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LIMIT DISTRIBUTION FOR 1-DIMENSIONAL DIFFUSION IN A REFLECTED BROWNIAN MEDIUM

By H. Tanaka

Introduction

In analogy with Sinai's problem [8] on a random walk in a random medium, Brox [1] considered the diffusion process $X(t)$ described by the stochastic differential equation

$$(1) \qquad dX(t) = dB(t) - \frac{1}{2}W'(X(t))dt \ , \quad X(0) = 0 \ ,$$

where $\{W(x),\ x\in\mathbb{R}\}$ is a Brownian medium independent of another Brownian motion $B(t)$, and proved that $(\log t)^{-2}X(t)$ converges in distribution as $t \to \infty$. Similar results in the case of a considerably wider class of self-similar random media were obtained by Schumacher [7]. Recently Kesten [5] obtained the exact form of the limit distribution for Sinai's random walk as well as for a diffusion in a Brownian medium. See also [2] for a related problem.

In this paper we substitute $W(x)$ in (1) by a nonnegative reflected Brownian medium and find the corresponding limit distribution. The result was already anounced in [9] without proof but the Laplace transform of the limit distribution given in [9: §3] is not correct. We give here a full proof to the whole result of [9: §3] with a correction (see Theorem 1 and 2 below). Our method is similar to that of [1].

<u>Theorem 1</u>. Let $X(t)$ be a solution of (1) where $W_+ = \{W(x), x \geq 0\}$ and $W_- = \{W(-x),\ x \geq 0\}$ are independent reflected Brownian motions on the half line $[0, \infty)$ starting from 0 which are also independent of the Brownian motion $B(t)$. Then the distribution of $(\log t)^{-2}X(t)$ converges as $t \to \infty$ to the distribution μ defined by

$$(2) \qquad \mu = \int m_W Q(dW)$$

where m_W is the probability measure on $\mathbb{R}$ defined by (3.1) and Q is the probability measure on the space of media $\mathbb{W} = C(\mathbb{R}\to [0,\infty)) \cap \{W: W(0)=0\}$ such that $W_\pm$ are independent reflected Brownian motions on $[0, \infty)$.

<u>Theorem 2</u>. μ has a symmetric density and for $\lambda > 0$

$$(3) \qquad \int_0^\infty e^{-\lambda x}\, \mu(dx) = \int_0^\infty \frac{\sinh\sqrt{2\lambda}}{\sqrt{2\lambda}\,\cosh\sqrt{2\lambda} + t\,\sinh\sqrt{2\lambda}} \cdot \frac{dt}{(1+t)^2} \quad .$$

The present case is not contained in the framework of [7] since the nonnegative reflected medium $W(x)$ has (uncountably) many points giving its minimum. The case of a nonpositive reflected Brownian medium was discussed in [9] . Some generalizations will be discussed in [5] .

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§1. Preliminaries and exit times from valleys

Let $\mathbb{W}$ and Q be defined as in Theorem 1. For each $W \in \mathbb{W}$ solutions of the stochastic differential equation (1) define a diffusion process in $\mathbb{R}$ with generator

$$\frac{1}{2}\, e^{W(x)} \frac{d}{dx}\left(e^{-W(x)} \frac{d}{dx}\right) . \tag{1.1}$$

Such a diffusion can be constructed from a Brownian motion $B(t)$ [1] as follows ([4]). Let Ω be the space of continuous functions $\omega : [0, \infty) \to \mathbb{R}$ with $\omega(0) = 0$, and denote by P the Wiener measure on Ω. Denote the value of ω at time t by $\omega(t)$ or by $B(t)$ and put

$$L(t,x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[x,x+\varepsilon)}(B(s))ds \qquad \text{(local time)},$$

$$S(x) = \int_0^x e^{W(y)}dy ,$$

$$A(t) = \int_0^t e^{-2W(S^{-1}(B(s)))}ds = \int_{\mathbb{R}} e^{-2W(S^{-1}(x))}L(t,x)dx , \quad t \geq 0 ,$$

S^{-1}, A^{-1} = the inverse functions .

Then the process $X(t, W) = S^{-1}(B(A^{-1}(t)))$ defined on the probability space (Ω, P) is a diffusion process with generator (1.1) starting at 0. If we set $(W^x)(\cdot) = W(\cdot + x)$, then $X^x(t, W) = x + X(t, W^x)$ is a diffusion process with generator (1.1) starting at x . Let

$$T(x_1, x_2) = \inf \{t \geq 0 : B(t) \notin (x_1, x_2)\} ,$$

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1) The Brownian motion here is not the same as the one in (1) but we use the same notation $B(t)$.

$$L(x_1, x_2, x) = L(T(x_1, x_2), x) , \quad x \in \mathbb{R} ,$$

$$S_\lambda(x) = \int_0^x e^{\lambda W(y)} dy ,$$

$$X_\lambda(t) = X(t, \lambda W) , \quad X_\lambda^x(t) = x + X(t, \lambda W^x) .$$

Next we define a valley. Given $W \in \mathbb{W}$, a quartet $V = (a, b_1, b_2, c)$ is called a valley of W if

(i) $a < b_1 < 0 < b_2 < c$,

(ii) $W(b_1) = W(b_2) = 0, \quad W(a) = W(c) = D$,

(iii) $0 < W(x) < W(a) \quad \text{for} \quad a < x < b_1$,

$0 < W(x) < W(c) \quad \text{for} \quad b_2 < x < c$,

(iv) $A_- = \sup \{W(y) - W(x) : a < x < y < b_2\} < D$,

$A_+ = \sup \{W(x) - W(y) : b_1 < x < y < c\} < D$.

We call D (resp. $A = A_- \vee A_+$ [2]) the depth (resp. the inner directed ascent) of V. It is clear that there exist valleys of W with $A < 1 < D$ for almost all reflected Brownian media W .

In what follows let $W \in \mathbb{W}$ be given and $V = (a, b_1, b_2, c)$ be a valley of W with the depth D and the inner directed ascent A . We put

$$T_\lambda^x = T_\lambda^x(a, c) = \inf \{t \geq 0 : X_\lambda^x(t) \notin (a, c)\} .$$

The following three lemmas were proved in [1] .

Lemma 1. For $a < x < c$

$$T_\lambda^x(a, c) \overset{d}{=} \int_a^c L(\hat{S}_\lambda(a), \hat{S}_\lambda(c), \hat{S}_\lambda(y)) e^{-\lambda W(y)} dy ,$$

where

$$\hat{S}_\lambda(y) = \int_x^y e^{\lambda W(z)} dz$$

and $\overset{d}{=}$ means the equality in distribution.

Lemma 2. For each $\lambda > 0$

$$\{L(\lambda x_1, \lambda x_2, \lambda x), x \in \mathbb{R}\} \overset{d}{=} \{\lambda L(x_1, x_2, x), x \in \mathbb{R}\}.$$

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2) $a \vee b = \max \{a, b\}$, $a \wedge b = \min \{a, b\}$.

Lemma 3. For $\lambda > 0$ and $W \in \mathbb{W}$

(1.2) $$\{X(t, \lambda W_\lambda), t \geq 0\} \overset{d}{=} \{\lambda^{-2}X(\lambda^4 t, W), t \geq 0\} ,$$

where W_λ ($\in \mathbb{W}$) is defined by

$$W_\lambda(x) = \lambda^{-1}W(\lambda^2 x) , \quad x \in \mathbb{R} .$$

The following lemma plays an essential role in our discussions.

Lemma 4. For any $\lambda > 0$ and $[u, v] \subset (a, c)$

$$\inf_{u \leq x \leq v} P\left\{e^{\lambda(D-\delta)} < T_\lambda^x < e^{\lambda(D+\delta)}\right\} \to 1 , \quad \lambda \to \infty .$$

Proof. The proof is similar to that of the corresponding lemma of [1] but even much simpler. Let $x \in [u, v]$ be fixed. Setting

$$s_\lambda(y) = \hat{S}_\lambda(y)/\hat{S}_\lambda(c) = \int_x^y e^{\lambda W(z)}dz \Big/ \int_x^c e^{\lambda W(z)}dz$$

and applying Lemma 1 and 2, we have

$$T_\lambda^x \overset{d}{=} \hat{S}_\lambda(c)\int_a^c L(s_\lambda(a), 1, s_\lambda(y))e^{-\lambda W(y)}dy .$$

Since

$$\hat{S}_\lambda(c) \leq (c - x)\exp\left\{\lambda \max_{[x,c]} W\right\} \quad ^{3)}$$

$$T_\lambda^x \overset{d}{\leq} (c - x)(c - a)\exp\left\{\lambda \max_{[x,c]} W - \lambda \min_{[a,c]} W\right\} L' \leq (c - a)^2 L' e^{\lambda D},$$

$$L' = \max_{y \leq 1} L(-\infty, 1, y) ,$$

we have

$$\begin{aligned} &P\left\{T_\lambda^x > e^{\lambda(D+\delta)}\right\} \\ &\leq P\left\{(c - a)^2 L' e^{\lambda D} > e^{\lambda(D+\delta)}\right\} \\ &= P\left\{L' > e^{\lambda\delta}/(c - a)^2\right\} \to 0 , \quad \lambda \to \infty . \end{aligned}$$

To obtain an estimate from below first we notice that

(1.3) $$\lim_{\lambda\to\infty} \lambda^{-1}\log C_\lambda = D ,$$

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3) $\max_I W = \max\{W(x), x \in I\}$, $\min_I W = \min\{W(x), x \in I\}$.

where

$$C_\lambda = |\hat{S}_\lambda(a)| \wedge |\hat{S}_\lambda(c)| ,$$

and the convergence is uniform in $x \in [u, v]$. Next, for given $\delta > 0$ we set

$$a_1 = \sup\{x < b_1 : W(x) = \delta/4\} ,$$
$$\hat{s}_\lambda(y) = \hat{S}_\lambda(y)/C_\lambda ,$$
$$L_\lambda = \min\{L(-1, 1, y) : \hat{s}_\lambda(a_1) \le y \le \hat{s}_\lambda(b_1)\} .$$

Then applying Lemma 1 and 2 we have

$$\begin{aligned} T_\lambda^x &\overset{d}{=} C_\lambda \int_a^c L(\hat{s}_\lambda(a), \hat{s}_\lambda(c), \hat{s}_\lambda(y)) e^{-\lambda W(y)} dy \\ &\ge C_\lambda \int_{a_1}^{b_1} L(-1, 1, \hat{s}_\lambda(y)) e^{-\lambda W(y)} dy \\ &\ge e^{\lambda(D-\frac{\delta}{4})}(b_1 - a_1) L_\lambda \exp\left\{-\lambda \max_{[a_1, b_1]} W\right\} \\ &= (b_1 - a_1) L_\lambda e^{\lambda(D-\frac{\delta}{2})} . \end{aligned}$$

Since $\lambda^{-1}\log|\hat{s}_\lambda(a_1)|$ and $\lambda^{-1}\log|\hat{s}_\lambda(b_1)|$ converges to $\max\limits_{[x\wedge a_1, x\vee a_1]} W - D$, $\max\limits_{[x\wedge b_1, x\vee b_1]} W - D$, respectively, which are both negative, we have

$$\lim_{\lambda\to\infty} \hat{s}_\lambda(a_1) = \lim_{\lambda\to\infty} \hat{s}_\lambda(b_1) = 0 ,$$

the convergence being uniform in $x \in [u, v]$. Therefore

$$P\left\{T_\lambda^x < e^{\lambda(D-\delta)}\right\} \le P\left\{L_\lambda < (b_1 - a_1)^{-1} e^{-\lambda\delta/2}\right\} \to 0 , \quad \lambda \to \infty$$

uniformly in $x \in [u, v]$, because $\lim\limits_{\lambda\to\infty} L_\lambda = L(-1, 1, 0) > 0$.

§2. The limit distribution of $X(e^{\lambda r}, \lambda W)$

In this section we change the notation slightly. Given $W \in \mathbb{W}$ and a valley $V = (a, b_1, b_2, c)$ of W, we set

$$\Omega = C([0, \infty) \to \mathbb{R}) ,$$
$$\hat{\Omega} = C([0, \infty) \to [a, c]) ,$$

and denote by P_λ^x, $x \in \mathbb{R}$ (resp. $\hat{P}_\lambda^y$, $y \in [a, c]$) the probability measure on Ω (resp. $\hat{\Omega}$) induced by the diffusion process with generator

$$\frac{1}{2} e^{\lambda W(x)} \frac{d}{dx}\left(e^{-\lambda W(x)} \frac{d}{dx}\right) \tag{2.1}$$

(resp. the diffusion process on $[a, c]$ with (local) generator (2.1) and with reflecting barriers at a and c). The latter diffusion has the invariant probability measure m_λ given by

$$m_\lambda(dy) = e^{-\lambda W(y)} dy \Big/ \int_a^c e^{-\lambda W(z)} dz .$$

For any interval $[u, v] \subset [a, c]$

$$m_\lambda([u, v]) = \frac{\int_0^\infty e^{-\lambda\xi} K([u, v], \xi) d\xi}{\int_0^\infty e^{-\lambda\xi} K([a, c], \xi) d\xi}$$

where, for an interval I in $\mathbb{R}$, $K(I, \xi)$ is the local time at ξ for the reflected Brownian medium, i.e.,

$$K(I, \xi) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_I \mathbb{1}_{[\xi, \xi+\varepsilon)}(W(s)) ds . \tag{2.2}$$

Therefore

$$m_\lambda([u, v]) = \frac{\int_0^\infty e^{-\xi} K([u, v], \lambda^{-1}\xi) d\xi}{\int_0^\infty e^{-\xi} K([a, c], \lambda^{-1}\xi) d\xi} \tag{2.3}$$

$$\to \frac{K([u, v], 0)}{K([a, c], 0)} \equiv m([u, v]) , \quad \lambda \to \infty .$$

Next we set

$$\hat{P}_\lambda = \int_a^b m_\lambda(dy) \hat{P}_\lambda^y , \quad \mathbb{P}_\lambda^{x,y} = P_\lambda^x \otimes \hat{P}_\lambda^y , \quad \mathbb{P}_\lambda^x = P_\lambda^x \otimes \hat{P}_\lambda .$$

$$R = R(\omega, \hat{\omega}) = \inf\{t \geq 0 : \omega(t) = \hat{\omega}(t)\} .$$

Lemma 5. For any $\delta > 0$

$$\lim_{\lambda\to\infty} \mathbb{P}_\lambda^0\left\{R < e^{\lambda(A+\delta)}\right\} = 1 .$$

Proof. First we prove that

$$\lim_{\lambda\to\infty} \mathbb{P}_\lambda^x\left\{R < e^{\lambda(A+\delta)}\right\} = 1 \quad \text{holds for} \quad x = b_1 \text{ and } b_2 . \tag{2.4}$$

Without loss of generality we may consider the case $x = b_2$. We write b instead of b_2 for simplicity. For any $\delta > 0$ such that $A + \delta < D$ we define $a_1 \in (a, b_1)$, $a_2 \in (a, b_1)$, $c_2 \in (b_2, c)$ by

$$a_1 = \max\left\{x < b_1 : W(x) = A + \frac{\delta}{4}\right\},$$
$$a_2 = \max\left\{x < b_1 : W(x) = A + \frac{\delta}{2}\right\},$$
$$c_2 = \min\left\{x > b_2 : W(x) = A + \frac{\delta}{2}\right\},$$

and set

$$T_0 = T_0(\omega) = \inf\left\{t \geq 0 : w(t) = a_1\right\},$$
$$T_1 = T_1(\omega) = \inf\left\{t \geq 0 : w(t) \notin (a_1, c_2)\right\},$$
$$T_2 = T_2(\omega) = \inf\left\{t \geq 0 : w(t) \notin (a_2, c_2)\right\}.$$

Then we can prove easily that

$$(2.5) \qquad P_\lambda^b\left\{T_0 < \infty\right\} \geq P_\lambda^b\left\{T_0 = T_1\right\} = \frac{S_\lambda(c_2) - S_\lambda(b)}{S_\lambda(c_2) - S_\lambda(a_1)} \to 1, \quad \lambda \to \infty,$$

and hence

$$(2.6) \qquad \begin{aligned} &\mathbb{P}_\lambda^b\left\{R \leq T_0\right\} \\ &\geq \mathbb{P}_\lambda^b\left\{\hat{\omega}(0) \in [a, b], \hat{\omega}(T_0) \in [a_1, c]\right\} \\ &\geq \mathbb{P}_\lambda^b\left\{\hat{\omega}(0) \in [a, b]\right\} + P_\lambda^b\left\{\hat{\omega}(T_0) \in [a_1, c]\right\} - 1 \\ &= m_\lambda([a, b]) + \int_0^\infty \hat{P}_\lambda\left\{\hat{\omega}(t) \in [a_1, c]\right\} P_\lambda^b\left\{T_0 \in dt\right\} - 1 \\ &\to 1, \quad \lambda \to \infty, \end{aligned}$$

by (2.3) because $m(\{x \in (a, c) : W(x) = 0\}) = 1$. On the other hand Lemma 4 applied to the valley (a_2, b_1, b_2, c_2) whose depth is $A + (\delta/2)$ implies

$$(2.7) \qquad P_\lambda^b\left\{T_1 < e^{\lambda(A+\delta)}\right\} \geq P_\lambda^b\left\{T_2 < e^{\lambda(A+\delta)}\right\} \to 1, \quad \lambda \to \infty,$$

and so

$$\begin{aligned} &\mathbb{P}_\lambda^x\left\{R < e^{\lambda(A+\delta)}\right\} \\ &\geq P_\lambda^x\left\{T_0 < e^{\lambda(A+\delta)}\right\} - o(1) && \text{(by (2.6))} \\ &\geq P_\lambda^x\left\{T_1 < e^{\lambda(A+\delta)}, T_1 = T_0\right\} - o(1) \end{aligned}$$

$$\geq P_\lambda^x\left\{T_1 < e^{\lambda(A+\delta)}\right\} - o(1) \qquad \text{(by (2.5))}$$

$$\longrightarrow 1 , \quad \text{as } \lambda \to \infty \qquad \text{(by (2.7))} .$$

Next , to consider the case where the diffusion starts at 0 we shall consider three diffusion processes starting at 0 , b_1 and b_2 , respectively. By making use of the comparison theorem in one-dimensional diffusion processes (for example, see [3 : p.352]) we can construct, on a suitable probability space $(\widetilde{\Omega}_\lambda, \widetilde{P}_\lambda)$, three processes $\widetilde{X}_0(t)$, $\widetilde{X}_1(t)$ and $\widetilde{X}_2(t)$ such that the probability measure on Ω induced by $\widetilde{X}_0(t)$ (resp. $\widetilde{X}_1(t)$, $\widetilde{X}_2(t)$) coincides with P_λ^0 (resp. $P_\lambda^{b_1}$, $P_\lambda^{b_2}$) and

$$\text{(2.8)} \qquad \widetilde{X}_1(t) \leq \widetilde{X}_0(t) \leq \widetilde{X}_2(t) , \qquad \forall t \geq 0 , \quad \widetilde{P}_\lambda\text{-a.s.}$$

Put

$$\widetilde{\mathbb{P}}_\lambda = \widetilde{P}_\lambda \otimes \widehat{P}_\lambda ,$$

$$\widetilde{R}_i = \inf\left\{t \geq 0 : \widetilde{X}_i(t) = \widehat{\omega}(t)\right\} , \quad i = 0, 1, 2 .$$

Since $\widetilde{R}_0 \leq \widetilde{R}_1 \vee \widetilde{R}_2$ by (2.8), we have

$$\begin{aligned} P_\lambda^0\left\{R<e^{\lambda(A+\delta)}\right\} &= \widetilde{\mathbb{P}}_\lambda\left\{\widetilde{R}_0 < e^{\lambda(A+\delta)}\right\} \\ &\geq \widetilde{\mathbb{P}}_\lambda\left\{\widetilde{R}_1 \vee \widetilde{R}_2 < e^{\lambda(A+\delta)}\right\} \\ &\geq \mathbb{P}_\lambda^{b_1}\left\{R < e^{\lambda(A+\delta)}\right\} + \mathbb{P}^{b_2}\left\{R < e^{\lambda(A+\delta)}\right\} - 1 \\ &\longrightarrow 1 , \quad \lambda \to \infty \end{aligned}$$

by (2.4), completing the proof of Lemma 5.

<u>Lemma 6</u>. For any r_1 , r_2 with $A<r_1<r_2<D$ and for any interval $[u, v]$ in $\mathbb{R}$

$$\lim_{\lambda\to\infty} P_\lambda^0\left\{\omega(e^{\lambda r}) \in [u, v]\right\} = m([u, v]\cap[b_1, b_2])$$

uniformly in $r \in [r_1, r_2]$, where m is defined in (2.3).

Proof. Denote by T (resp. $\widehat{T}$) the exit time of (a, c) for $\omega(t)$ (resp. $\widehat{\omega}(t)$), and by $\widetilde{T}_R$ (resp. $\widehat{T}_R$) the exit time of (a, c) for $\omega(t)$ (resp. $\widehat{\omega}(t)$) after the collision time R . Since $m_\lambda(U) \to 1$ as $\lambda \to \infty$ for any open set U containing $\left\{x \in (a, c) : W(x) = 0\right\}$, it follows from Lemma 4 that

$$\widehat{P}_\lambda\left\{e^{\lambda(D-\delta)} < \widehat{T} < e^{\lambda(D+\delta)}\right\}$$

$$= \int_a^c m_\lambda(dx) P_\lambda^x \left\{ e^{\lambda(D-\delta)} < T < e^{\lambda(D+\delta)} \right\}$$

$$\longrightarrow 1 , \lambda \to \infty .$$

This combined with Lemma 5 implies

$$\begin{aligned} p_\lambda \; := \mathbb{P}_\lambda^0 &\left\{ R < e^{\lambda r_1} < e^{\lambda r_2} < \widehat{T}_R \right\} \\ \geq \mathbb{P}_\lambda^0 &\left\{ R < e^{\lambda r_1} < e^{\lambda r_2} < \widehat{T} \right\} \quad (\because \widehat{T} \leq \widehat{T}_R) \\ &\longrightarrow 1 , \lambda \to \infty . \end{aligned}$$

Therefore for $r \in [r_1, r_2]$

$$\begin{aligned} (2.9) \qquad & P_\lambda^0 \left\{ \omega(e^{\lambda r}) \in [u, v] \right\} \\ & \geq \mathbb{P}_\lambda^0 \left\{ R < e^{\lambda r_1}, \omega(e^{\lambda r}) \in [u, v], e^{\lambda r_2} < \widetilde{T}_R \right\} \\ & = \mathbb{P}_\lambda^0 \left\{ R < e^{\lambda r_1}, \widehat{\omega}(e^{\lambda r}) \in [u, v], e^{\lambda r_2} < \widehat{T}_R \right\} \\ & \geq p_\lambda + m_\lambda([u, v]) - 1 \\ & \longrightarrow m([u , v] \cap [b_1, b_2]) , \lambda \to \infty ; \end{aligned}$$

as for the above equality we used the strong Markov property. Similarly we have

$$\varliminf_{\lambda \to \infty} P_\lambda^0 \left\{ \omega(e^{\lambda r}) \in [u, v]^c \right\} \geq m([u, v]^c \cap [b_1, b_2]) ,$$

which combined with (2.9) implies

$$P_\lambda^0 \left\{ \omega(e^{\lambda r}) \in [u, v] \right\} \to m([u, v] \cap [b_1, b_2]) , \lambda \to \infty .$$

The uniform convergence in $r \in [r_1, r_2]$ is also clear.

§3. Proof of Theorem 1

Let $V = (a, b_1, b_2, c)$ be a valley of W such that $A < 1 < D$. Such a valley exists with Q-probability 1. In fact, b_1 and b_2 are taken as

$$b_1 = \text{the smallest root of } W(x) = 0 \text{ in } (a', 0)$$

$$b_2 = \text{the largest root of } W(x) = 0 \text{ in } (0, c')$$

where $a' = \sup\{x < 0 : W(x) = 1\}$ and $c' = \inf\{x > 0 : W(x) = 1\}$. The endpoints a and c can be chosen suitably so that $a < a'$, $c > c'$ and

$V = (a, b_1, b_2, c)$ is a valley with $A<1<D$. In what follows $V = (a, b_1, b_2, c)$ denotes such a valley of W. We denote by m_W the probability measure on $\mathbb{R}$ defined by

$$(3.1) \qquad m_W([u, v]) = \frac{K([u', v'], 0)}{K([b_1, b_2], 0)}$$

where $[u', v'] = [u, v] \cap [b_1, b_2]$. Then, in the notation of §1 Lemma 6 reads as follows: For any interval I in $\mathbb{R}$ and for any family $\{r(\lambda), \lambda>0\}$ satisfying $\lim_{\lambda\to\infty} r(\lambda) = 1$,

$$(3.2) \qquad \lim_{\lambda\to\infty} P\{X(e^{\lambda r(\lambda)}, \lambda W) \in I\} = m_W(I)$$

for almost all W with respect to Q. Now we define $\mathbb{P} = P \otimes Q$ and $\mu = \int m_W Q(dW)$. Integrating both sides of (3.2) with respect to Q we have

$$(3.3) \qquad \lim_{\lambda\to\infty} \mathbb{P}\{X(e^{\lambda r(\lambda)}, \lambda W) \in I\} = \mu(I) .$$

Next, define W_λ as in Lemma 3. Then $\{W_\lambda(x), x \in \mathbb{R}\}$ is again a reflected Brownian medium. Therefore (3.3) yields

$$(3.4) \qquad \lim_{\lambda\to\infty} \mathbb{P}\{X(e^{\lambda r(\lambda)}, \lambda W_\lambda) \in I\} = \mu(I) .$$

We now apply the scaling relation (1.2) to (3.4); the result is

$$\lim_{\lambda\to\infty} \mathbb{P}\{\lambda^{-2} X(\lambda^4 e^{\lambda r(\lambda)}, W) \in I\} = \mu(I) .$$

Taking $r(\lambda) = 1 - 4\lambda^{-1}\cdot\log\lambda$ in the above, we obtain

$$\lim_{\lambda\to\infty} \mathbb{P}\{\lambda^{-2} X(e^{\lambda}, W) \in I\} = \mu(I) .$$

This completes the proof of Theorem 1.

§4. Proof of Theorem 2

The absolute continuity of μ can be proved easily. In fact, if μ_n is the measure in $\mathbb{R}$ defined by

$$\mu_n(I) = E^Q\left\{\frac{K(I \cap [b_1, b_2])}{K([b_1, b_2])} ; K([b_1, b_2]) > \frac{1}{n}\right\},$$

then μ_n is absolutely continuous because

$$\begin{aligned}\mu_n(I) &\le n E^Q\{K(I \cap [b_1, b_2])\} \\ &= 2n \int_I p(|x|, 0, 0)dx ,\end{aligned}$$

where $p(t, \xi, \eta)$ is the transition density of the Brownian motion with absorbing barriers at ± 1. Thus μ is absolutely continuous because $\mu_n \uparrow \mu$ as $n \uparrow \infty$.

We proceed to the proof of (3). Let $K(I) = K(I, 0)$ be the local time at 0 for the reflected Brownian medium as defined by (2.2) with $\xi = 0$ and consider the number of times $d_\varepsilon(t)$ that the reflected Brownian path $\{W(u) : u \geq 0\}$ crosses down from $\varepsilon > 0$ to 0 before time t. Then as found in[4: p.48]

$$(4.1) \qquad Q\left\{\lim_{\varepsilon \downarrow 0} 2\varepsilon d_\varepsilon(t) = K([0, t]),\ t \geq 0\right\} = 1 .$$

Let a', c', b_1 and b_2 be defined as in the beginning of §3.

Lemma 7. For $\alpha, \beta > 0$

$$(4.2) \qquad E^Q\left\{e^{-\alpha K([0,b_2])-\beta c'}\right\} = \frac{1}{2\alpha + c(\beta)} \cdot \frac{2\sqrt{2\beta}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} ,$$

where

$$c(\beta) = \frac{e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \cdot \sqrt{2\beta} .$$

In Particular, $K([0, b_2])$ is exponentially distributed:

$$(4.3) \qquad E^Q\left\{e^{-\alpha K([0,b_2])}\right\} = \frac{1}{2\alpha + 1} .$$

Proof. Since $c(\beta) \sim 1$ as $\beta \downarrow 0$, (4.3) follows from (4.2) by letting $\beta \downarrow 0$. To prove (4.2) we first apply (4.1) to write down

$$\begin{aligned}
(4.4) \qquad & E^Q\left\{e^{-\alpha K([0,b_2])-\beta c'}\right\} \\
&= E^Q\left\{e^{-\alpha K([0,c'])-\beta c'}\right\} \\
&= \lim_{\varepsilon \downarrow 0} E^Q\left\{e^{-2\alpha\varepsilon d_\varepsilon(c')-\beta c'}\right\} \\
&= \lim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} e^{-2\alpha\varepsilon n} E^Q\left\{e^{-\beta T_\varepsilon}\right\}^{n+1} E^Q_\varepsilon\left\{e^{-\beta T_0}; T_0 < T_1\right\}^n E^Q_\varepsilon\left\{e^{-\beta T_1}; T_1 < T_0\right\} ,
\end{aligned}$$

where E^Q_ε denotes the expectation with respect to the probability measure of the reflected Brownian motion starting at ε and

$$T_x = \inf\{u \geq 0 : W(u) = x\} .$$

If we set

$$A_\varepsilon = e^{-2\alpha\varepsilon} E^Q\left\{e^{-\beta T_\varepsilon}\right\} E^Q_\varepsilon\left\{e^{-\beta T_0};\ T_0 < T_1\right\},$$

$$B_\varepsilon = E^Q\left\{e^{-\beta T_\varepsilon}\right\} E^Q_\varepsilon\left\{e^{-\beta T_1};\ T_1 < T_0\right\},$$

then (4.4) yields

$$(4.5) \qquad E^Q\left\{e^{-\alpha K([0,b_2])-\beta c'}\right\} = \lim_{\varepsilon\downarrow 0} B_\varepsilon \sum_{n=0}^{\infty} A_\varepsilon^n$$

$$= \lim_{\varepsilon\downarrow 0} \frac{B_\varepsilon}{1 - A_\varepsilon}.$$

Next we make use of the well-known formula

$$E_x\left\{e^{-\alpha T_a}; T_a < T_b\right\} = \frac{e^{\sqrt{2\alpha}(b-x)} - e^{-\sqrt{2\alpha}(b-x)}}{e^{\sqrt{2\alpha}(b-a)} - e^{-\sqrt{2\alpha}(b-a)}}, \quad a \le x \le b,$$

where E_x denotes the expectation with respect to the probability measure of a standard Brownian motion starting at x. We then have

$$(4.6) \qquad E^Q\left\{e^{-\beta T_\varepsilon}\right\} = 2E_0\left\{e^{-\beta T_\varepsilon};\ T_\varepsilon < T_{-\varepsilon}\right\}$$

$$= \frac{2(e^{\varepsilon\sqrt{2\beta}} - e^{-\varepsilon\sqrt{2\beta}})}{e^{2\varepsilon\sqrt{2\beta}} - e^{-2\varepsilon\sqrt{2\beta}}}$$

$$= 1 + 0(\varepsilon^2), \quad \varepsilon\downarrow 0;$$

$$(4.7) \qquad E^Q_\varepsilon\left\{e^{-\beta T_0}; T_0 < T_1\right\} = \frac{e^{\sqrt{2\beta}(1-\varepsilon)} - e^{-\sqrt{2\beta}(1-\varepsilon)}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}$$

$$= 1 - \frac{\sqrt{2\beta}(e^{\sqrt{2\beta}} + e^{-\sqrt{2\beta}})}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \cdot \varepsilon + o(\varepsilon), \quad \varepsilon\downarrow 0;$$

$$(4.8) \qquad E^Q_\varepsilon\left\{e^{-\beta T_1};\ T_1 < T_0\right\} = \frac{e^{\sqrt{2\beta}\,\varepsilon} - e^{-\sqrt{2\beta}\,\varepsilon}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}}$$

$$\sim \frac{2\sqrt{2\beta}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \cdot \varepsilon, \quad \varepsilon\downarrow 0.$$

From (4.6), (4.7) and (4.8) we obtain

$$\frac{B_\varepsilon}{1 - A_\varepsilon} \sim \frac{1}{2\alpha + c(\beta)} \cdot \frac{2\sqrt{2\beta}}{e^{\sqrt{2\beta}} - e^{-\sqrt{2\beta}}} \;, \quad \varepsilon \downarrow 0 \;,$$

which combined with (4.5) proves the lemma.

Given $x > 0$ we set

$$K_1 = K([b_1, 0]), \quad K_2 = K([0,x]), \quad K_3 = K([x, b_2]) \;.$$

Lemma 8. For $x > 0$ and $t > 0$

$$E^Q\left\{K_3 e^{-t(K_1+K_2+K_3)} ;\; x < b_2\right\} = \frac{2}{(2t+1)^3} E^Q\left\{(1-W(x)) e^{-tK([0,x])} ;\; x < c'\right\} \;. \tag{4.9}$$

Proof. The left hand side of (4.9) equals

$$E^Q\left\{e^{-tK_1}\right\} E^Q\left\{K_3 e^{-t(K_2+K_3)} ;\; x < b_2\right\} \;.$$

Since $E^Q\left\{e^{-tK_1}\right\} = (2t+1)^{-1}$ by Lemma 7, for the proof of the lemma it is enough to show

$$E^Q\left\{K_3 e^{-t(K_2+K_3)} ;\; x < b_2\right\} = \frac{2}{(2t+1)^2} E^Q\left\{(1 - W(x)) e^{-tK_2} ;\; x < c'\right\} \;. \tag{4.10}$$

To prove this we introduce the smallest σ-field $\mathcal{F}_x$ on $\mathbb{W}$ which makes $W(s)$, $0 \le s \le x$, measurable and consider the event Γ that the shifted trajectory $W(\cdot + x)$ hits 0 before it hits 1. Then first using the strong Markov property of the reflected Brownian motion and then (4.3), we have

$$\begin{aligned} & E^Q\left\{K_3 e^{-tK_3} \mathbb{1}_\Gamma / \mathcal{F}_x\right\} \\ &= \left\{1 - W(x)\right\} E^Q\left\{K([0, b_2]) e^{-tK([0,b_2])}\right\} \\ &= \frac{2}{(2t+1)^2}\left\{1 - W(x)\right\} \;, \quad \text{a.s.} \end{aligned}$$

Since $\{x < b_2\} = \{x < c'\} \cap \Gamma$ and $\{x < c'\} \in \mathcal{F}_x$, we have

$$\begin{aligned} & E^Q\left\{K_3 e^{-t(K_2+K_3)} ;\; x < b_2\right\} \\ &= E^Q\left(e^{-tK_2} \mathbb{1}_{\{x<c'\}} E^Q\left\{K_3 e^{-tK_3} \mathbb{1}_\Gamma / \mathcal{F}_x\right\}\right) \end{aligned}$$

$$= \frac{2}{(2t+1)^2} E^Q\left\{(1 - W(x))e^{-tK_2};\ x < c'\right\} ,$$

proving (4.10) and hence the lemma.

Lemma 9. For $\lambda > 0$ and $t > 0$

$$\int_0^\infty e^{-\lambda x} E^Q\left\{(1 - W(x))e^{-tK([0,x])};\ x < c'\right\} dx \tag{4.11}$$
$$= \frac{1}{\lambda}\left\{1 - \frac{(2t+1)S}{C + 2tS}\right\},$$

where

$$C = \cosh\sqrt{2\lambda} \quad , \quad S = \frac{\sinh\sqrt{2\lambda}}{\sqrt{2\lambda}} .$$

Proof. Let $\varphi(x) = 1 - |x|$. Consulting with [4: Chapter 5] , we see that the left hand side of (4.11) equals $f_\lambda(0)$ where f_λ is the continuous solution of

$$\begin{cases} \lambda f - \frac{1}{2}f'' = \varphi & \text{in } (-1,0)\cup(0,1) \\ \frac{1}{2}\{f'(0+) - f'(0-)\} = 2tf(0) \\ f(-1) = f(1) = 0 . \end{cases} \tag{4.12}$$

To solve (4.12) we first find the solution g_λ of $\lambda f - \frac{1}{2}f'' = \varphi$ in $(-1,1)$ with boundary condition $f(-1) = f(1) = 0$ and then express f_λ as follows:

$$f_\lambda(x) = \begin{cases} g_\lambda(x) + c\sinh\sqrt{2\lambda}\,(1+x) & \text{for } x \in (-1, 0) \\ g_\lambda(x) + c\sinh\sqrt{2\lambda}\,(1-x) & \text{for } x \in (0, 1) \end{cases} .$$

If we determine c so that the above f_λ satisfies the second condition of (4.12), then the f_λ is a solution of (4.12). Thus $f_\lambda(0)$ can be computed and we obtain (4.11).

Now Theorem 2 can be proved as follows. By Lemma 8 we have

$$\mu((x,\infty)) = E^Q\left\{\frac{K((x, x\vee b_2])}{K([b_1, b_2])}\right\}$$
$$= E^Q\left\{\frac{K_3}{K_1 + K_2 + K_3} ;\ x < b_2\right\}$$
$$= \int_0^\infty E^Q\left\{K_3 e^{-t(K_1+K_2+K_3)};\ x < b_2\right\}dt$$

$$= \int_0^\infty \frac{2}{(2t+1)^3} E^Q\left\{(1 - W(x))e^{-tK([0,x])} \; ; \; x < c'\right\}dt$$

and hence by Lemma 9

$$\int_0^\infty e^{-\lambda x} \mu((x,\infty))dx = \int_0^\infty \frac{2}{(2t+1)^3}\cdot\frac{1}{\lambda}\left\{1 - \frac{(2t+1)S}{C+2tS}\right\}dt$$

$$= \frac{1}{2\lambda} - \frac{1}{\lambda}\int_0^\infty \frac{2}{(2t+1)^2}\cdot\frac{S}{C+2tS}\,dt \; .$$

Thus integration by parts yields

$$\int_0^\infty e^{-\lambda x}\mu(dx) = \frac{1}{2} - \lambda\int_0^\infty e^{-\lambda x}\mu((x,\infty))dx \quad \text{(notice that } \mu((0,\infty)) = \tfrac{1}{2}\text{)}$$

$$= \int_0^\infty \frac{2S}{(2t+1)^2(C+2tS)}\,dt$$

$$= \int_0^\infty \frac{Sdt}{(t+1)^2(C+tS)} \;,$$

and this proves (3).

REFERENCES

[1] T.Brox, A one-dimensional diffusion process in a Wiener medium, to appear in Ann. Probab.

[2] A.O.Golosov, The limit distributions for random walks in random environments, Soviet Math. Dokl., 28(1983), 18-22.

[3] N.Ikeda and S.Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, 1981.

[4] K.Itô and H.P.McKean, Diffusion Processes and Their Sample Paths, Springer-Verlag, 1965.

[5] K.Kawazu, Y.Tamura and H.Tanaka, One-dimensional diffusions and random walks in random environments, in preparation.

[6] H.Kesten, The limit distribution of Sinai's random walk in random environment, to appear in Physica.

[7] S.Schumacher, Diffusions with random coefficients, Contemporary Math. (Particle Systems, Random Media and Large Deviations, ed. by R. Durrett), 41(1985), 351-356.

[8] Y.G.Sinai, The limiting behavior of a one-dimensional random walk in a random medium, Theory of Probab. and its Appl. 27(1982), 256-268.
[9] H.Tanaka, Limit distributions for one-dimensional diffusion processes in self-similar random environments, to appear in the Proc. of the workshop on Hydrodynamic behavior and interacting particle systems and applications held at IMA, University of Minnesota, March 1986.

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