

SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

HAYA KASPI

BERNARD MAISONNEUVE

Predictable local times and exit systems

Séminaire de probabilités (Strasbourg), tome 20 (1986), p. 95-100

<http://www.numdam.org/item?id=SPS_1986__20__95_0>

© Springer-Verlag, Berlin Heidelberg New York, 1986, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

PREDICTABLE LOCAL TIMES
AND EXIT SYSTEMS

Haya Kaspi
Department of Industrial Engineering
Technion, Haifa 32000
ISRAEL

Bernard Maisonneuve
I. M. S. S.
47-X 38040 Grenoble Cedex
FRANCE

1. INTRODUCTION.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^X)$ be the canonical realization of a Hunt semi-group (P_t) on a state space (E, \mathcal{E}) and let M be the closure of the random set $\{t > 0 : X_t \in B\}$, where B is in \mathcal{E} . We set $R = \inf\{t > 0 : t \in M\} = \inf\{t > 0 : X_t \in B\}$. If M has no isolated point a.s., the predictable additive functional with 1-potential $P \cdot (e^{-R})$ is a local time of M (the set of its increase points is M a.s. by [5], p.66). This restriction on M is essential, as proved by the following example of Azéma. Consider a process which stays at 0 for an exponential time and then jumps to 1 and moves to the right with speed 1. For $B = \{1\}$, R is totally inaccessible and $M = \{R\}$ cannot have a predictable local time.

One can always define an optional local time for M , as recalled in section 2. One unpleasant feature of such a local time is that it may jump at times t where $X_t \notin \bar{B}$, so that the associated time changed process is not necessarily \bar{B} valued. Nevertheless, one can construct a local time which avoids this unpleasant feature by using the methods of [4] (see Remark 2). Here we shall give a direct construction by taking the (\mathcal{F}_{D_t}) dual predictable projection of the process Λ_t of §2, where as usual

$$D_t = \inf\{s > t : s \in M\}.$$

We shall also prove the existence of a related (\mathcal{F}_{D_t}) predictable exit system in full generality, whereas the existence of an (\mathcal{F}_t) predictable exit system requires some special assumptions as noted by Gettoor and Sharpe [2] (see V of [8] for sufficient conditions). From this one can deduce conditioning formulae like in the optional case ([8]).

2. THE (\mathcal{F}_{D_t}) PREDICTABLE LOCAL TIME.

Let X be like previously and let M be an optional random closed set, homogeneous in $(0, \infty)$ and such that $M = \overline{M \setminus \{0\}}$. The following notations are taken from [6]:

$$\begin{aligned}
R &= \inf\{s > 0 : s \in M\} \quad (\inf \emptyset = +\infty), \\
R_t &= R \circ \theta_t, \quad D_t = t + R_t, \quad \hat{\mathcal{F}}_t = \mathcal{F}_{D_t}, \\
F &= \{x \in E : P^x\{R=0\} = 1\}, \\
G &= \{t > 0 : R_{t-} = 0, R_t > 0\}, \\
G^r &= \{t \in G : X_t \in F\}, \\
G^i &= \{t \in G : X_t \notin F\}.
\end{aligned}$$

For every homogeneous subset Γ of G we shall set

$$\Lambda_t^\Gamma = \sum_{\substack{s \in \Gamma \\ s \leq t}} (1 - e^{-R_s}), \quad L_t^\Gamma = \sum_{\substack{s \in \Gamma \\ s \leq t}} P^{X_s}(1 - e^{-R}).$$

The process (Λ_t) defined by

$$\Lambda_t = \int_0^t 1_M(s) ds + \Lambda_t^G, \quad t \geq 0,$$

is an $(\hat{\mathcal{F}}_t)$ adapted additive functional with support (or set of increase) M . Its (\mathcal{F}_t) dual optional projection (L_t^0) is a local time for M (i.e. an (\mathcal{F}_t) adapted additive functional with support M). Its jump part is $(L_t^{G^i})$, as it follows easily from [6] for example. But this jump part is too big with respect to the discussion of section 1.

THEOREM 1. 1) The set I of isolated points of M ($I \subset G$) is (\mathcal{F}_t) optional and $(\hat{\mathcal{F}}_t)$ predictable. Each (\mathcal{F}_t) stopping time T in $I \cup \{\infty\}$ is $(\hat{\mathcal{F}}_t)$ predictable and satisfies $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$.

2) The set $G^{-i} = \{t \in G \setminus I : X_{t-} \notin F\}$ is (\mathcal{F}_t) predictable. For each (\mathcal{F}_t) predictable stopping time T in $G^{-i} \cup \{\infty\}$ one has $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$.

3) The set $G^{-r} = \{t \in G \setminus I : X_{t-} \in F\}$ is (a countable union of graphs of) $(\hat{\mathcal{F}}_t)$ totally inaccessible (stopping times).

THEOREM 2. There exists an (\mathcal{F}_t) adapted local time (L_t) for M which is, under each measure P^μ , the $(\hat{\mathcal{F}}_t)$ dual predictable projection of (Λ_t) . Its jump part is $L^d = L^{I \cup G^{-i}}$.

It will be convenient in the sequel to write simply o., p., s.t., d.p. for optional, predictable, stopping time(s), dual projection(s).

Remark 1. We know that $T \notin G^r$ a.s. for each s.t. T . Hence $I \cup G^{-i} \subset G^i$

a.s. by Theorem 1, and L^d is less than the jump part of L^0 . When M is related to a Borel set B like in § 1, we have $X_t \in \bar{B}$ for $t \in I \cup G^{-i}$ a.s., since $X_T = X_{T-} \in \bar{B}$ a.s. on $\{T < \infty\}$ for each p.s.t. T in $G^{-i} \cup \{\infty\}$. Therefore our local time L is really local.

Proof. (a) The set I is (\mathcal{F}_t) optional (see (3.3) of [7]) and can be written as a countable union of graphs of (\mathcal{F}_t) s.t. . Let T be one of these s.t. and let $g_T = \sup\{s < T : s \in M\}$ ($\sup \emptyset = 0$). By (2.4) of [7], g_T is an $(\hat{\mathcal{F}}_t)$ s.t. . Consider $T_n = \inf\{t \geq g_T : R_t \leq \frac{1}{n}\}$ for $n \in \mathbb{N}$. Since $T_n < T$ on $\{T < \infty\}$ and $T_n \uparrow T$, T is $(\hat{\mathcal{F}}_t)$ predictable (it is announced by the sequence $(T_n \wedge n)$). In addition $\hat{\mathcal{F}}_{T-} = \bigvee_n \hat{\mathcal{F}}_{T_n \wedge n} = \bigvee_n \hat{\mathcal{F}}_{T_n} = \bigvee_n \mathcal{F}_{D_{T_n}}$ and $D_{T_n} = T$ on $\{T < \infty\}$, so that $\hat{\mathcal{F}}_{T-} \cap \{T < \infty\} = \mathcal{F}_T \cap \{T < \infty\}$ and $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$. The first part of Theorem 1 is established.

(b) Let T be an $(\hat{\mathcal{F}}_t)$ p.s.t. which is a left accumulation point of M on $\{T < \infty\}$. If T is announced by a sequence (T_n) , it is also announced by the sequence (D_{T_n}) of (\mathcal{F}_t) s.t., so that T is (\mathcal{F}_t) predictable and satisfies $\hat{\mathcal{F}}_{T-} = \bigvee_n \hat{\mathcal{F}}_{T_n} = \bigvee_n \mathcal{F}_{D_{T_n}} = \mathcal{F}_{T-} = \mathcal{F}_T$ the last equality following from the quasi-left continuity of (\mathcal{F}_t) .

(c) Consider the (\mathcal{F}_t) p. part $G^{i,P}$ and the (\mathcal{F}_t) totally inaccessible part $G^{i,i}$ of the (\mathcal{F}_t) o. set $G^i \setminus I$:

$$G^{i,P} = \{t \in G^i \setminus I : X_{t-} = X_t\},$$

$$G^{i,i} = \{t \in G^i \setminus I : X_{t-} \neq X_t\}.$$

It follows from b) that $\hat{\mathcal{F}}_{T-} = \mathcal{F}_T$ for each (\mathcal{F}_t) p.s.t. in $G^{i,P} \cup \{\infty\}$ and that $G^{i,i}$ is $(\hat{\mathcal{F}}_t)$ totally inaccessible.

(d) It follows from (a), (c) that L^I and $L^{G^{i,P}}$ are the $(\hat{\mathcal{F}}_t)$ d.p.p. of Λ^I and $\Lambda^{G^{i,P}}$ under each measure P^μ . Now consider under P^μ , the $(\hat{\mathcal{F}}_t)$ d.p.p. of $\Lambda^{G^r \cup G^{i,i}}$: it is continuous since G^r and $G^{i,i}$ are $(\hat{\mathcal{F}}_t)$ totally inaccessible (for G^r see (3.2) of [7]) and carried by M (recall that $M \setminus \{0\} = \{t > 0 : R_{t-} = 0\}$ is $(\hat{\mathcal{F}}_t)$ p.), hence it is (\mathcal{F}_t) adapted ([5], p. 56 or [9], p. 229) and thus it is P^μ -indistinguishable from the continuous additive functional (K_t) which is the (\mathcal{F}_t) d.p.p. of $\Lambda^{G^r \cup G^{i,i}}$. Therefore the (\mathcal{F}_t) adapted additive functional

$$L_t = \int_0^t 1_M(s) ds + K_t + L_t^{I \cup G^{i,P}}$$

is the $(\hat{\mathcal{F}}_t)$ d.p.p. of (Λ_t) under P^μ . Since the support of Λ is the $(\hat{\mathcal{F}}_t)$ p. set M , the support of L is M a.s. The proof of both theorems will be complete if we

show that $G^r \cup G^{i,i} = G^{-r}$ a.s. and $G^{i,P} = G^{-i}$ a.s. But the continuous part L^c of L is carried by F since $\{t \in M : X_t \notin F\}$ is a.s. countable. Therefore $X_{t-} \in F$ for $t \in G^r \cup G^{i,i}$ a.s. ; on the other hand $X_{t-} = X_t \notin F$ for $t \in G^{i,P}$ a.s. ■

Remark 2. We indicate here how to construct a local time by using the methods of [4]. Consider the local time of equilibrium of order 1 (\bar{L}_t) (see [5]) for the perfect kernel of M , and define $\bar{G}^i = \{t \in G, \Delta \bar{L}_t > 0 \text{ or } t \in \bar{I}^G\}$, where \bar{I}^G is the left closure of I . Then $L^i = \bar{L}^c + L\bar{G}^i$ is a local time such that $\{t : t \notin I, \Delta L_t^i > 0\}$ is (\mathcal{F}_t) predictable and thus is good with respect to the discussion of §1. One can even show that L^c is absolutely continuous with respect to \bar{L}^c , and that $I \cup G^{-i}$ and \bar{G}^i are indistinguishable.

3. THE (\mathcal{F}_{D_t}) PREDICTABLE EXIT SYSTEM.

In this section we shall assume that R is \mathcal{F}^* measurable, where \mathcal{F}^* is the universal completion of $\mathcal{F}^0 = \sigma(X_t, t \in \mathbb{R}_+)$. The universal completion of \mathcal{E} will be denoted by \mathcal{E}^* .

THEOREM 3. There exists an \mathcal{E}^* measurable positive function ϱ on E , carried by F , and a kernel *P from (E, \mathcal{E}^*) to (Ω, \mathcal{F}^*) such that (L is defined as in Theorem 2)

- (i) $\int_0^t 1_M(s) ds = \int_0^t \varrho \circ X_s dL_s$,
- (ii) $P \cdot \sum_{s \in G} Z_s f \circ \theta_s = P \cdot \int_0^\infty Z_s {}^*P^{X_s}(f) dL_s$

for all positive $(\hat{\mathcal{F}}_t)$ predictable Z and \mathcal{F}^* measurable f ,

- (iii) $\varrho + {}^*P \cdot (1 - e^{-R}) \equiv 1$ on E and ${}^*P \cdot \equiv P \cdot / P \cdot (1 - e^{-R})$ on $E \setminus F$.

The system $(L, {}^*P)$ will be called the (\mathcal{F}_{D_t}) predictable "exit system" (according to the terminology of [6]). Note that in (ii) X_s can be replaced by Y_{s-} , where $Y_s = X_{D_s}$.

Proof. - Let ${}^*P \cdot$ be defined on $E \setminus F$ as in (iii). The equality (ii) is immediate with $I \cup G^{-i}$ and L^d instead of G and L , due to Theorem 1. By the arguments of [6] we then establish the existence of a kernel N from (E, \mathcal{E}^*) into (Ω, \mathcal{F}^*) such

that $N\{R=0\} = 0$ and

$$P \cdot \sum_{s \in G-r} Z_s ((1-e^{-R})f) \circ \theta_s = P \cdot \int_0^\infty Z_s N^X_s(f) dL_s^c$$

for all positive (\mathcal{F}_t) p.z. This formula extends to positive $(\hat{\mathcal{F}}_t)$ p.z. by the argument of (d) of Section 2. If ϱ is a Motoo density of $(\int_0^t 1_M(s) ds)$ relative to (L_t^c) , the kernel N can be modified in such a way that $\varrho + N^*(1) = 1$. We can also assume that ϱ is carried by F . Setting $*P^*(f) = N^*(\frac{f}{1-e^{-R}})$ on F , we get (ii) with G^{-r} and L^c instead of G and L and the proof is complete.

From this result one can extend some results of [8] and [3] (based on the (\mathcal{F}_t) p. exit system). For analogous results without duality see Boutabia's thesis [1].

REFERENCES.

- [1] BOUTABIA, H., "Sur les lois conditionnelles des excursions d'un processus de Markov". Thèse de 3e cycle, Grenoble, 1985.
- [2] GETOOR, R.K., SHARPE, M.J., Last exit decompositions and distributions. Indiana Univ., Math. J., 23, 377-404 (1973).
- [3] GETOOR, R.K., SHARPE, M.J., Excursions of dual processes. Adv. Math., 45 No. 3, 259-309 (1982).
- [4] KASPI, H., Excursions of Markov processes : an approach via Markov additive processes. Z. Wahrsch. verw. Geb. 64, 251-268 (1983).
- [5] MAISONNEUVE, B., Systèmes Régénératifs. Astérisque 15 (Soc. Math. France) 1974.
- [6] MAISONNEUVE, B., Exit Systems. Ann. Prob. 3, 399-411 (1975).
- [7] MAISONNEUVE, B., Entrance-Exit results for semi-regenerative processes. Z. Wahrsch. verw. Geb. 32, 81-94 (1975).
- [8] MAISONNEUVE, B., On the structure of certain excursions of a Markov process. Z. Wahrsch. verw. Geb. 47, 61-67 (1979).
- [9] MAISONNEUVE, B., MEYER, P.A., Ensembles aléatoires markoviens homogènes IV. Séminaire de Probabilités VIII. Lecture Notes 381. Springer 1974.

Note. There is an error in Theorem V.3 of p. 64 of [5]. The functional (A_t) should be assumed $(\hat{\mathcal{F}}_t)$ p. and the condition $H_U^\lambda \hat{\mathcal{F}}(y) < \hat{\mathcal{F}}(y)$ should be required for each $(\hat{\mathcal{F}}_t)$ s.t. U such that $P^Y\{U>0\} > 0$. For the proof of the converse part (1.3 of p. 65) one considers the predictable s.t. $T = S_{\{A_S>0\}}$ and a sequence (T_n) that announces T . One has $A_{T_n \wedge S} \leq A_{S-} = 0$. Hence $H_{T_n \wedge S}^\lambda \hat{\mathcal{F}}(y) = \hat{\mathcal{F}}(y)$ by (13) and $T_n \wedge S = 0$ P^Y -a.s. by assumption. Since $T_n \wedge S \uparrow T \wedge S = S$, we have $S = 0$ P^Y -a.s. and the proof is complete. Note also that Definition V.7, should be modified accordingly.
