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A Maximal Inequality for Martingale Local Times

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1. Introduction

Let M and N be continuous local martingales, let \hat{M}, \hat{N} denote $M-M_0$ and $N-N_0$ respectively, and let $L_t^a(M), L_t^a(N)$ denote the local times of M and N respectively.

It was shown in [3] that

$$K_p \left\| \sup_a \sup_t |L_t^a(M) - L_t^a(N)| \right\|_p \geq \left\| \langle \hat{M} - \hat{N} \rangle_\infty^{\frac{1}{2}} \right\|_p,$$

or equivalently,

$$c_p \left\| \sup_a \sup_t |L_t^a(M) - L_t^a(N)| \right\|_p \geq \left\| (\hat{M} - \hat{N})_\infty^* \right\|_p \quad (1.1)$$

for all $p \in (0, \infty)$, whilst Barlow and Yor established in [2] that

$$\left\| \sup_a \sup_t |L_t^a(M) - L_t^a(N)| \right\|_p \leq c_p \left\| (\hat{M} - \hat{N})_\infty^* \right\|_p^{\frac{1}{2}} \left\| (M_\infty^* + N_\infty^*) \right\|_p^{\frac{1}{2}} \left(1 \vee \ln \left\{ \frac{\left\| (M_\infty^* + N_\infty^*) \right\|_p}{\left\| (\hat{M} - \hat{N})_\infty^* \right\|_p} \right\} \right)^{\frac{1}{2}}.$$

In this note we prove the following:

Theorem 1 For all $p \in (1, \infty)$ there is a universal constant c_p such that for all continuous martingales $M, N \in H^1$

$$\left\| \sup_a \sup_\tau |L_\tau^a(M) - L_\tau^a(N)| \right\|_p \leq C_p \left\| \sup_a |L_\infty^a(M) - L_\infty^a(N)| \right\|_p.$$

2. Some preliminaries. We recall some properties of local times.

For a continuous semi-martingale $(X_\tau; \tau \geq 0)$ we may define (c.f. [1]) its family of local times by means of Tanaka's formula:

$$|X_\tau - a| = |X_0 - a| + \int_{0+}^\tau \text{sgn}(X_s) dX_s + L_\tau^a(X)$$

where

$$\text{sgn}(x) = \begin{cases} 1 & : x > 0 \\ -1 & : x \leq 0 \end{cases}$$

Note that $L_\tau^a(X)$ is increasing in τ and increases only on $\{\tau: X_\tau = a\}$ (c.f. [4]).

Furthermore it has been shown in [5] that if X is a continuous local martingale then $L_\tau^a(X)$ has a bi-continuous version and we shall assume, without loss of generality, that we are working with such a version.

To simplify notation we fix M and N , two continuous martingales, and their filtration $(F_\tau; \tau \geq 0)$ and define

$$U(a, \tau) = (L_\tau^a(M) - L_\tau^a(N))$$

$$A_\tau = \sup_a (L_\tau^a(M) - L_\tau^a(N)) = \sup_a U(a, \tau)$$

$$B_{\tau} = \sup_a (L_{\tau}^a(N) - L_{\tau}^a(M)) = - \inf_a U(a, \tau)$$

$$D_{\tau} = \sup_a |L_s^a(M) - L_s^a(N)|$$

and for any $(X_{\tau}; \tau \geq 0)$

$$X_{\tau}^* = \sup_{s \leq \tau} |X_s|, \hat{X}_{\tau} = X_{\tau} - X_0.$$

3. Proof of Theorem 1. The crucial result is contained in the following lemma:

Lemma 2 Define

$$\sigma_x = \inf\{\tau \geq 0: A_{\tau} \geq 2x\}$$

$$\tau_x = \inf\{\tau \geq \sigma_x : U(M_{\sigma_x}, \tau) \leq x\}$$

where, as is usual $\inf \phi$ is taken as ∞ : then, if M and N are in H^1 ,

$$\mathbb{E}[(2(\hat{M}-\hat{N})_{\infty}^* + A_{\infty})I_{(\sigma_x < \infty, \tau_x = \infty)}] \geq x \mathbb{P}(\sigma_x < \infty) \quad (3.1)$$

Proof It was shown in [3] that A_{τ} is continuous, so on $(\sigma_x < \infty)$, $A_{\sigma_x} = 2x$. Now M and N are in H^1 so $M_{\infty}^*, N_{\infty}^* < \infty$ a.s., so a.s. $U(a, \sigma_x)$ is zero off a compact set (since $L_{\tau}^a(X)$ only increases when X is at a) and continuous and we may conclude that $\sup_a U(a, \sigma_x)$ is attained.

We may deduce that, on $(\sigma_x < \infty)$, $\sup_a U(a, \sigma_x)$ is attained at $a = M_{\sigma_x}$ for, suppose not, then $\exists b \neq M_{\sigma_x}$ s.t. $2x = U(b, \sigma_x) > U(b, \tau)$ for all $\tau < \sigma_x$ but,

since $b \neq M_{\sigma_x}$, $\exists \tau < \sigma_x$ s.t. $L_{\tau}^b(M) = L_{\sigma_x}^b(M)$ whilst (since $L_s^b(N)$ is increasing in s) $L_{\tau}^b(N) \leq L_{\sigma_x}^b(N)$ so that $U(b, \tau) \geq U(b, \sigma_x)$ which contradicts the definition of σ_x . We conclude that, on $(\sigma_x < \infty)$, $U(M_{\sigma_x}, \sigma_x) = 2x$ whilst M is in H^1 so has a limit variable M_{∞} and so

$$\mathbb{E}[U(M_{\sigma_x}, \sigma_x) - U(M_{\sigma_x}, \tau_x)] = \mathbb{E}[(2x - U(M_{\sigma_x}, \tau_x))I_{(\sigma_x < \infty)}] \quad (3.2)$$

(since $\tau_x \geq \sigma$ so, on $(\sigma_x = \infty)$, $\sigma_x = \tau_x = \infty$).

Similarly, we may see that, on $(\tau_x < \infty)$, $U(M_{\sigma_x}, \tau_x) = x$ so that (3.2) is

$$\mathbb{E}[2xI_{(\sigma_x < \infty)} - xI_{(\tau_x < \infty)} - U(M_{\sigma_x}, \tau_x)I_{(\sigma_x < \infty, \tau_x = \infty)}] \quad (3.3)$$

Conversely, (3.2) is

$$\mathbb{E}[(L_{\tau_x}^{\sigma_x}(N) - L_{\sigma_x}^{\sigma_x}(N)) - (L_{\tau_x}^{\sigma_x}(M) - L_{\sigma_x}^{\sigma_x}(M))] \quad (3.4)$$

Applying Tanaka's formula to the two $(F_{\sigma_x + \tau} : \tau \geq 0)$ martingales,

$m_{\tau} = M_{\sigma_x + \tau}$ and $n_{\tau} = N_{\sigma_x + \tau}$, we obtain the formulae

$$\begin{aligned} L_{\tau_x}^{\sigma_x}(M) - L_{\sigma_x}^{\sigma_x}(M) &= L_{\tau_x - \sigma_x}^{\sigma_x}(m) \\ &= |M_{\tau_x} - M_{\sigma_x}| + \int_{\sigma_x}^{\tau_x} \text{sgn}(M_s - M_{\sigma_x}) dM_s \end{aligned} \quad (3.5.i)$$

$$\begin{aligned} L_{\tau_x}^{\sigma_x}(N) - L_{\sigma_x}^{\sigma_x}(N) &= L_{\tau_x - \sigma_x}^{\sigma_x}(n) \\ &= |N_{\tau_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}| \\ &\quad + \int_{\sigma_x}^{\tau_x} \text{sgn}(N_s - M_{\sigma_x}) dN_s \end{aligned} \quad (3.5.ii)$$

Now M and N are in H^1 and $|\text{sgn}(x)| = 1$ so the two stochastic integrals in (3.5) are uniformly integrable and so we may apply the optional sampling theorem to obtain:

$$\mathbb{E}[L_{\tau_x}^{M, \sigma_x}(M) - L_{\sigma_x}^{M, \sigma_x}(M)] = \mathbb{E}[M_{\tau_x} - M_{\sigma_x}] \quad (3.6.i)$$

$$\mathbb{E}[L_{\tau_x}^{M, \sigma_x}(N) - L_{\sigma_x}^{M, \sigma_x}(N)] = \mathbb{E}[(|N_{\tau_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}|)] \quad (3.6.ii)$$

Substituting equations (3.6) in (3.4), and equating (3.2), (3.3) and (3.4) we see that

$$\begin{aligned} & \mathbb{E}[2xI_{(\sigma_x < \infty)} - xI_{(\tau_x < \infty)} - U(M_{\sigma_x}, \tau_x)I_{(\sigma_x < \infty, \tau_x = \infty)}] \\ &= \mathbb{E}[|N_{\tau_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}| - |M_{\tau_x} - M_{\sigma_x}|] \end{aligned} \quad (3.7)$$

Now, by a similar argument to that given above, we may see that, on $(\tau_x < \infty)$, $N_{\tau_x} = M_{\sigma_x}$, so on $(\tau_x < \infty)$ the term inside the expectation on the RHS of (3.7) is non-positive whilst on $(\sigma_x = \infty)$ it disappears so that the RHS is dominated by

$$\mathbb{E}[(|N_{\infty} - M_{\infty}| - |N_{\sigma_x} - M_{\sigma_x}|) I_{(\sigma_x < \infty, \tau_x = \infty)}]$$

Observing that $|X_{\infty}| - |X_{\sigma_x}| \leq 2\hat{X}_{\infty}^*$ and rearranging terms in (3.7) we achieve the inequality:

$$\begin{aligned} & \mathbb{E}[(U(M_{\sigma_x}, \tau_x) + 2(\hat{M} - \hat{N})_{\infty}^*) I_{(\sigma_x < \infty, \tau_x = \infty)}] \\ & \geq 2x \mathbb{P}(\sigma_x < \infty) - x \mathbb{P}(\tau_x < \infty) \end{aligned} \quad (3.8)$$

All that remains, to complete the proof, is to see that, since

$$\tau_x \geq \sigma_x, \quad \mathbb{P}(\tau_x < \infty) \leq \mathbb{P}(\sigma_x < \infty), \text{ whilst on } (\tau_x = \infty)$$

$$U(M_{\sigma_x}, \tau_x) = U(M_{\sigma_x}, \infty) \leq A_\infty. \quad \square$$

Lemma 3 If M and N are martingales in H^1

$$\mathbb{E}(2(\hat{M}-\hat{N})_\infty^* + A_\infty)I_{(A_\infty \geq x)} \geq x \mathbb{P}(A_\infty^* \geq 2x) \quad (3.9)$$

Proof On $(\sigma_x < \infty, \tau_x = \infty)$, $A_\infty \geq x$ whilst $(\sigma_x < \infty) = (A_\infty^* \geq 2x)$ so (3.9) follows immediately from (3.1). \square

We may now establish the theorem:

Proof of the theorem: multiplying both sides of (3.9) by px^{p-2} and integrating with respect to x we obtain, by Fubini's theorem:

$$\frac{p}{p-1} \mathbb{E}(2(\hat{M}-\hat{N})_\infty^* + A_\infty)A_\infty^{p-1} \geq \mathbb{E}(A_\infty^*)^p/2^p \quad (3.10)_A$$

whilst reversing the roles of M and N in (3.9) we obtain:

$$\frac{p}{p-1} \mathbb{E}(2(\hat{M}-\hat{N})_\infty^* + B_\infty)B_\infty^{p-1} \geq \mathbb{E}(B_\infty^*)^p/2^p \quad (3.10)_B$$

Clearly $D_\tau = A_\tau \vee B_\tau$, so that, since A_τ and B_τ are non-negative,

$$2D_\tau^p \geq A_\tau^p + B_\tau^p \geq D_\tau^p.$$

Thus, adding (3.10)_A and (3.10)_B,

$$\frac{2p}{(p-1)} \mathbb{E}[(2(M-N)_\infty^* + D_\infty)D_\infty^{p-1}] \geq \mathbb{E}(D_\infty^*)^p/2^p$$

Applying Holder's inequality to the first term on the left, we obtain,

$$\frac{2^{p+1}}{(p-1)} \mathbb{E} \left(\left\| (\hat{M}-\hat{N})_\infty^* \right\|_p \left(\left\| D_\infty \right\|_p \right)^{p-1} + \mathbb{E} D_\infty^p \right) \geq \mathbb{E}(D_\infty^*)^p \quad (3.11)$$

Now, by (1.1), $\left\| (\hat{M}-\hat{N})_\infty^* \right\|_p \leq c_p \left\| D_\infty^* \right\|_p$, so substituting this inequality in (3.11):

$$\frac{2^{p+1}}{(p-1)} \left(\left\| D_\infty \right\|_p^p + 2c_p \left\| D_\infty^* \right\|_p \left\| D_\infty \right\|_p^{p-1} \right) \geq \left\| D_\infty^* \right\|_p^p, \quad (3.12)$$

and dividing both sides of (3.12) by $\left\| D_\infty \right\|_p^p$ we obtain the result that

$$\left\| D_\infty^* \right\|_p \leq K_p \left\| D_\infty \right\|_p$$

where K_p is the largest zero of

$$f_p(x) = x^p - \frac{2^{p+1}}{(p-1)} (2c_p x+1) \quad \square$$

Corollary 4 If M is in H^1 then for all $p \in (1, \infty)$, $a \in \mathbb{R}$

$$\left\| (M-M_0)_\infty^* \right\|_p \leq \frac{K_p}{2} \inf_{x \in \mathbb{R}_+} \left\| \sup_a L_\infty^a(M) - L_\infty^{x-a}(M) \right\|_p$$

This follows immediately from theorem 1 and (1.1) by setting $N = x-M$.

Remarks

(1) Theorem 8 of [1] enables us to extend the range of p in Theorem 1 to $(1, \infty]$.

(2) Corollary 4 is a specific case of the more general result that

$$\|(\hat{M}-\hat{N})_{\infty}^*\|_p \leq K_p \inf_{x \in \mathbb{R}} \left\| \sup_a |L_{\infty}^a(M) - L_{\infty}^{a-x}(N)| \right\|_p .$$

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References

- [1] AZÉMA, J. and YOR, M. En guise d'introduction. *Temps Locaux Astérisque* 52-53, 3-16 (1978).
- [2] BARLOW, M.T. and YOR, M. Semimartingale Inequalities via the Garsia-Rodemich-Rumsey Lemma. *J. Funct. Anal.*, 49, 198-229 (1982).
- [3] JACKA, S.D. A Local Time Inequality for Martingales. *Sém. Probab. XVII, Lecture Notes in Maths 986*. Berlin-Heidelberg-New York: Springer (1983).
- [4] YOR, M. Rappels et préliminaires généraux. *Temps Locaux Astérisque* 52-53, 17-22 (1978).
- [5] YOR, M. Sur la continuité des temps locaux associés à certaines semimartingales. *Ibid.* 23-36.