DANIEL W. STROOCK

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HOMOGENEOUS CHAOS REVISITED

Daniel W. Stroock*

Let $(\Theta, H, \mathcal{W})$ be an abstract Wiener space. That is: Θ is a separable real Banach space with norm $\|\cdot\|_\Theta$; H is a separable real Hilbert space with norm $\|\cdot\|_H$; $H \subseteq \Theta$, $\|h\|_\Theta \le C\|h\|_H$ for some $C < \infty$ and all $h \in H$, and H is $\|\cdot\|_\Theta$ - dense in Θ; and $\mathcal{W}$ is the probability measure on $(\Theta, \mathcal{B}_\Theta)$ with the property that, for each $\ell \in \Theta^*$, $\theta \in \Theta \to \langle \ell, \theta\rangle$ under $\mathcal{W}$ is a Gaussian random variable with mean zero and variance $\|\ell\|_H^2 \equiv \sup\{\langle \ell, h\rangle^2 : h \in H \text{ with } \|h\|_H = 1\}$.
Let $\{\ell^k : k \in Z^+\} \subseteq \Theta^*$ be an orthonormal basis in H; set $\mathcal{A} = \{\alpha \in \mathcal{N}^{Z^+} : |\alpha| = \sum_{k\in Z^+} \alpha_K < \infty\}$; and for $\alpha \in \mathcal{A}$, define

$$\mathcal{H}_\alpha(\theta) = \prod_{k\in Z^+} H_{\alpha_k}(\langle \ell^k, \theta\rangle), \quad \theta \in \Theta ,$$

where

$$H_m(\xi) = (-1)^m e^{\xi^2/2} \frac{d^m}{d\xi^m} (e^{-\xi^2/2}), \quad m \in \mathcal{N} \text{ and } \xi \in R^1 .$$

Then, $\{(\alpha!)^{-1/2} \mathcal{H}_\alpha : \alpha \in \mathcal{A}\}$ is an orthonormal basis in $L^2(\mathcal{W})$. Moreover, if, for $m \in \mathcal{N}$,

$$Z^{(m)} \equiv \overline{\text{span}\{\mathcal{H}_\alpha : |\alpha| = m\}}^{L^2(\mathcal{W})} ,$$

then: $Z^{(m)}$ is independent of the particular choice of the orthonormal basis $\{\ell^k : k \in Z^+\}$; $Z^{(m)} \perp Z^{(n)}$ for $m \ne n$; and $L^2(\mathcal{W}) = \bigoplus_{m=0}^{\infty} Z^{(m)}$. These facts were first proved by N. Wiener [6] and constitute the foundations on which his theory of <u>homogeneous chaos</u> is based.

The purpose of the present article is to explain how, for given $\Phi \in L^2(\mathcal{W})$, one can compute the orthogonal projection $\Pi_{Z^{(m)}}\Phi$ of Φ onto $Z^{(m)}$. In order to describe the procedure, it will be necessary to describe the elementary Sobolev theory associated with $(\Theta, H, \mathcal{W})$.

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To this end, let Y be a separable real Hilbert space and set $\mathscr{P}(Y) = \operatorname{span}\{\mathscr{H}_\alpha y : \alpha \in \mathscr{A} \text{ and } y \in Y\}$. Then $\mathscr{P}(Y)$ is dense in $L^2(\mathscr{W}; Y)$. Next, for $m \in \mathscr{N}$ and $\Phi \in \mathscr{P}(Y)$, define $\theta \to D^m\Phi(\theta) \in H^{\otimes^m} \otimes Y$ by

$$(D^m\Phi(\theta), h^1 \otimes \dots \otimes h^m \otimes y)_{H^{\otimes^m} \otimes Y} = \frac{\partial^m}{\partial t_1 \dots \partial t_m} (\Phi(\theta + \sum_{j=1}^m t_j h^j), y)_Y \Big|_{t_1 = \dots = t_m = 0}$$

for $h^1, \dots, h^m \in H$ and $y \in Y$. Then D^m maps $\mathscr{P}(Y)$ into $\mathscr{P}(H^{\otimes^m} \otimes Y)$ and $D^n = D^m \circ D^{n-m}$ for $0 \le m \le n$. Associated with the operator $D^m : \mathscr{P}(Y) \to \mathscr{P}(H^{\otimes^m} \otimes Y)$ is its adjoint operator ∂^m. Using the Cameron-Martin formula [1], one can easily prove the following lemma.

(1) <u>Lemma</u>: The operator ∂^m does not depend on the choice of orthonormal basis $\{\ell^k : k \in Z^+\}$, $\mathscr{P}(H^{\otimes^m} \otimes Y) \subseteq \operatorname{Dom}(\partial^m)$, and $\partial^m : \mathscr{P}(H^{\otimes^m} \otimes Y) \to \mathscr{P}(Y)$. Moreover, if $m \in Z^+$, $K = (k_1, \dots, k_m)$ $(Z^+)^m$, and $\ell^K = \ell^{k_1} \otimes \dots \otimes \ell^{k_m}$, then

$$\partial^m \ell^K = \mathscr{H}_{\alpha(K)} \tag{2}$$

where $\alpha(K)$ is the element of $\mathscr{A}$ defined by

$$(\alpha(K))_k = \operatorname{card}\{1 \le j \le m : k_j = k\}, \ k \in Z^+.$$

In particular, $H^{\otimes^m} \subseteq \operatorname{Dom}(\partial^m)$.

Since ∂^m is densely defined, it has a well-defined adjoint $(\partial^m)^*$. Set $W_m^2(Y) = \operatorname{Dom}((\partial^M)^*)$ and use $\|\cdot\|_{W_m^2(Y)}$ to denote the associated graph norm on $W_m^2(Y)$. The following lemma is an easy application of inequalities proved by M and P. Kree [3].

(3) <u>Lemma</u>: $W_m^2(H^{\otimes^m} \otimes Y) \subseteq \operatorname{Dom}(\partial^m)$, $\|\partial^m\Psi\|_{L^2(\mathscr{W};Y)} \le C_m \|\Psi\|_{W_m^2(h^{\otimes^m} \otimes Y)}$, and $\partial^m = ((\partial^m)^*)^*$. Moreover, $\mathscr{P}(Y)$ is $\|\cdot\|_{W_m^2(Y)}$-dense in $W_m^2(Y)$.

Finally, $W^2_{m+1}(Y) \subseteq W^2_m(Y)$ and $\|\cdot\|_{W^2_m(Y)} \le C_m \|\cdot\|_{W^{m+1}_2(Y)}$ for all $m \ge 0$.

Warning: In view of the preceding, the use of D^m to denote its own closure $(\partial^m)^*$ is only a mild abuse of notation. Because it simplifies the notation, this abuse of notation will be used throughout what follows.

Now set $\mathcal{W}^2_{-m}(Y) = \mathcal{W}^2_m(Y)^*$, $m \ge 0$, and $\mathcal{W}^2_\infty(Y) = \bigcap_{m=0}^{\infty} \mathcal{W}^2_m(Y)$. Then, when $W^2_\infty(Y)$ is given the Fréchet topology determined by $\{\|\cdot\|_{W^2_m(Y)} : m \ge 0\}$, $(W^2_\infty(Y))^*$ is $W^2_{-\infty}(Y) \equiv \bigcup_{m=0}^{\infty} W^2_{-m}(Y)$. Moreover, $L^2(\mathcal{W};Y)$ becomes a subspace of $W^2_{-\infty}(Y)$ when $\Phi \in L^2(\mathcal{W};Y)$ is identified with the linear functional $\Psi \in W^2_\infty(Y) \to E^{\mathcal{W}}[(\Phi,\Psi)_Y]$; and in this way $W^2_\infty(Y)$ becomes a dense subspace of $W^2_{-\infty}(Y)$. Finally, D^m has a unique continuous extension as a map from $W^2_{-\infty}(Y)$ into $W^2_{-\infty}(H^{\otimes^m} \otimes Y)$. In particular, for $T \in W^2_{-\infty}(R^1)$, there is a unique $D^mT(1) \in H^{\otimes^m}$ defined by:

(4)
$$(D^mT(1),\ h)_{H^{\otimes^m}} = T(\partial^m h),\ h \in H^{\otimes^m}.$$

Note that when $\Phi \in W^2_\infty(R^1)$,

(5)
$$D^m\Phi(1) = E^{\mathcal{W}}[D^m\Phi].$$

(6) Theorem: Let $\Phi \in L^2(\mathcal{W})$ be given. Then, for each $m \ge 0$:

(7)
$$\Pi_{Z(m)}\Phi = \frac{1}{m!}\,\partial^m(D^m\Phi(1)).$$

Hence,

(8)
$$\Phi = \sum_{m=0}^{\infty} \frac{1}{m!}\,\partial^m(D^m\Phi(1)) .$$

In particular, when $\Phi \in W^2_\infty(R^1)$:

(7')
$$\Pi_{Z(m)}\Phi = \frac{1}{m!}\,\partial^m E^{\mathcal{W}}[D^m\Phi]$$

and

(8')
$$\Phi = \sum_{m=0}^{\infty} \frac{1}{m!}\,\partial^m\ E^{\mathcal{W}}[D^m\Phi] .$$

<u>Proof:</u> Simply observe that, by Lemma (1):

$$\partial^m(D^m\Phi(1)) = \sum_{K\in(Z^+)^m} E^{\mathscr{W}}[\Phi\partial^m \ell^K]\partial^m \ell^K$$

$$= \sum_{|\alpha|=m} \binom{m}{\alpha} E^{\mathscr{W}}[\Phi\mathscr{H}_\alpha]\mathscr{H}_\alpha = m!\ \Pi_{Z^{(m)}}\Phi \quad .$$

The classic abstract Wiener space is the Wiener space associated with a Brownian motion on R^1. Namely, define $H_1(R^1)$ and $\Theta(R^1)$ to be, respectively, the completion of $C_0^\infty((0,\infty);\ R^1)$ with respect to

$$\|\psi\|_{H_1(R^1)} \equiv \left(\int_0^\infty |\psi'(t)|^2\, dt\right)^{1/2}$$

and

$$\|\psi\|_{\Theta(R^1)} \equiv \sup_{t\geq 0} \frac{1}{1+t}\ |\theta(t)| \ .$$

Then Wiener's famous existence theorem shows that there is a probability measure on $\Theta(R^1)$ such that $(\Theta(R^1),\ H_1(R^1),\ \mathscr{W})$ is an abstract Wiener space. For $(\Theta(R^1),\ H_1(R^1),\ \mathscr{W})$, K. Itô [2] showed how to cast Wiener's theory of homogeneous chaos in a particularly appealing form. To be precise, set $\square_m = [0,\infty)^m$; and, for $f \in L^2(\square_m)$, define

$$\int_{\square_m} f\ d^m\theta = \sum_{\sigma\in\Pi_m} \int_0^\infty d\theta(t_m) \int_0^{t_{m-1}} d\theta(t_{m-2}) \ \cdots$$

$$\int_0^{t_2} f(t_{\sigma(1)},\ldots,t_{\sigma(m)})d\theta(t_1)$$

where Π_m denotes the permutation group on $\{1,\ldots,m\}$ and the $d\theta(t)$-integrals are taken in the sense of Itô. What Itô discovered is that, for given $\Phi \in L^2(\mathscr{W})$, there exists a unique symmetric $f_\Phi^{(m)} \in L^2(\square_m)$ such that

(9)
$$\Pi_{Z^{(m)}}\Phi = \frac{1}{m!} \int_{\square_m} f_\Phi^{(m)}\ d^m\theta$$

In order to interpret Itô's result in terms of Theorem (5), let $\{\psi^k : k \in Z^+\} \subseteq C_0^\infty\ ((0,\infty);\ R^1)$ be an orthonormal basis in $L^2(\square_1)$ and define $\ell^k \in \square\,(R^1)^*$ by $\ell^k(dt) = (\int_0^t \psi^k(s)ds)dt$. Then $\langle \ell^k, \theta\rangle = \int_{\square_1} \psi^k d^1\theta$. Moreover, by using, on the one hand, the generating function for the Hermite polynomials and, on the other hand, the uniqueness of solutions to linear stochastic integral equations (cf. H. P. McKean [5]), one finds that for $K = (k_1,\dots,k_m) \in (Z^+)^m$:

$$\int_{\square_m} \psi^k\, d^m\theta = \mathscr{H}_{\alpha(K)}$$

where $\psi^K = \psi^{k_1} \otimes \dots \otimes \psi^{k_m}$ and $\alpha(K) \in \mathscr{A}$ is defined as in Lemma (1). Hence, by Lemma (1):

(10) $$\partial^m\, \ell^K = \int_{\square_m} \psi^K\, d^m\theta\ , \quad K \in (Z^+)^m\ .$$

Finally, for $(t_1,\dots,t_n) \in \square_m$, define $h_{(t_1,\dots,t_m)}(s_1,\dots s_m) = (s_1 \wedge t_1) \dots (s_m \wedge t_m)$. Then, for each $h \in H_1(R^1)^{\otimes^m}$, there is a unique $h' \in L^2(\square_m)$ such that $(h, h_{(t_1,\dots,t_m)})_{H_1(R^1)^{\otimes^m}}$

$\int_0^{t_m} \dots \int_0^{t_r} h'(s_1,\dots,s_m)ds_1,\dots,ds_m$ for all $(t_1,\dots,t_m) \in \square_m$

(11) <u>Theorem:</u> Given $\Phi \in L^2(\mathscr{W})$ and $m \geq 1$, the $f_\Phi^{(m)}$ in (9) is $(D^m\Phi(1))'$.

<u>Proof</u>: By (9):

$$\partial^m(D^m\Phi(1)) = \partial^m\Big[\sum_{K\in(Z^+)^m} (D^m\Phi(1),\ \ell^K)_{H_1(R^1)^{\otimes^m}}\ \ell^K\Big]$$
$$= \sum_{K\in(Z^+)^m} ((D^m\Phi(1))',\ \psi^K)_{L^2(\square_m)} \int_{\square_m} \psi^K\, d^m\theta$$
$$= \int_{\square_m} (D^m\Phi(1))'\ d^m\theta\ .$$

Thus, by (7):

$$\Pi_{Z^{(m)}} = \frac{1}{m!}\int_{\square_m} (D^m\Phi(1))'\ d^m\theta\ .$$

(12) <u>Remark</u>: It is intuitively clear that the $f_\Phi^{(m)}$ in (9) must be given by $f_\Phi^{(m)}(t_1,\dots,t_m) = E^{\mathscr{W}}[\Phi\dot\theta(t_1)\dots\dot\theta(t_m)]$, where $\dot\theta(t)$ is white noise. What Theorem (11) does is provide a rigorous meaning for this equation.

(13) Remark: Given $d \geq 2$, define $H_1(R^d)$ and $\Theta(R^d)$ by analogy with $H_1(R^1)$ and $\Theta(R^1)$. Then $(\Theta(R^d), H_1(R^d), \mathscr{W})$ becomes an abstract Wiener space when $\mathscr{W}$ is the Wiener measure associated with the Brownian motion in R^d. To provide an Itô interpretation in this case, let $\{\Psi^k : k \in Z^+\} \subset C_o^\infty((0,\infty); R^1)$ be chosen as before and set $\ell^{(k,i)} = \Psi^k e_i$, $k \in Z^+$ and $i \in \mathscr{D} \equiv \{1,\ldots,d\}$, where $\{e_1,\ldots,e_d\}$ is a standard basis for R^d. Next, for $f = \sum_{I\in\mathscr{D}^m} f_I e_I \in L^2(\square_1; (R^d)^{\otimes m})$, define

$$\int_{\square_m} f\, d^m\theta = \sum_{I\in\mathscr{D}^m} \int_{\square_m} f_I d^m\theta_I$$

where

$$\int_{\square_m} f_I d^m\theta_I =$$

$$\sum_{\sigma\in\Pi_m} \int_o^\infty d\theta_{i_m}(t_m) \int_o^{t_m} d\theta_{i_{m-1}}(t_{m-1}) \cdots \int_o^{t_2} f_I(t_{\sigma(1)},\ldots,t_{\sigma(m)}) d\theta_{i_1}(t_1)$$

for $I = (i_1,\ldots,i_m) \in \mathscr{D}^m$. One can then check that

$$\partial^m \ell^{(K,I)} = \int_{\square_m} \Psi^K d^m\theta_I .$$

Finally, after associating with each $h \in H_1(R^d)^{\otimes m}$ the unique $h' \in L^2(\square_i^m; (R^d)^{\otimes m})$ satisfying

$$h(t_1,\ldots,t_m) = \int_o^{t_m} \int_o^{t_1} h'(s_1,\ldots,s_m) ds_1 \ldots ds_m ,$$

we again arrive at the equation

$$\Pi_{Z^{(m)}} \Phi = \int_{\square_m} (D^m \Phi(1))' d^m\theta .$$

(14) Remark: Theorem (11) is little more than an exercise in formalism unless $\Phi \in W_\infty^2(R^1)$. Fortunately, many interesting functions are in $W_\infty^2(R^1)$. For example, let $\sigma : R^1 \to R^1$ and $b : R^1 \to R^1$ be smooth functions having bounded first derivatives and slowly increasing derivatives of all orders. Define $X(\cdot,x)$, $x \in R^1$, to be the solution to

$$X(T,x) = x + \int_0^T \sigma(X(t,x))d\theta(t) + \int_0^T b(X(t,x))dt, \ T \geq 0.$$

Then, for each $(\Gamma,x) \in (0,\infty) \times R^1$, $X(T,x) \in W^2_\infty(R^1)$. In fact, $DX(\cdot,x)$ satisfies:

$$DX(T,x) = \int_0^T \sigma'(X(t,x))\, DX(t,x)\, d\theta(t) + \int_0^T b'(X(t,x))DX(t,x)dt$$
$$+ \int_0^{\cdot \wedge T} \sigma(X(t,x))\, dt;$$

an equation which can be easily solved by the method of variation of parameters. Moreover, $D^mX(T,x)$, $m \geq 2$, can be found by iteration of the preceding.

(15) Remark: In many ways, the present paper should be viewed as an outgrowth of P. Malliavin's note [4]. Indeed, it was only after reading Malliavin's note that the ideas developed here occurred to the present author.

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