

# SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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*Séminaire de probabilités (Strasbourg)*, tome 21 (1987), p. 206-217

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# $L_p$ INEQUALITIES FOR FUNCTIONALS OF BROWNIAN MOTION

Richard Bass

## 1. Introduction

Let  $M_t$  be a continuous martingale. Let  $\langle M \rangle_t$  be the quadratic variation process, let  $M_t^* = \sup_{s \leq t} |M_s|$ , let  $L(t,x)$  be local time at  $x$ , and let  $L_t^* = \sup_x L(t,x)$ . Barlow and Yor [3,4] showed that in addition to the well-known equivalence in  $L_p$  norm between  $M_t^*$  and  $\langle M \rangle_T^{1/2}$ , one also had equivalence in  $L_p$  norm between  $L_T^*$  and  $\langle M \rangle_T^{1/2}$ . That is, if  $p \in (0, \infty)$ , there exist constants  $c_p$  and  $C_p$  depending only on  $p$  such that if  $T$  is any stopping time,

$$(1.1) \quad c_p E \langle M \rangle_T^{p/2} \leq E L_T^{*p} \leq C_p E \langle M \rangle_T^{p/2}.$$

Many other functionals of  $M$  have been found to be dominated in  $L_p$  norm by  $\langle M \rangle_T^{1/2}$ . These include various ratios of  $M^*$  and  $\langle M \rangle_T^{1/2}$  [4,6,9]; moduli of continuity of  $M$  and  $L(t,x)$  [4]; and number of upcrossings [1,2]. For example, if  $U_t(a, a+\epsilon)$  is the number of upcrossings of the interval  $[a, a+\epsilon]$  by  $M$  up to time  $t$ , and  $V_t = \sup_{\epsilon} \sup_a U_t(a, a+\epsilon)$ , the main result of [2] is that

$$(1.2) \quad E V_T^p \leq C_p E \langle M \rangle_T^{p/2},$$

$C_p$  a constant depending only on  $p$ .

The main purpose of this paper is to give some quite general and easily verifiable conditions for increasing functionals and ratios of increasing functionals of Brownian motion to dominate or be dominated in  $L_p$  norm by  $T^{1/2}$ . We state our results for Brownian motion, but these translate immediately via a time change argument to results for arbitrary continuous martingales. The results on  $L^*$ , ratios of  $M^*$  and  $\langle M \rangle_T^{1/2}$ , moduli of continuity, and upcrossings mentioned above then become

special cases of our general theorems. In particular, our proofs of Theorems 1 and 2 give a new and very simple demonstration of the main results of [3], while the proof of Theorem 4 gives a very simple demonstration of the result of [2].

As another application of Theorem 4, we also prove a new inequality. Let  $N_t(a, \epsilon)$  be the number of excursions of Brownian motion at level  $a$  of length longer than  $\epsilon$  that are completed by time  $t$ . Let  $S_t = \sup_{\epsilon} \sup_a \epsilon^{1/2} N_t(a, \epsilon)$ . We then show that there exists  $C_p$  depending on  $p \in (0, \infty)$  such that

$$(1.3) \quad ES_T^p \leq C_p E T^{p/2}.$$

Section 2 contains the results on increasing continuous functionals of Brownian motion plus some examples, while Section 3 contains the results on ratios of increasing continuous functionals. To handle upcrossings, we also need to consider discontinuous functionals, and this is done in Section 4.

I would like to thank Marc Yor for suggesting this problem and for his continued interest.

## 2. Increasing functionals

Suppose  $(B_t, P^x, \theta_t)$  is canonical Brownian motion. That is,  $\Omega = C([0, \infty), \mathbb{R})$ , the continuous functions from  $[0, \infty)$  to  $\mathbb{R}$ , and  $B_t(\omega) = \omega(t)$ , the coordinate map.  $P^x$  is Wiener measure on  $\Omega$  with  $P^x(B_0 = x) = 1$ . When  $x = 0$ , we will usually write just  $P$ . Denote the natural filtration by  $\mathbb{F}_t$ . Finally  $\theta_t : \Omega \rightarrow \Omega$  are the translation operators defined by  $(\theta_t(\omega))(s) = \omega(s+t)$ .

Suppose  $\Phi$  is increasing, continuous,  $\Phi(0) = 0$ , and of moderate growth:

$$(2.1) \quad \sup_{\lambda > 0} \frac{\Phi(a\lambda)}{\Phi(\lambda)} < a^p \quad \text{for all } a > 2, \text{ for some } p \in (0, \infty).$$

The functions  $x^p$ ,  $p \in (0, \infty)$  obviously satisfy these hypotheses.

Suppose  $F_t$  is a continuous adapted nondecreasing functional of  $\omega$  satisfying

$$(2.2) \quad (i) \quad (\text{Uniform scaling near } \infty) \quad \sup_{x, \lambda} p^x (F_{\frac{2}{\lambda}} > b\lambda) \rightarrow 0 \quad \text{as } b \rightarrow \infty;$$

(ii) (Subadditivity) There exists a constant  $K_1$  such that for all  $s, t$ ,

$$F_t - F_s \leq K_1 F_{t-s} \circ \theta_s.$$

Suppose  $G_t$  is a nondecreasing adapted functional of  $\omega$  satisfying

$$(2.3) \quad (i) \quad (\text{Uniform scaling near } 0) \quad \sup_{x, \lambda} p^x (G_{\frac{2}{\lambda}} < b\lambda) \rightarrow 0 \quad \text{as } b \rightarrow 0;$$

(ii) There exists a constant  $K_2$  such that for all  $s, t$ ,

$$G_{t-s} \circ \theta_s \leq K_2 G_t.$$

Note we do not require  $G$  to be continuous. A consequence of (2.3) (i) is that  $G_t > 0$ , a.s. for  $t > 0$ .

Our first two results are the following:

Theorem 1. Suppose  $F$  satisfies (2.2). There exists a constant  $C_\Phi$  such that if  $T$  is any stopping time, then

$$E\Phi(F_T) \leq C_\Phi E\Phi(T^{1/2}).$$

Theorem 2. Suppose  $G$  satisfies (2.3). There exists a constant  $C_\Phi$  such that if  $T$  is any stopping time, then

$$E\Phi(T^{1/2}) \leq C_\Phi E\Phi(G_T).$$

Before proving Theorems 1 and 2, we give some examples. The first

example is  $M_t^* = \sup_{s \leq t} |B_s - B_0|$ . The  $P^x$  distribution of  $M_t^*$  does not depend on  $x$ , and by scaling, we get (2.2) (i) and (2.3) (i). The subadditivity (2.2) (ii) is just the triangle inequality. Since  $M_{t-s}^* \circ \theta_s = \sup_{s \leq r \leq t} |B_r - B_s| \leq 2M_t^*$ , we have (2.3) (ii). Thus  $M^*$  satisfies both (2.2) and (2.3), and observing that  $P(B_0=0) = 1$ , we recover from Theorems 1 and 2 the well-known Burkholder-Davis-Gundy inequalities.

A more interesting example is  $L_t^* = \sup_x L(t,x)$ . Because of the supremum in  $x$ , the  $P^x$  distribution of  $L_t^*$  does not depend on  $x$ . By scaling and the well-known fact that  $0 < L_1^* < \infty$ , a.s., we get (2.2) (i) and (2.3) (i). Since  $L(t,x)$  is an additive functional,

$$(2.4) \quad L(t,x) = L(s,x) + L(t-s,x) \circ \theta_s.$$

Taking suprema over  $x$  leads to (2.2) (ii). Since by (2.4),

$$L(t-s,x) \circ \theta_s \leq L(t,x),$$

taking suprema over  $x$  again gives (2.3) (ii). Thus  $L^*$  satisfies both (2.2) and (2.3).

Two other examples satisfying both (2.2) and (2.3) that can be treated similarly are

$$C_t^B = \left[ \sup_{0 \leq r < s \leq t} \frac{|B_s - B_r|}{|s - r|^{1/2 - \epsilon}} \right]^{\epsilon/2}$$

and

$$C_t^L = \left[ \sup_{a \neq b} \frac{|L(t,a) - L(t,b)|}{|a - b|^{1/2 - \epsilon}} \right]^{\frac{2}{1+2\epsilon}}.$$

To show  $C_t^B$  is continuous, one needs to use the fact that

$$\limsup_{\delta \downarrow 0, |s-r| \leq \delta, r,s \in [0,t]} \frac{|B_s - B_r|}{|t - s|^{1/2 - \epsilon/2}} = 0, \text{ a.s.,}$$

with a similar comment for  $C_t^L$ .

We now prove Theorems 1 and 2.

**Proof of Theorem 1.** Trivially we may assume  $T < \infty$ , a.s. Let  $\beta > 1$ ,  $\delta < 1$ , and let  $U = \inf \{t: F_t > \lambda\}$ . Using the strong Markov property of Brownian motion at  $U$ ,

$$\begin{aligned}
 P(F_T > \beta\lambda, T^{1/2} < \delta\lambda) &< P(F_T - F_U > (\beta-1)\lambda, T < \delta^2\lambda^2, U < T) \\
 &< P(F_{U+\delta^2\lambda^2} - F_U > (\beta-1)\lambda, U < T) \\
 &< P(F_{\delta^2\lambda^2} \circ \theta_U > (\beta-1)\lambda / K_1, U < T) \\
 &= E[P(F_{\delta^2\lambda^2} \circ \theta_U > (\beta-1)\lambda / K_1 | \mathbb{F}_U); U < T] \\
 &= E[P^{B_U}(F_{\delta^2\lambda^2} > (\beta-1)\lambda / K_1); U < T] \\
 &< \sup_x P^x(F_{\delta^2\lambda^2} > (\beta-1)\lambda / K_1) P(U < T) \\
 &< \sup_{x, \lambda} P^x(F_{\lambda^2} > \frac{\beta-1}{2K_1\delta} \lambda) P(F_T > \lambda).
 \end{aligned}$$

By taking  $\delta$  sufficiently small, using (2.2) (i), and appealing to Lemma 7.1 of [5], the proof is complete.  $\square$

**Proof of Theorem 2.** Suppose  $\beta > 1$ ,  $\delta < 1$ . Using the Markov property at the fixed time  $\lambda^2$ , we have

$$\begin{aligned}
 P(T^{1/2} > \beta\lambda, G_T < \delta\lambda) &< P(T > \beta^2\lambda^2, G_{T-\lambda^2} \circ \theta_{\lambda^2} < K_2\delta\lambda) \\
 &< P(T > \lambda^2, G_{\beta^2\lambda^2-\lambda^2} \circ \theta_{\lambda^2} < K_2\delta\lambda) \\
 &= E[P(G_{(\beta^2-1)\lambda^2} \circ \theta_{\lambda^2} < K_2\delta\lambda | \mathbb{F}_{\lambda^2}); T > \lambda^2] \\
 &= E[P^{B_{\lambda^2}}(G_{(\beta^2-1)\lambda^2} < K_2\delta\lambda); T > \lambda^2] \\
 &< \sup_{x, \lambda} P^x(G_{(\beta^2-1)\lambda^2} < 2K_2\delta\lambda) P(T^{1/2} > \lambda).
 \end{aligned}$$

Again, take  $\delta$  sufficiently small and use (2.3)(i) and [5, Lemma 7.1] to complete the proof.  $\square$

### 3. Ratios of functionals

Our result here is

**Theorem 3** Suppose  $\alpha > 0$ . Suppose  $F$  satisfies (2.2),  $G$  satisfies (2.3), and moreover  $G$  is a continuous functional of  $\omega$ . Then there exists  $C_\Phi$  such that if  $T$  is any strictly positive stopping time,

$$E\Phi \left[ \frac{F_T^{\alpha+1}}{G_T^\alpha} \right] < C_\Phi E\Phi(G_T).$$

We make the obvious remark that if  $G_t$  satisfies (2.2) as well as (2.3), we can replace  $G_T$  on the right side of the above equation by  $T^{1/2}$ .

Proof. We start with

$$P \left[ \frac{F_T^{\alpha+1}}{G_T^\alpha} > \beta\lambda, G_T < \delta\lambda \right] = \sum_{n=0}^{\infty} P(F_T^{\alpha+1} > \beta\lambda G_T^\alpha, \delta 2^{-(n+1)}\lambda < G_T < \delta 2^{-n}\lambda) \\ < \sum_{n=0}^{\infty} p_n,$$

where

$$p_n = P(F_T > \beta' \zeta 2^{-n\gamma} \lambda, G_T < \delta 2^{-n}\lambda),$$

$$\gamma = \alpha/(\alpha+1),$$

$$\beta' = \beta^{1/(\alpha+1)},$$

$$\text{and } \zeta = \delta^\gamma 2^{-\gamma}.$$

Let

$$(3.1) \quad U_n = \inf \{t: F_t > 2^{-n\gamma} \zeta \lambda\},$$

$$V_n = \inf \{t: G_t > 2K_2 \delta 2^{-n}\lambda\},$$

$$\text{and } W_n = U_n + V_n \circ \theta_{U_n} = \inf \{t > U_n: G_{t-U_n} \circ \theta_{U_n} > 2K_2 \delta 2^{-n}\lambda\}.$$

Observe that by (2.3)(ii) we have  $W_n > T$  on the set  $(U_n < T, G_T < \delta 2^{-n}\lambda)$ .

Then by the strong Markov property at  $U_n$ ,

$$\begin{aligned}
 (3.2) \quad p_n &< P(F_T - F_{U_n} > (\beta' - 1)\zeta 2^{-n\gamma} \lambda, U_n < T, G_T < \delta 2^{-n} \lambda) \\
 &< P(F_{W_n} - F_{U_n} > (\beta' - 1)\zeta 2^{-n\gamma} \lambda, U_n < T) \\
 &< P(F_{V_n} \circ \theta_{U_n} > K_1^{-1}(\beta' - 1)\zeta 2^{-n\gamma} \lambda, U_n < T) \\
 &= E[P^{B_{U_n}}(F_{V_n} > K_1^{-1}(\beta' - 1)\zeta 2^{-n\gamma} \lambda); U_n < T]
 \end{aligned}$$

For any  $x$ , any  $r > 0$ ,

$$(3.3) \quad P^x(F_{V_n} > K_1^{-1}(\beta' - 1)\zeta 2^{-n\gamma} \lambda) \leq c E^x F_{V_n}^r 2^{rn\gamma} \lambda^{-r},$$

where here and in the remainder of the proof  $c$  denotes a constant whose value is unimportant and may change from place to place and which depends on  $\beta$ ,  $\alpha$ ,  $\delta$ , and  $r$ , but not  $\lambda$  or  $n$ . Using Theorems 1 and 2 with  $P$  replaced by  $P^x$ , the right side of (3.3) is

$$\leq c E^x V_n^{r/2} 2^{rn\gamma} \lambda^{-r} \leq c E^x G_{V_r}^r 2^{rn\gamma} \lambda^{-r}.$$

Since  $G_{V_r} \leq 2K_2 \delta 2^{-n} \lambda$ , we then have

$$\begin{aligned}
 p_r &\leq c 2^{rn(\gamma-1)} P(U_n < T) \\
 &\leq c 2^{rn(\gamma-1)} P(F_T > \zeta 2^{-n\gamma} \lambda).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (3.4) \quad P \left[ \frac{F_T^{\alpha+1}}{\beta G_T^\alpha} > \lambda \right] &\leq P \left[ \frac{F_T^{\alpha+1}}{G_T^\alpha} > \beta \lambda, G_T \leq \delta \lambda \right] + P(G_T > \delta \lambda) \\
 &\leq c \sum_{n=0}^{\infty} 2^{rn(\gamma-1)} P \left[ \frac{2^{n\gamma} F_T}{\zeta} > \lambda \right] + P \left[ \frac{G_T}{\delta} > \lambda \right].
 \end{aligned}$$

We integrate (3.4) against  $d\Phi(\lambda)$  and use integration by parts to get

$$\begin{aligned}
 E\Phi \left[ \frac{F_T^{\alpha+1}}{\beta G_T^\alpha} \right] &\leq c \sum_{n=0}^{\infty} 2^{rn(\gamma-1)} E\Phi \left[ \frac{2^{n\gamma} F_T}{\zeta} \right] + E\Phi \left[ \frac{G_T}{\delta} \right] \\
 &\leq c \sum_{n=0}^{\infty} 2^{rn(\gamma-1)+n\gamma} \zeta^{-p} E\Phi(F_T) + \delta^{-p} E\Phi(G_T).
 \end{aligned}$$



Since  $\gamma - 1 < 0$ , the infinite series will be summable provided we choose  $r$  larger than  $p\gamma/(1-\gamma)$ . Another application of Theorems 1 and 2 to handle  $E\Phi(F_T)$  completes the proof.  $\square$

#### 4. Discontinuous functionals

To handle the results on upcrossings of [2], we need to consider discontinuous functionals.

Suppose  $H_t$  is a nondecreasing adapted functional of  $\omega$  satisfying

$$(4.1)(i) \text{ (Uniform scaling near } \infty) \sup_{x, \lambda} P^x(H_2 > b\lambda) \rightarrow 0 \text{ as } b \rightarrow \infty.$$

(ii) There exists a continuous adapted nondecreasing functional  $F$  satisfying (2.2) such that

$$(a) \text{ (Bounded jumps) } \sup_{s < t} |\Delta H_s| < F_t \text{ for all } s, t;$$

$$(b) \text{ (Partial subadditivity) } H_t - H_s < K_3 H_{t-s} \circ \theta_s + F_t \text{ for all } s, t.$$

For such  $H$  we have

**Theorem 4** Suppose  $H$  satisfies (4.1). There exists a constant  $C_\Phi$  such that if  $T$  is any stopping time, then

$$E\Phi(H_T) < C_\Phi E\Phi(T^{1/2})$$

and

**Theorem 5** Suppose  $H$  satisfies (4.1). Suppose  $G$  satisfies (2.3) and moreover is a continuous functional of  $\omega$ . Suppose  $\alpha > 0$ . Then there exists a constant  $C_\Phi$  such that for any strictly positive stopping time  $T$

$$E\Phi \left[ \frac{H_T^{\alpha+1}}{G_T^\alpha} \right] < C_\Phi E\Phi(G_T).$$

Before proceeding to the proofs, let us look at some examples. First consider  $V_t = \sup_{a, \epsilon} U_t(a, a+\epsilon)$ , where  $U_t(a, a+\epsilon)$  is the number of

upcrossings of the interval  $[a, a+\epsilon]$  by time  $t$ . The  $P^x$  distribution of  $V_t$  is independent of  $x$  because of the supremum in  $a$ , and the scaling in  $\lambda$  follows easily from that of the Brownian motion. Provided we know  $P(V_1 < \infty) = 1$  (which we will show shortly), we then have (4.1)(i).

Let  $F_t = 2M_t^*$  and observe that we cannot have an upcrossing before time  $t$  of size larger than  $2M_t^*$ . This gives (4.1)(iia). It is not hard to see that

$$U_t(a, a+\epsilon) \leq U_s(a, a+\epsilon) + U_{t-s}(a, a+\epsilon) \circ \theta_s + 1_{(2M_t^* > \epsilon)}.$$

Multiplying by  $\epsilon$  and taking suprema over  $a$  and  $\epsilon$  gives (4.1)(iib).

It remains to show  $P(V_1 < \infty) = 1$ . Let  $\tau_r = \inf \{t: L_t^* > r\}$  and let  $T_r(x) = \inf \{t: L(t, x) > r\}$ . Let  $\epsilon_n = 2^{-n}$ . Fix  $M$  and let

$$W_n = \sup \{\epsilon_n U_{T_r}(a, a+\epsilon_n): |a| \leq M, a/\epsilon_n \text{ an integer}\}.$$

Since  $L(\tau_r, x) \leq r$ , then  $T_r(x) \geq \tau_r$ , and so  $U_{T_r}(a, a+\epsilon_n) \leq U_{T_r(a)}(a, a+\epsilon_n)$ . If  $N$  is the number of excursions at level  $a$  whose maxima exceed  $a+\epsilon_n$  by time  $T_r(a)$ , then  $U_{T_r(a)}(a, a+\epsilon_n) \leq N + 1$ . By Ito's theory of excursions,  $N$  is a Poisson random variable, and the parameter is  $r/2\epsilon_n$  (see [8]). By standard estimates for the tail of the Poisson distribution, if  $\beta > 3r$ ,

$$P(N > \beta/\epsilon_n) \leq \exp(-cr/\epsilon_n),$$

where  $c$  is a constant whose value is unimportant. From this follows

$$P(W_n > \beta+1) \leq 2M\epsilon_n^{-1} \exp(-cr/\epsilon_n).$$

This is summable in  $n$ , and by Borel-Cantelli,  $P(W_n > \beta+1 \text{ i.o.}) = 0$ . Each  $W_n < \infty$ , a.s. by the continuity of Brownian paths, and so we conclude that  $W = \sup_n W_n < \infty$ , a.s.

Given  $a$  and  $\epsilon$ , we can find  $n$  and  $x$  such that  $a \leq x \leq x+\epsilon_n \leq a + \epsilon$ ,  $x$  is

an integer multiple of  $\epsilon_n$ , and  $\epsilon/8 < \epsilon_n < \epsilon$ . So

$$\epsilon U_{T_r}(a, a+\epsilon) < 8\epsilon_n U_{T_r}(x, x+\epsilon_n).$$

Hence

$$\sup_{|a| < M/2, \epsilon} \epsilon U_{T_r}(a, a+\epsilon) < 8W < \infty, \text{ a.s.}$$

Finally,  $M$  and  $r$  are arbitrary; that  $V_1 < \infty$ , a.s. follows easily.

For a second example, consider  $S_t = \sup_{\epsilon, a} \epsilon^{1/2} N_t(a, \epsilon)$ , where  $N_t(a, \epsilon)$  is the number of excursions at level  $a$  whose length exceeds  $\epsilon$  and which are completed by time  $t$ . Let  $F_t = t^{1/2}$ . It is trivial that  $F$  satisfies (2.2). It is impossible to have completed an excursion of length longer than  $\epsilon$  by time  $t$  if  $\epsilon > t$ , and so (4.1)(ia) is immediate. The argument for (4.1)(ib) is similar to the one for  $V_t$ , and by scaling, we will have (4.1)(i) as soon as we know  $S_1 < \infty$ , a.s.

Since  $N_t(a, \epsilon) < t/\epsilon$  for  $\epsilon < t$  and  $= 0$  for  $\epsilon > t$ , it suffices to show  $\limsup_{\epsilon \downarrow 0} \sup_a \epsilon^{1/2} N_t(a, \epsilon) < \infty$ , a.s. But this follows from a result of Perkins [7].

We now prove Theorem 4.

Proof of Theorem 4 Let  $\beta > 3$ . Let  $U = \inf \{t: H_t > \lambda\}$ . By (4.1)(ia),  $H_U < \lambda + F_U$ . Then

$$\begin{aligned} P(H_T > \beta\lambda, T^{1/2} < \delta\lambda) &< P(H_T > \beta\lambda, T < \delta^2\lambda^2, F_T < \lambda) + P(F_T > \lambda) \\ &< P(H_T - H_U > (\beta-2)\lambda, U < T, F_T < \lambda, T < \delta^2\lambda^2) + P(F_T > \lambda) \\ &< P(H_{\delta^2\lambda^2} - \theta_U > (\beta-3)\lambda/K_3, U < T) + P(F_T > \lambda) \\ &= E[P_{\theta_U}^{B_U}(H_{\delta^2\lambda^2} > (\beta-3)\lambda/K_3); U < T] + P(F_T > \lambda) \\ &< \epsilon(\delta, \beta) P(H_T > \lambda) + P(F_T > \lambda), \end{aligned}$$

where  $\epsilon(\delta, \beta) = \sup_{x, \lambda} P^x(H_{\delta^2\lambda^2} > (\beta-3)\lambda/K_3)$ .

Next,

$$(4.2) \quad P \left[ \frac{H_T}{\beta} > \lambda \right] \leq P(H_T > \beta\lambda, T^{1/2} \leq \delta\lambda) + P(T^{1/2} > \delta\lambda) \\ \leq \epsilon(\delta, \beta) P(H_T > \lambda) + P(F_T > \lambda) + P \left[ \frac{T^{1/2}}{\delta} > \lambda \right].$$

Suppose for the moment that  $H_T$  is bounded. Integrating from 0 to  $\infty$  with respect to  $d\Phi(\lambda)$ ,

$$E\Phi \left[ \frac{H_T}{\beta} \right] \leq \epsilon(\delta, \beta) E\Phi(H_T) + E\Phi(F_T) + E\Phi \left[ \frac{T^{1/2}}{\delta} \right]$$

and so

$$(4.3) \quad E\Phi(H_T) \leq \beta^p E\Phi \left[ \frac{H_T}{\beta} \right] \leq \beta^p \epsilon(\delta, \beta) E\Phi(H_T) + \beta^p E\Phi(F_T) + \beta^p \delta^{-p} E\Phi(T^{1/2}).$$

Choose  $\delta$  sufficiently small so that  $\beta^p \epsilon(\delta, \beta) < 1/2$ . Subtracting  $\beta^p \epsilon(\delta, \beta) E\Phi(H_T)$  from both sides of (4.3), multiplying by  $[1 - \beta^p \epsilon(\delta, \beta)]^{-1}$ , and using Theorem 1 completes the proof when  $H_T$  is bounded.

If  $H_T$  is not bounded, note that (4.2) holding for  $H_T$  implies (4.2) holds for  $H_T \wedge N$ , for all  $N > 0$ . Arguing as above, we get

$$E\Phi(H_T \wedge N) \leq C_\Phi E\Phi(T^{1/2}),$$

$C_\Phi$  independent of  $N$ . Now let  $N \rightarrow \infty$ .  $\square$

Since the proof of Theorem 5 is very similar to that of Theorem 3, we omit the proof.

Note: B. Davis (in this volume) has independently discovered a simple proof of the main result of [3], and also an extension to the case of stable processes.

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