(3.2) $$\xi = \int_0^\infty L_{T(u)} m \, du + L_T \xi,$$

where $T \equiv \uparrow \lim_{u \uparrow \infty} T(u)$. If $\xi \leq r \cdot m$ for some $r > 0$, then

(3.3) $$\xi = \int_0^r L_{T(u)} m \, du.$$

To prove (3.1) we adapt an argument that Heath [10] ascribes to Mokobodzki. We first recall some potential theory; [1] and [3] are good sources for this material. If μ is a measure on $(E, \mathcal{E})$ dominated by some element of Exc, then the *réduite* $R\mu \in$ Exc is defined by

(3.4) $$R\mu = \wedge\{\xi \in \text{Exc}: \xi \geq \mu\}.$$

Evidently $\mu \mapsto R\mu$ is increasing, positive homogeneous, subadditive, and additive on Exc. If $m \in$ Exc, then $m = Rm$, and $R_A m = R(1_A \cdot m)$, $A \in \mathcal{E}$.

In the sequel, if γ and Γ are σ-finite measures, then an inclusion $\{\epsilon\gamma \leq \Gamma\} \subset A$ $(0 < \epsilon < 1,\ A \in \mathcal{E})$ should be interpreted as $\lambda(\{\epsilon g \leq G\} \backslash A) = 0$, where λ is a σ-finite measure dominating both γ and Γ, and where $g = d\gamma/d\lambda$, $G = d\Gamma/d\lambda$. We refer the reader to [1] or [3] for proofs of the following two lemmas, due to Mokobodzki.

(3.5) Lemma. *Let Γ be a measure on E such that $R\Gamma$ exists. Write $\gamma = R\Gamma$ and suppose that $\{\epsilon\gamma \leq \Gamma\} \subset A$ where $0 < \epsilon < 1$, $A \in \mathcal{E}$. Then $R_A\gamma = \gamma$.*

For the next lemma let ξ and m be excessive measures, and let μ be the smallest σ-finite measure dominating both ξ and m. Since $\mu \geq m$ there is a unique σ-finite measure ν such that $\mu = m + \nu$. We write $(\xi - m)_+$ for ν, and note that $R(\xi - m)_+$ exists since $(\xi - m)_+ \leq \xi$. In fact, $R(\xi - m)_+ = \wedge\{\gamma \in \text{Exc}: \gamma + m \geq \xi\}$.

(3.6) Lemma. *Let $\gamma = R(\xi - m)_+$ where ξ and m are excessive measures. Then there is a unique $\rho \in$ Exc such that $\gamma + \rho = \xi$. Moreover, $\rho \leq m$.*

We now proceed with the proof of (3.1). Fix ξ and m in Exc, and for $u \geq 0$ define

$$\gamma_u = R(\xi - u \cdot m)_+ \tag{3.7}$$

Clearly $u \mapsto \gamma_u$ is decreasing and the limit

$$\gamma_\infty = \downarrow \lim_{u\uparrow\infty} \gamma_u \tag{3.8}$$

is an excessive measure. Set $\Gamma_u = (\xi - u \cdot m)_+$ and note that if $f \in \mathcal{E}^+$ with $\xi(f) < \infty$, then $u \mapsto \Gamma_u(f)$ is decreasing and convex. Since R is "sublinear," $u \mapsto \gamma_u(f)$ is likewise convex. These facts in hand, it is not hard to produce $\mathcal{E}$-measurable, finite-valued densities $g_u = d\gamma_u/d(\xi + m)$, $G_u = d\Gamma_u/d(\xi + m)$, such that $u \mapsto g_u(x)$ and $u \mapsto G_u(x)$ are decreasing and convex in $u \geq 0$, for each $x \in E$. Set $b = dm/d(\xi + m)$ aand for $u \geq 0$, $\epsilon > 0$ define

$$A(u, \epsilon) = \{(1 + \epsilon u)g_0 \geq u \cdot b + g_u\} \supset \{g_0 \geq u \cdot b + (1 - \epsilon u)g_u\}.$$

Because of (3.5) we have

$$\gamma_v = R_{A(v,\epsilon)}\gamma_v = R_{A(v,\epsilon)}\gamma_u, \quad 0 \leq u \leq v. \tag{3.9}$$

Clearly $A(u, \epsilon)$ is increasing in ϵ and decreasing in u (the latter since $u \mapsto g_u(x)$ is convex). Thus we may define

$$\begin{aligned} \delta_u &= \downarrow \lim_{\epsilon\downarrow 0} R_{A(u,\epsilon)}m, \\ T(u) &= \uparrow \lim_{\epsilon\downarrow 0} \Pi_{A(u,\epsilon)}, \end{aligned} \tag{3.10}$$

where $\Pi_{A(u,\epsilon)}$ is the Lebesgue penetration time of $A(u,\epsilon)$ as in section 2. The family $\{T(u): u \geq 0\}$ has the properties listed in Theorem (3.1). Also, by (2.7) and (3.10),

$$\delta_u = L_{T(u)}m, \qquad u \geq 0. \tag{3.11}$$

Now if $0 \leq u < v$ then $\Gamma_v \leq \Gamma_u + (v-u)m$; hence $\gamma_v \leq \gamma_u + (v-u)m$ upon applying R. Applying $R_{A(v,\epsilon)}$ and using (3.9) we obtain $\gamma_v \leq \gamma_u + (v-u)R_{A(v,\epsilon)}m$. Letting $\epsilon \downarrow 0$, it follows that $\gamma_v \leq \gamma_u + (v-u)\delta_v$. On the other hand, on $A(u,\epsilon)$ we have $\gamma_v + (v-u)m + \epsilon u\xi \geq (1+\epsilon u)\xi - um \geq \gamma_u$; applying $R_{A(u,\epsilon)}$ we find that $\gamma_v + (v-u)R_{A(u,\epsilon)}m + \epsilon u\xi \geq \gamma_u$. Letting $\epsilon \downarrow 0$ we obtain $\gamma_v + (v-u)\delta_u \geq \gamma_u$. Thus

$$\delta_v \leq -(\gamma_v - \gamma_u)/(v-u) \leq \delta_u, \quad 0 \leq u < v. \tag{3.12}$$

Letting $v \downarrow u$ in (3.12) we see that if $f \in \mathcal{E}^+$ with $(\xi+m)(f) < \infty$, then $\delta_{u+}(f) \leq -(d^+/du^+)\gamma_u(f) \leq \delta_u(f)$ with equality except possibly for u in some countable set, since δ_u is decreasing in u. Since $u \mapsto \gamma_u(f)$ is convex, it follows that

$$\xi(f) = \gamma_v(f) + \int_0^v L_{T(u)}m\, du, \qquad v > 0, \tag{3.13}$$

first if $(\xi+m)(f) < \infty$, and then for all $f \in \mathcal{E}^+$ by monotone convergence. Now (3.2) will obtain upon letting $v \uparrow \infty$ in (3.13), once we identify the limit γ_∞ with $L_T\xi$. For this, note that $L_Tm = \downarrow \lim_{u\to\infty} L_{T(u)}m = 0$ since the integral in (3.2) is dominated by ξ. Let $\epsilon \downarrow 0$ in (3.9) to obtain $\gamma_v = L_{T(v)}\gamma_u$ if $0 \leq u \leq v$; now let $v \uparrow \infty$ to see that $\gamma_\infty = L_T\gamma_u$. Finally, apply L_T to both sides of (3.13) (noting that $L_TL_{T(u)}m \leq L_Tm = 0$) to obtain $L_T\xi = \gamma_\infty$ as required. If $\xi \leq r \cdot m$ then $\gamma_v = 0$ for $v > r$ and (3.3) follows from (3.2) since $L_T\xi = \gamma_\infty = 0$. The proof of (3.1) is complete.

(3.14) Remark. The family $\{T(u) : u \geq 0\}$ is not unique but the particular family produced in the proof of (3.1) enjoys a certain extremal property. Indeed, if $\xi = \int_0^\infty \delta_u^* du + \gamma_\infty^*$ is a second decomposition of ξ (where $\delta_s^* \leq m$, and $\delta_s^*, \gamma_\infty^*$ are excessive) then

$$\gamma_\infty \leq \gamma_\infty^* \quad \text{and} \quad \int_0^v L_{T(u)}m\, du \geq \int_0^v \delta_u^* du\,, \quad \text{all} \quad v > 0. \tag{3.15}$$

Using (3.15) one can check that $R(\gamma_\infty - u \cdot m)_+ = \gamma_\infty$ for all $u > 0$.

An important case of (3.1) occurs when $\gamma_\infty = L_T\xi = 0$. Following section 6 of [6] we write $\xi \leftarrow m$ in this case, and say that ξ is *weakly* dominated by m. When $\xi \leftarrow m$, (3.2) exhibits ξ

as a "randomized balayage" of m. The relation $\leftarrow$ is transitive but it is only a preorder since $m \leftarrow 2m \leftarrow m$. We offer two characterizations of $\leftarrow$. The first of these is from [6]; its proof is left to the reader as an exercise.

(3.16) Proposition. *Fix ξ and m in* Exc. *Then $\xi \leftarrow m$ if and only if $\xi = \sum_{n=1}^{\infty} \xi_n$ where $\xi_n \in$ Exc and $\xi_n \leq m$ for all n.*

The second characterization of $\leftarrow$ is a variant of a result found in [6].

(3.17) Proposition. *Let ξ and m be excessive measures. Then $\xi \leftarrow m$ if and only if $\xi << m$ and $R_{\{\psi>u\}}\xi \downarrow 0$ as $u \uparrow \infty$, where $\psi \in \mathcal{E}^+$ is any version of $d\xi/dm$.*

Proof. It is clear from (3.1) that $\xi \leftarrow m$ if and only if $\gamma_u \equiv R(\xi - u \cdot m)_+ \downarrow 0$ as $u \uparrow \infty$. Also, if $\xi \leftarrow m$ then certainly $\xi << m$. In view of these remarks the proposition follows from

$$(u/u+v)R_{\{\psi>u+v\}}\xi \leq \gamma_v \leq R_{\{\psi>v\}}\xi \quad u, v > 0, \quad \psi = d\xi/dm. \tag{3.18}$$

For the left hand inequality in (3.18) use (3.6) to produce $\rho_v \in$ Exc with $\xi = \rho_v + \gamma_v$, $\rho_v \leq v \cdot m$. Then, using the fact that $(u+v)m \leq \xi$ on $\{\psi > u+v\}$ for the second equality below

$$\begin{aligned} R_{\{\psi>u+v\}}\xi &\leq vR_{\{\psi>u+v\}}m + \gamma_v \\ &\leq (v/u+v)R_{\{\psi>u+v\}}\xi + \gamma_v. \end{aligned} \tag{3.19}$$

We obtain the first inequality in (3.18) by rearranging (3.19). For the second inequality in (3.18) note that

$$\xi \leq v\, 1_{\{\psi\leq v\}}m + 1_{\{\psi>v\}}\xi \leq v \cdot m + R_{\{\psi>v\}}\xi$$

so that $\gamma_v = R(\xi - v \cdot m)_+ \leq R_{\{\psi>v\}}\xi$ as desired. ∎

(3.20) Remark. Letting $u \uparrow \infty$, then $v \uparrow \infty$ in (3.18) we see that if $\xi << m$, then $\gamma_\infty = L_T\xi = \lim_{v\uparrow\infty} R_{\{\psi>v\}}\xi$.

Finally, let us interpret (3.1) as a Skorohod stopping theorem. Let $\xi \in$ Exc and let $m = \mu U$ be a potential with $\xi \leftarrow m$. Let $\{T(u): u \geq 0\}$ be the family of stopping times provided by (3.1). If $\mathcal{F}_t^*$ denotes the universal completion of $\mathcal{F}_t^0$, then the restrictions $S(u) \equiv T(u)\,|_\Omega$ form an increasing family of $(\mathcal{F}_{t+}^*)$-stopping times. Moreover, each $S(u)$ is a terminal time since the $T(u)$ satisfy (2.5). Arguing as in [4] one shows that $L_{T(u)}(\mu U) = \mu P_{S(u)}U$ where $P_{S(u)}$ is the hitting operator for $S(u)$.

(3.21) Proposition. *Let $\xi \in$ Exc, $\mu U \in$ Exc with $\xi \leftarrow \mu U$. Then $\xi = \nu U$ where $\nu = \int_0^\infty \mu P_{S(u)}\, du$, and where $\{S(u): u \geq 0\}$ is as described above.*

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