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LIMIT DISTRIBUTION FOR 1-DIMENSIONAL DIFFUSION IN A REFLECTED BROWNIAN MEDIUM

By H. Tanaka

Introduction

In analogy with Sinai's problem [8] on a random walk in a random medium, Brox [1] considered the diffusion process $X(t)$ described by the stochastic differential equation

$$(1) \quad dX(t) = dB(t) - \frac{1}{2} W'(X(t))dt, \quad X(0) = 0,$$

where $\{W(x), x \in \mathbb{R}\}$ is a Brownian medium independent of another Brownian motion $B(t)$, and proved that $(\log t)^{-2}X(t)$ converges in distribution as $t \rightarrow \infty$. Similar results in the case of a considerably wider class of self-similar random media were obtained by Schumacher [7]. Recently Kesten [5] obtained the exact form of the limit distribution for Sinai's random walk as well as for a diffusion in a Brownian medium. See also [2] for a related problem.

In this paper we substitute $W(x)$ in (1) by a nonnegative reflected Brownian medium and find the corresponding limit distribution. The result was already announced in [9] without proof but the Laplace transform of the limit distribution given in [9: §3] is not correct. We give here a full proof to the whole result of [9: §3] with a correction (see Theorem 1 and 2 below). Our method is similar to that of [1].

Theorem 1. Let $X(t)$ be a solution of (1) where $W_+ = \{W(x), x \geq 0\}$ and $W_- = \{W(-x), x \geq 0\}$ are independent reflected Brownian motions on the half line $[0, \infty)$ starting from 0 which are also independent of the Brownian motion $B(t)$. Then the distribution of $(\log t)^{-2}X(t)$ converges as $t \rightarrow \infty$ to the distribution μ defined by

$$(2) \quad \mu = \int m_W Q(dW)$$

where m_W is the probability measure on $\mathbb{R}$ defined by (3.1) and Q is the probability measure on the space of media $W = C(\mathbb{R} \rightarrow [0, \infty)) \cap \{W: W(0)=0\}$ such that $W_{\pm}$ are independent reflected Brownian motions on $[0, \infty)$.

Theorem 2. μ has a symmetric density and for $\lambda > 0$

$$(3) \quad \int_0^{\infty} e^{-\lambda x} \mu(dx) = \int_0^{\infty} \frac{\sinh \sqrt{2\lambda}}{\sqrt{2\lambda} \cosh \sqrt{2\lambda} + t \sinh \sqrt{2\lambda}} \cdot \frac{dt}{(1+t)^2}.$$

The present case is not contained in the framework of [7] since the nonnegative reflected medium $W(x)$ has (uncountably) many points giving its minimum. The case of a nonpositive reflected Brownian medium was discussed in [9]. Some generalizations will be discussed in [5].

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§1. Preliminaries and exit times from valleys

Let W and Q be defined as in Theorem 1. For each $W \in \mathcal{W}$ solutions of the stochastic differential equation (1) define a diffusion process in $\mathbb{R}$ with generator

$$(1.1) \quad \frac{1}{2} e^{W(x)} \frac{d}{dx} (e^{-W(x)} \frac{d}{dx}) .$$

Such a diffusion can be constructed from a Brownian motion $B(t)$ ¹⁾ as follows ([4]). Let Ω be the space of continuous functions $\omega : [0, \infty) \rightarrow \mathbb{R}$ with $\omega(0) = 0$, and denote by P the Wiener measure on Ω . Denote the value of ω at time t by $\omega(t)$ or by $B(t)$ and put

$$\begin{aligned} L(t, x) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{[x, x+\varepsilon)}(B(s)) ds && \text{(local time),} \\ S(x) &= \int_0^x e^{W(y)} dy , \\ A(t) &= \int_0^t e^{-2W(S^{-1}(B(s)))} ds = \int_{\mathbb{R}} e^{-2W(S^{-1}(x))} L(t, x) dx , \quad t \geq 0 , \\ S^{-1}, A^{-1} &= \text{the inverse functions .} \end{aligned}$$

Then the process $X(t, W) = S^{-1}(B(A^{-1}(t)))$ defined on the probability space (Ω, P) is a diffusion process with generator (1.1) starting at 0. If we set $(W^x)(\cdot) = W(\cdot + x)$, then $X^x(t, W) = x + X(t, W^x)$ is a diffusion process with generator (1.1) starting at x . Let

$$T(x_1, x_2) = \inf \{t \geq 0 : B(t) \notin (x_1, x_2)\} ,$$

1) The Brownian motion here is not the same as the one in (1) but we use the same notation $B(t)$.

$$L(x_1, x_2, x) = L(T(x_1, x_2), x), \quad x \in \mathbb{R},$$

$$S_\lambda(x) = \int_0^x e^{\lambda W(y)} dy,$$

$$X_\lambda(t) = X(t, \lambda W), \quad X_\lambda^x(t) = x + X(t, \lambda W^x).$$

Next we define a valley. Given $W \in \mathbb{W}$, a quartet $V = (a, b_1, b_2, c)$ is called a valley of W if

- (i) $a < b_1 < 0 < b_2 < c$,
- (ii) $W(b_1) = W(b_2) = 0$, $W(a) = W(c) = D$,
- (iii) $0 < W(x) < W(a)$ for $a < x < b_1$,
 $0 < W(x) < W(c)$ for $b_2 < x < c$,
- (iv) $A_- = \sup \{W(y) - W(x) : a < x < y < b_2\} < D$,
 $A_+ = \sup \{W(x) - W(y) : b_1 < x < y < c\} < D$.

We call D (resp. $A = A_- \vee A_+$)²⁾ the depth (resp. the inner directed ascent) of V . It is clear that there exist valleys of W with $A < 1 < D$ for almost all reflected Brownian media W .

In what follows let $W \in \mathbb{W}$ be given and $V = (a, b_1, b_2, c)$ be a valley of W with the depth D and the inner directed ascent A . We put

$$T_\lambda^x = T_\lambda^x(a, c) = \inf \{t \geq 0 : X_\lambda^x(t) \notin (a, c)\}.$$

The following three lemmas were proved in [1].

Lemma 1. For $a < x < c$

$$T_\lambda^x(a, c) \stackrel{d}{=} \int_a^c L(\hat{S}_\lambda(a), \hat{S}_\lambda(c), \hat{S}_\lambda(y)) e^{-\lambda W(y)} dy,$$

where

$$\hat{S}_\lambda(y) = \int_x^y e^{\lambda W(z)} dz$$

and $\stackrel{d}{=}$ means the equality in distribution.

Lemma 2. For each $\lambda > 0$

$$\{L(\lambda x_1, \lambda x_2, \lambda x), x \in \mathbb{R}\} \stackrel{d}{=} \{\lambda L(x_1, x_2, x), x \in \mathbb{R}\}.$$

2) $a \vee b = \max \{a, b\}$, $a \wedge b = \min \{a, b\}$.

Lemma 3. For $\lambda > 0$ and $W \in \mathbb{W}$

$$(1.2) \quad \{X(t, \lambda W_\lambda), t \geq 0\} \stackrel{d}{=} \{\lambda^{-2} X(\lambda^4 t, W), t \geq 0\},$$

where $W_\lambda (\in \mathbb{W})$ is defined by

$$W_\lambda(x) = \lambda^{-1} W(\lambda^2 x), \quad x \in \mathbb{R}.$$

The following lemma plays an essential role in our discussions.

Lemma 4. For any $\lambda > 0$ and $[u, v] \subset (a, c)$

$$\inf_{u \leq x \leq v} P \left\{ e^{\lambda(D-\delta)} < T_\lambda^x < e^{\lambda(D+\delta)} \right\} \rightarrow 1, \quad \lambda \rightarrow \infty.$$

Proof. The proof is similar to that of the corresponding lemma of [1] but even much simpler. Let $x \in [u, v]$ be fixed. Setting

$$s_\lambda(y) = \widehat{S}_\lambda(y) / \widehat{S}_\lambda(c) = \int_x^y e^{\lambda W(z)} dz / \int_x^c e^{\lambda W(z)} dz$$

and applying Lemma 1 and 2, we have

$$T_\lambda^x \stackrel{d}{=} \widehat{S}_\lambda(c) \int_a^c L(s_\lambda(a), 1, s_\lambda(y)) e^{-\lambda W(y)} dy.$$

Since

$$\begin{aligned} \widehat{S}_\lambda(c) &\leq (c - x) \exp \left\{ \lambda \max_{[x, c]} W \right\} \quad 3) \\ T_\lambda^x &\stackrel{d}{\leq} (c - x)(c - a) \exp \left\{ \lambda \max_{[x, c]} W - \lambda \min_{[a, c]} W \right\} L' \leq (c - a)^2 L' e^{\lambda D}, \\ L' &= \max_{y \leq 1} L(-\infty, 1, y), \end{aligned}$$

we have

$$\begin{aligned} &P \left\{ T_\lambda^x > e^{\lambda(D+\delta)} \right\} \\ &\leq P \left\{ (c - a)^2 L' e^{\lambda D} > e^{\lambda(D+\delta)} \right\} \\ &= P \left\{ L' > e^{\lambda \delta} / (c - a)^2 \right\} \rightarrow 0, \quad \lambda \rightarrow \infty. \end{aligned}$$

To obtain an estimate from below first we notice that

$$(1.3) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-1} \log C_\lambda = D,$$

$$3) \quad \max_I W = \max\{W(x), x \in I\}, \quad \min_I W = \min\{W(x), x \in I\}.$$

where

$$C_\lambda = |\widehat{S}_\lambda(a)| \wedge |\widehat{S}_\lambda(c)| ,$$

and the convergence is uniform in $x \in [u, v]$. Next, for given $\delta > 0$ we set

$$\begin{aligned} a_1 &= \sup\{x < b_1 : W(x) = \delta/4\} , \\ \widehat{s}_\lambda(y) &= \widehat{S}_\lambda(y)/C_\lambda , \\ L_\lambda &= \min\{L(-1, 1, y) : \widehat{s}_\lambda(a_1) \leq y \leq \widehat{s}_\lambda(b_1)\} . \end{aligned}$$

Then applying Lemma 1 and 2 we have

$$\begin{aligned} T_\lambda^x &\stackrel{d}{=} C_\lambda \int_a^c L(\widehat{s}_\lambda(a), \widehat{s}_\lambda(c), \widehat{s}_\lambda(y)) e^{-\lambda W(y)} dy \\ &\geq C_\lambda \int_{a_1}^{b_1} L(-1, 1, \widehat{s}_\lambda(y)) e^{-\lambda W(y)} dy \\ &\geq e^{\lambda(D-\frac{\delta}{4})} (b_1 - a_1) L_\lambda \exp\left\{-\lambda \max_{[a_1, b_1]} W\right\} \\ &= (b_1 - a_1) L_\lambda e^{\lambda(D-\frac{\delta}{2})} . \end{aligned}$$

Since $\lambda^{-1} \log |\widehat{s}_\lambda(a_1)|$ and $\lambda^{-1} \log |\widehat{s}_\lambda(b_1)|$ converges to $\max_{[x \wedge a_1, x \vee a_1]} W - D$,
 $\max_{[x \wedge b_1, x \vee b_1]} W - D$, respectively, which are both negative, we have

$$\lim_{\lambda \rightarrow \infty} \widehat{s}_\lambda(a_1) = \lim_{\lambda \rightarrow \infty} \widehat{s}_\lambda(b_1) = 0 ,$$

the convergence being uniform in $x \in [u, v]$. Therefore

$$P\left\{T_\lambda^x < e^{\lambda(D-\delta)}\right\} \leq P\left\{L_\lambda < (b_1 - a_1)^{-1} e^{-\lambda\delta/2}\right\} \rightarrow 0 , \quad \lambda \rightarrow \infty$$

uniformly in $x \in [u, v]$, because $\lim_{\lambda \rightarrow \infty} L_\lambda = L(-1, 1, 0) > 0$.

§2. The limit distribution of $X(e^{\lambda r}, \lambda W)$

In this section we change the notation slightly. Given $W \in \mathbb{W}$ and a valley $V = (a, b_1, b_2, c)$ of W , we set

$$\begin{aligned} \Omega &= C([0, \infty) \rightarrow \mathbb{R}) , \\ \widehat{\Omega} &= C([0, \infty) \rightarrow [a, c]) , \end{aligned}$$

and denote by P_λ^x , $x \in \mathbb{R}$ (resp. $\widehat{P}_\lambda^y$, $y \in [a, c]$) the probability measure

on Ω (resp. $\hat{\Omega}$) induced by the diffusion process with generator

$$(2.1) \quad \frac{1}{2} e^{\lambda W(x)} \frac{d}{dx} (e^{-\lambda W(x)} \frac{d}{dx})$$

(resp. the diffusion process on $[a, c]$ with (local) generator (2.1) and with reflecting barriers at a and c). The latter diffusion has the invariant probability measure m_λ given by

$$m_\lambda(dy) = e^{-\lambda W(y)} dy / \int_a^c e^{-\lambda W(z)} dz .$$

For any interval $[u, v] \subset [a, c]$

$$m_\lambda([u, v]) = \frac{\int_0^\infty e^{-\lambda \xi} K([u, v], \xi) d\xi}{\int_0^\infty e^{-\lambda \xi} K([a, c], \xi) d\xi}$$

where, for an interval I in $\mathbb{R}$, $K(I, \xi)$ is the local time at ξ for the reflected Brownian medium, i.e.,

$$(2.2) \quad K(I, \xi) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_I \mathbb{I}_{[\xi, \xi+\varepsilon)}(W(s)) ds .$$

Therefore

$$(2.3) \quad m_\lambda([u, v]) = \frac{\int_0^\infty e^{-\xi} K([u, v], \lambda^{-1} \xi) d\xi}{\int_0^\infty e^{-\xi} K([a, c], \lambda^{-1} \xi) d\xi} \\ \rightarrow \frac{K([u, v], 0)}{K([a, c], 0)} \equiv m([u, v]) , \quad \lambda \rightarrow \infty .$$

Next we set

$$\hat{P}_\lambda = \int_a^b m_\lambda(dy) \hat{P}_\lambda^y , \quad \mathbb{P}_\lambda^{x,y} = P_\lambda^x \otimes \hat{P}_\lambda^y , \quad \mathbb{P}_\lambda^x = P_\lambda^x \otimes \hat{P}_\lambda .$$

$$R = R(\omega, \hat{\omega}) = \inf\{t \geq 0 : \omega(t) = \hat{\omega}(t)\} .$$

Lemma 5. For any $\delta > 0$

$$\lim_{\lambda \rightarrow \infty} P_\lambda^0 \left\{ R < e^{\lambda(A+\delta)} \right\} = 1 .$$

Proof. First we prove that

$$(2.4) \quad \lim_{\lambda \rightarrow \infty} \mathbb{P}_\lambda^x \left\{ R < e^{\lambda(A+\delta)} \right\} = 1 \quad \text{holds for } x = b_1 \text{ and } b_2 .$$

Without loss of generality we may consider the case $x = b_2$. We write b instead of b_2 for simplicity. For any $\delta > 0$ such that $A + \delta < D$ we define $a_1 \in (a, b_1)$, $a_2 \in (a, b_1)$, $c_2 \in (b_2, c)$ by

$$\begin{aligned} a_1 &= \max \left\{ x < b_1 : W(x) = A + \frac{\delta}{4} \right\}, \\ a_2 &= \max \left\{ x < b_1 : W(x) = A + \frac{\delta}{2} \right\}, \\ c_2 &= \min \left\{ x > b_2 : W(x) = A + \frac{\delta}{2} \right\}, \end{aligned}$$

and set

$$\begin{aligned} T_0 &= T_0(\omega) = \inf \{ t \geq 0 : w(t) = a_1 \}, \\ T_1 &= T_1(\omega) = \inf \{ t \geq 0 : w(t) \notin (a_1, c_2) \}, \\ T_2 &= T_2(\omega) = \inf \{ t \geq 0 : w(t) \notin (a_2, c_2) \}. \end{aligned}$$

Then we can prove easily that

$$(2.5) \quad P_\lambda^b \{ T_0 < \infty \} \geq P_\lambda^b \{ T_0 = T_1 \} = \frac{S_\lambda(c_2) - S_\lambda(b)}{S_\lambda(c_2) - S_\lambda(a_1)} \rightarrow 1, \lambda \rightarrow \infty,$$

and hence

$$\begin{aligned} (2.6) \quad & P_\lambda^b \{ R \leq T_0 \} \\ & \geq P_\lambda^b \{ \hat{w}(0) \in [a, b], \hat{w}(T_0) \in [a_1, c] \} \\ & \geq P_\lambda^b \{ \hat{w}(0) \in [a, b] \} + P_\lambda^b \{ \hat{w}(T_0) \in [a_1, c] \} - 1 \\ & = m_\lambda([a, b]) + \int_0^\infty \hat{P}_\lambda \{ \hat{w}(t) \in [a_1, c] \} P_\lambda^b \{ T_0 \in dt \} - 1 \\ & \rightarrow 1, \lambda \rightarrow \infty, \end{aligned}$$

by (2.3) because $m(\{x \in (a, c) : W(x) = 0\}) = 1$. On the other hand Lemma 4 applied to the valley (a_2, b_1, b_2, c_2) whose depth is $A + (\delta/2)$ implies

$$(2.7) \quad P_\lambda^b \{ T_1 < e^{\lambda(A+\delta)} \} \geq P_\lambda^b \{ T_2 < e^{\lambda(A+\delta)} \} \rightarrow 1, \lambda \rightarrow \infty,$$

and so

$$\begin{aligned} & P_\lambda^x \{ R < e^{\lambda(A+\delta)} \} \\ & \geq P_\lambda^x \{ T_0 < e^{\lambda(A+\delta)} \} - o(1) \quad (\text{by (2.6)}) \\ & \geq P_\lambda^x \{ T_1 < e^{\lambda(A+\delta)}, T_1 = T_0 \} - o(1) \end{aligned}$$

$$\begin{aligned} & \geq P_{\lambda}^x \{T_1 < e^{\lambda(A+\delta)}\} - o(1) && \text{(by (2.5))} \\ & \rightarrow 1, \quad \text{as } \lambda \rightarrow \infty && \text{(by (2.7)) .} \end{aligned}$$

Next, to consider the case where the diffusion starts at 0 we shall consider three diffusion processes starting at 0, b_1 and b_2 , respectively. By making use of the comparison theorem in one-dimensional diffusion processes (for example, see [3: p.352]) we can construct, on a suitable probability space $(\tilde{\Omega}_{\lambda}, \tilde{\mathbb{P}}_{\lambda})$, three processes $\tilde{X}_0(t)$, $\tilde{X}_1(t)$ and $\tilde{X}_2(t)$ such that the probability measure on Ω induced by $\tilde{X}_0(t)$ (resp. $\tilde{X}_1(t)$, $\tilde{X}_2(t)$) coincides with P_{λ}^0 (resp. $P_{\lambda}^{b_1}$, $P_{\lambda}^{b_2}$) and

$$(2.8) \quad \tilde{X}_1(t) \leq \tilde{X}_0(t) \leq \tilde{X}_2(t), \quad \forall t \geq 0, \quad \tilde{\mathbb{P}}_{\lambda}\text{-a.s.}$$

Put

$$\tilde{\mathbb{P}}_{\lambda} = \tilde{\mathbb{P}}_{\lambda} \otimes \hat{\mathbb{P}}_{\lambda},$$

$$\tilde{R}_i = \inf\{t \geq 0 : \tilde{X}_i(t) = \hat{\omega}(t)\}, \quad i = 0, 1, 2.$$

Since $\tilde{R}_0 \leq \tilde{R}_1 \vee \tilde{R}_2$ by (2.8), we have

$$\begin{aligned} P_{\lambda}^0 \{R < e^{\lambda(A+\delta)}\} &= \tilde{\mathbb{P}}_{\lambda} \{\tilde{R}_0 < e^{\lambda(A+\delta)}\} \\ &\geq \tilde{\mathbb{P}}_{\lambda} \{\tilde{R}_1 \vee \tilde{R}_2 < e^{\lambda(A+\delta)}\} \\ &\geq P_{\lambda}^{b_1} \{R < e^{\lambda(A+\delta)}\} + P_{\lambda}^{b_2} \{R < e^{\lambda(A+\delta)}\} - 1 \\ &\rightarrow 1, \quad \lambda \rightarrow \infty \end{aligned}$$

by (2.4), completing the proof of Lemma 5.

Lemma 6. For any r_1, r_2 with $A < r_1 < r_2 < D$ and for any interval $[u, v]$ in $\mathbb{R}$

$$\lim_{\lambda \rightarrow \infty} P_{\lambda}^0 \{\omega(e^{\lambda r}) \in [u, v]\} = m([u, v] \cap [b_1, b_2])$$

uniformly in $r \in [r_1, r_2]$, where m is defined in (2.3).

Proof. Denote by T (resp. $\hat{T}$) the exit time of (a, c) for $\omega(t)$ (resp. $\hat{\omega}(t)$), and by $\tilde{T}_R$ (resp. $\hat{\tilde{T}}_R$) the exit time of (a, c) for $\omega(t)$ (resp. $\hat{\omega}(t)$) after the collision time R . Since $m_{\lambda}(U) \rightarrow 1$ as $\lambda \rightarrow \infty$ for any open set U containing $\{x \in (a, c) : W(x) = 0\}$, it follows from Lemma 4 that

$$\hat{P}_{\lambda} \{e^{\lambda(D-\delta)} < \hat{T} < e^{\lambda(D+\delta)}\}$$

$$\begin{aligned}
&= \int_a^c m_\lambda(dx) P_\lambda^x \left\{ e^{\lambda(D-\delta)} < T < e^{\lambda(D+\delta)} \right\} \\
&\rightarrow 1, \lambda \rightarrow \infty.
\end{aligned}$$

This combined with Lemma 5 implies

$$\begin{aligned}
p_\lambda &:= \mathbb{P}_\lambda^0 \left\{ R < e^{\lambda r_1} < e^{\lambda r_2} < \hat{T}_R \right\} \\
&\geq \mathbb{P}_\lambda^0 \left\{ R < e^{\lambda r_1} < e^{\lambda r_2} < \hat{T} \right\} \quad (\because \hat{T} \leq \hat{T}_R) \\
&\rightarrow 1, \lambda \rightarrow \infty.
\end{aligned}$$

Therefore for $r \in [r_1, r_2]$

$$\begin{aligned}
(2.9) \quad &P_\lambda^0 \left\{ \omega(e^{\lambda r}) \in [u, v] \right\} \\
&\geq \mathbb{P}_\lambda^0 \left\{ R < e^{\lambda r_1}, \omega(e^{\lambda r}) \in [u, v], e^{\lambda r_2} < \tilde{T}_R \right\} \\
&= \mathbb{P}_\lambda^0 \left\{ R < e^{\lambda r_1}, \hat{\omega}(e^{\lambda r}) \in [u, v], e^{\lambda r_2} < \hat{T}_R \right\} \\
&\geq p_\lambda + m_\lambda([u, v]) - 1 \\
&\rightarrow m([u, v] \cap [b_1, b_2]), \lambda \rightarrow \infty;
\end{aligned}$$

as for the above equality we used the strong Markov property. Similarly we have

$$\lim_{\lambda \rightarrow \infty} P_\lambda^0 \left\{ \omega(e^{\lambda r}) \in [u, v]^c \right\} \geq m([u, v]^c \cap [b_1, b_2]),$$

which combined with (2.9) implies

$$P_\lambda^0 \left\{ \omega(e^{\lambda r}) \in [u, v] \right\} \rightarrow m([u, v] \cap [b_1, b_2]), \lambda \rightarrow \infty.$$

The uniform convergence in $r \in [r_1, r_2]$ is also clear.

§3. Proof of Theorem 1

Let $V = (a, b_1, b_2, c)$ be a valley of W such that $A < 1 < D$. Such a valley exists with Q -probability 1. In fact, b_1 and b_2 are taken as

b_1 = the smallest root of $W(x) = 0$ in $(a', 0)$

b_2 = the largest root of $W(x) = 0$ in $(0, c')$

where $a' = \sup\{x < 0 : W(x) = 1\}$ and $c' = \inf\{x > 0 : W(x) = 1\}$. The endpoints a and c can be chosen suitably so that $a < a'$, $c > c'$ and

$V = (a, b_1, b_2, c)$ is a valley with $A < 1 < D$. In what follows $V = (a, b_1, b_2, c)$ denotes such a valley of W . We denote by m_W the probability measure on $\mathbb{R}$ defined by

$$(3.1) \quad m_W([u, v]) = \frac{K([u', v'], 0)}{K([b_1, b_2], 0)}$$

where $[u', v'] = [u, v] \cap [b_1, b_2]$. Then, in the notation of §1 Lemma 6 reads as follows: For any interval I in $\mathbb{R}$ and for any family $\{r(\lambda), \lambda > 0\}$ satisfying $\lim_{\lambda \rightarrow \infty} r(\lambda) = 1$,

$$(3.2) \quad \lim_{\lambda \rightarrow \infty} P\{X(e^{\lambda r(\lambda)}), \lambda W \in I\} = m_W(I)$$

for almost all W with respect to Q . Now we define $\mathbb{P} = P \otimes Q$ and $\mu = \int m_W Q(dW)$. Integrating both sides of (3.2) with respect to Q we have

$$(3.3) \quad \lim_{\lambda \rightarrow \infty} \mathbb{P}\{X(e^{\lambda r(\lambda)}), \lambda W \in I\} = \mu(I).$$

Next, define W_λ as in Lemma 3. Then $\{W_\lambda(x), x \in \mathbb{R}\}$ is again a reflected Brownian medium. Therefore (3.3) yields

$$(3.4) \quad \lim_{\lambda \rightarrow \infty} \mathbb{P}\{X(e^{\lambda r(\lambda)}), \lambda W_\lambda \in I\} = \mu(I).$$

We now apply the scaling relation (1.2) to (3.4); the result is

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}\{\lambda^{-2} X(\lambda^4 e^{\lambda r(\lambda)}), W \in I\} = \mu(I).$$

Taking $r(\lambda) = 1 - 4\lambda^{-1} \cdot \log \lambda$ in the above, we obtain

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}\{\lambda^{-2} X(e^\lambda), W \in I\} = \mu(I).$$

This completes the proof of Theorem 1.

§4. Proof of Theorem 2

The absolute continuity of μ can be proved easily. In fact, if μ_n is the measure in $\mathbb{R}$ defined by

$$\mu_n(I) = E^Q \left\{ \frac{K(I \cap [b_1, b_2])}{K([b_1, b_2])} ; K([b_1, b_2]) > \frac{1}{n} \right\},$$

then μ_n is absolutely continuous because

$$\begin{aligned} \mu_n(I) &\leq n E^Q \{ K(I \cap [b_1, b_2]) \} \\ &= 2n \int_I p(|x|, 0, 0) dx, \end{aligned}$$

where $p(t, \xi, \eta)$ is the transition density of the Brownian motion with absorbing barriers at ± 1 . Thus μ is absolutely continuous because $\mu_n \uparrow \mu$ as $n \uparrow \infty$.

We proceed to the proof of (3). Let $K(I) = K(I, 0)$ be the local time at 0 for the reflected Brownian medium as defined by (2.2) with $\xi = 0$ and consider the number of times $d_\varepsilon(t)$ that the reflected Brownian path $\{W(u) : u \geq 0\}$ crosses down from $\varepsilon > 0$ to 0 before time t . Then as found in [4: p.48]

$$(4.1) \quad \mathbb{Q} \left\{ \lim_{\varepsilon \downarrow 0} 2\varepsilon d_\varepsilon(t) = K([0, t]), t \geq 0 \right\} = 1.$$

Let a', c', b_1 and b_2 be defined as in the beginning of §3.

Lemma 7. For $\alpha, \beta > 0$

$$(4.2) \quad \mathbb{E}^Q \left\{ e^{-\alpha K([0, b_2]) - \beta c'} \right\} = \frac{1}{2\alpha + c(\beta)} \cdot \frac{2\sqrt{2\beta}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}},$$

where

$$c(\beta) = \frac{e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \cdot \sqrt{2\beta}.$$

In Particular, $K([0, b_2])$ is exponentially distributed:

$$(4.3) \quad \mathbb{E}^Q \left\{ e^{-\alpha K([0, b_2])} \right\} = \frac{1}{2\alpha + 1}.$$

Proof. Since $c(\beta) \sim 1$ as $\beta \downarrow 0$, (4.3) follows from (4.2) by letting $\beta \downarrow 0$. To prove (4.2) we first apply (4.1) to write down

$$(4.4) \quad \begin{aligned} & \mathbb{E}^Q \left\{ e^{-\alpha K([0, b_2]) - \beta c'} \right\} \\ &= \mathbb{E}^Q \left\{ e^{-\alpha K([0, c']) - \beta c'} \right\} \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}^Q \left\{ e^{-2\alpha \varepsilon d_\varepsilon(c') - \beta c'} \right\} \\ &= \lim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} e^{-2\alpha \varepsilon n} \mathbb{E}^Q \left\{ e^{-\beta T_\varepsilon} \right\}^{n+1} \mathbb{E}_\varepsilon^Q \left\{ e^{-\beta T_0}; T_0 < T_1 \right\}^n \mathbb{E}_\varepsilon^Q \left\{ e^{-\beta T_1}; T_1 < T_0 \right\}, \end{aligned}$$

where $\mathbb{E}_\varepsilon^Q$ denotes the expectation with respect to the probability measure of the reflected Brownian motion starting at ε and

$$T_x = \inf \{ u \geq 0 : W(u) = x \}.$$

If we set

$$A_{\varepsilon} = e^{-2\alpha\varepsilon} E^Q \left\{ e^{-\beta T_{\varepsilon}} \right\} E^Q_{\varepsilon} \left\{ e^{-\beta T_0}; T_0 < T_1 \right\},$$

$$B_{\varepsilon} = E^Q \left\{ e^{-\beta T_{\varepsilon}} \right\} E^Q_{\varepsilon} \left\{ e^{-\beta T_1}; T_1 < T_0 \right\},$$

then (4.4) yields

$$(4.5) \quad E^Q \left\{ e^{-\alpha K([0, b_2]) - \beta c'} \right\} = \lim_{\varepsilon \downarrow 0} B_{\varepsilon} \sum_{n=0}^{\infty} A_{\varepsilon}^n$$

$$= \lim_{\varepsilon \downarrow 0} \frac{B_{\varepsilon}}{1 - A_{\varepsilon}}.$$

Next we make use of the well-known formula

$$E_x \left\{ e^{-\alpha T_a; T_a < T_b} \right\} = \frac{e^{\sqrt{2\alpha}(b-x)} - e^{-\sqrt{2\alpha}(b-x)}}{e^{\sqrt{2\alpha}(b-a)} - e^{-\sqrt{2\alpha}(b-a)}}, \quad a \leq x \leq b,$$

where E_x denotes the expectation with respect to the probability measure of a standard Brownian motion starting at x . We then have

$$(4.6) \quad E^Q \left\{ e^{-\beta T_{\varepsilon}} \right\} = 2E_0 \left\{ e^{-\beta T_{\varepsilon}}; T_{\varepsilon} < T_{-\varepsilon} \right\}$$

$$= \frac{2(e^{\varepsilon\sqrt{2\beta}} - e^{-\varepsilon\sqrt{2\beta}})}{e^{2\varepsilon\sqrt{2\beta}} - e^{-2\varepsilon\sqrt{2\beta}}}$$

$$= 1 + o(\varepsilon^2), \quad \varepsilon \downarrow 0;$$

$$(4.7) \quad E^Q_{\varepsilon} \left\{ e^{-\beta T_0}; T_0 < T_1 \right\} = \frac{e^{\sqrt{2\beta}(1-\varepsilon)} - e^{-\sqrt{2\beta}(1-\varepsilon)}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}$$

$$= 1 - \frac{\sqrt{2\beta}(e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}})}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \cdot \varepsilon + o(\varepsilon), \quad \varepsilon \downarrow 0;$$

$$(4.8) \quad E^Q_{\varepsilon} \left\{ e^{-\beta T_1}; T_1 < T_0 \right\} = \frac{e^{\sqrt{2\beta}\varepsilon} - e^{-\sqrt{2\beta}\varepsilon}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}$$

$$\sim \frac{2\sqrt{2\beta}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \cdot \varepsilon, \quad \varepsilon \downarrow 0.$$

From (4.6), (4.7) and (4.8) we obtain

$$\frac{B_\varepsilon}{1 - A_\varepsilon} \sim \frac{1}{2\alpha + c(\beta)} \cdot \frac{2\sqrt{2\beta}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}, \quad \varepsilon \downarrow 0,$$

which combined with (4.5) proves the lemma.

Given $x > 0$ we set

$$K_1 = K([b_1, 0]), \quad K_2 = K([0, x]), \quad K_3 = K([x, b_2]).$$

Lemma 8. For $x > 0$ and $t > 0$

$$(4.9) \quad \begin{aligned} & E^Q \left\{ K_3 e^{-t(K_1 + K_2 + K_3)}; x < b_2 \right\} \\ &= \frac{2}{(2t + 1)^3} E^Q \left\{ (1 - W(x)) e^{-tK([0, x])}; x < c' \right\}. \end{aligned}$$

Proof. The left hand side of (4.9) equals

$$E^Q \left\{ e^{-tK_1} \right\} E^Q \left\{ K_3 e^{-t(K_2 + K_3)}; x < b_2 \right\}.$$

Since $E^Q \left\{ e^{-tK_1} \right\} = (2t + 1)^{-1}$ by Lemma 7, for the proof of the lemma it is enough to show

$$(4.10) \quad \begin{aligned} & E^Q \left\{ K_3 e^{-t(K_2 + K_3)}; x < b_2 \right\} \\ &= \frac{2}{(2t + 1)^2} E^Q \left\{ (1 - W(x)) e^{-tK_2}; x < c' \right\}. \end{aligned}$$

To prove this we introduce the smallest σ -field $\mathcal{F}_x$ on W which makes $W(s)$, $0 \leq s \leq x$, measurable and consider the event Γ that the shifted trajectory $W(\cdot + x)$ hits 0 before it hits 1. Then first using the strong Markov property of the reflected Brownian motion and then (4.3), we have

$$\begin{aligned} & E^Q \left\{ K_3 e^{-tK_3} \mathbb{1}_\Gamma / \mathcal{F}_x \right\} \\ &= \left\{ 1 - W(x) \right\} E^Q \left\{ K([0, b_2]) e^{-tK([0, b_2])} \right\} \\ &= \frac{2}{(2t + 1)^2} \left\{ 1 - W(x) \right\}, \quad \text{a.s.} \end{aligned}$$

Since $\{x < b_2\} = \{x < c'\} \cap \Gamma$ and $\{x < c'\} \in \mathcal{F}_x$, we have

$$\begin{aligned} & E^Q \left\{ K_3 e^{-t(K_2 + K_3)}; x < b_2 \right\} \\ &= E^Q \left[e^{-tK_2} \mathbb{1}_{\{x < c'\}} E^Q \left\{ K_3 e^{-tK_3} \mathbb{1}_\Gamma / \mathcal{F}_x \right\} \right] \end{aligned}$$

$$= \frac{2}{(2t+1)^2} E^Q \left\{ (1 - W(x)) e^{-tK_2}; x < c' \right\},$$

proving (4.10) and hence the lemma.

Lemma 9. For $\lambda > 0$ and $t > 0$

$$(4.11) \quad \int_0^\infty e^{-\lambda x} E^Q \left\{ (1 - W(x)) e^{-tK([0, x])}; x < c' \right\} dx \\ = \frac{1}{\lambda} \left\{ 1 - \frac{(2t+1)S}{C + 2tS} \right\},$$

where

$$C = \cosh \sqrt{2\lambda}, \quad S = \frac{\sinh \sqrt{2\lambda}}{\sqrt{2\lambda}}.$$

Proof. Let $\varphi(x) = 1 - |x|$. Consulting with [4: Chapter 5], we see that the left hand side of (4.11) equals $f_\lambda(0)$ where f_λ is the continuous solution of

$$(4.12) \quad \begin{cases} \lambda f - \frac{1}{2}f'' = \varphi & \text{in } (-1, 0) \cup (0, 1) \\ \frac{1}{2}\{f'(0+) - f'(0-)\} = 2tf(0) \\ f(-1) = f(1) = 0. \end{cases}$$

To solve (4.12) we first find the solution g_λ of $\lambda f - \frac{1}{2}f'' = \varphi$ in $(-1, 1)$ with boundary condition $f(-1) = f(1) = 0$ and then express f_λ as follows:

$$f_\lambda(x) = \begin{cases} g_\lambda(x) + c \sinh \sqrt{2\lambda}(1+x) & \text{for } x \in (-1, 0) \\ g_\lambda(x) + c \sinh \sqrt{2\lambda}(1-x) & \text{for } x \in (0, 1) \end{cases}.$$

If we determine c so that the above f_λ satisfies the second condition of (4.12), then the f_λ is a solution of (4.12). Thus $f_\lambda(0)$ can be computed and we obtain (4.11).

Now Theorem 2 can be proved as follows. By Lemma 8 we have

$$\begin{aligned} \mu((x, \infty)) &= E^Q \left\{ \frac{K((x, x \vee b_2])}{K([b_1, b_2])} \right\} \\ &= E^Q \left\{ \frac{K_3}{K_1 + K_2 + K_3}; x < b_2 \right\} \\ &= \int_0^\infty E^Q \left\{ K_3 e^{-t(K_1 + K_2 + K_3)}; x < b_2 \right\} dt \end{aligned}$$

$$= \int_0^{\infty} \frac{2}{(2t+1)^3} E^Q \left\{ (1 - W(x)) e^{-tK([0,x])} ; x < c' \right\} dt$$

and hence by Lemma 9

$$\begin{aligned} \int_0^{\infty} e^{-\lambda x} \mu((x, \infty)) dx &= \int_0^{\infty} \frac{2}{(2t+1)^3} \cdot \frac{1}{\lambda} \left\{ 1 - \frac{(2t+1)S}{c+2tS} \right\} dt \\ &= \frac{1}{2\lambda} - \frac{1}{\lambda} \int_0^{\infty} \frac{2}{(2t+1)^2} \cdot \frac{S}{c+2tS} dt. \end{aligned}$$

Thus integration by parts yields

$$\begin{aligned} \int_0^{\infty} e^{-\lambda x} \mu(dx) &= \frac{1}{2} - \lambda \int_0^{\infty} e^{-\lambda x} \mu((x, \infty)) dx \quad (\text{notice that } \mu((0, \infty)) = \frac{1}{2}) \\ &= \int_0^{\infty} \frac{2S}{(2t+1)^2(c+2tS)} dt \\ &= \int_0^{\infty} \frac{Sdt}{(t+1)^2(c+tS)}, \end{aligned}$$

and this proves (3).

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