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FUNCTIONALS ASSOCIATED WITH SELF-INTERSECTIONS OF THE PLANAR BROWNIAN MOTION¹

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ABSTRACT

For every $k=1,2,3,\dots$ and for a wide class of measures λ , we construct a one-parameter family $\mathcal{I}_k(\lambda, u)$, $u \geq 0$ of functionals of the planar Brownian motion (X_t, P_μ) related to its self-intersections of multiplicity k during the time interval $[0, u]$. We investigate various families of functionals which converge to $\mathcal{I}_k(\lambda, u)$ and we evaluate the moment functions $P_\mu[\mathcal{I}_{k_1}(\lambda_1, u) \dots \mathcal{I}_{k_n}(\lambda_n, u)]$.

1. MAIN RESULTS

1.1. We denote by (X_t, P_μ) the Brownian motion in $\mathbb{R}^2$ with the initial law μ (which can be any σ -finite measure on $\mathbb{R}^2$). If $0 < t_1 < \dots < t_n$, then the joint probability density for $X_{t_1}, \dots, X_{t_n}$ is given by the formula

$$(1.1) \quad p_\mu(t, x) = \int \mu(dx_0) p_{t_1}(x_1 - x_0) p_{t_2 - t_1}(x_2 - x_1) \dots p_{t_n - t_{n-1}}(x_n - x_{n-1}).$$

Here

$$(1.2) \quad p_t(x) = t^{-1} p(x/\sqrt{t}), \quad p(z) = (2\pi)^{-1} e^{-|z|^2/2}.$$

Put

$$(1.3) \quad G_r(x) = \int_0^\infty e^{-rt} p_t(x) dt,$$

$$g_r(x_0) = 1, \quad g_r(x_0, x_1, \dots, x_M) = G_r(x_1 - x_0) \dots G_r(x_M - x_{M-1}) \text{ for } M \geq 1.$$

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We drop the subscript r if it is equal to 1.

We write $f \approx g$ if $f(\epsilon) - g(\epsilon) = O(|\epsilon|^\alpha)$ for every $0 < \alpha < 2$ as $\epsilon \downarrow 0$. (If ϵ is a vector $(\epsilon_1, \dots, \epsilon_n)$, then $|\epsilon| = \max |\epsilon_i|$.) We also introduce an equivalence relation for Brownian functionals depending on parameters u and ϵ : $Y \approx Z$ means that

$$\int_0^\infty du e^{-ru} P_\mu[|Y_{\epsilon u} - Z_{\epsilon u}|^p] \approx 0$$

for every $r > 0$ and every $p \geq 2$. (Of course this relation depends on μ .)

A special role in our investigation is played by a function

$$(1.4) \quad h_\epsilon = \frac{1}{\pi} \ln \frac{1}{\epsilon}$$

and by a one-parameter group of fractional linear transformations in $\mathbb{R}$

$$(1.5) \quad \phi_h(w) = (w^{-1} + h)^{-1} = \frac{w}{1 + hw}.$$

1.2. We say that a pair of measures (μ, λ) on $\mathbb{R}^2$ is *admissible* if:

(a) λ has a bounded density;

(b) either μ is finite or λ is finite and μ has a bounded Hölder continuous density.

We consider finite sequences $b = (b_1, \dots, b_M)$ of elements taken from the set $\{1, \dots, n\}$ subject to the condition: $b_j \neq b_{j+1}$ for $j = 1, \dots, M-1$. We call them *routes*. We note that if (μ, λ_1) is an admissible pair for $i = 1, \dots, n$, then:

1.2.A. For every route b and every $r > 0$

$$(1.6) \quad g_{rb}(\mu, \lambda_1, \dots, \lambda_n) = \int \mu(dx_0) \lambda_1(dx_1) \dots \lambda_n(dx_n) g_r(x_0, x_{b_1}, \dots, x_{b_M}) < \infty.$$

We put

$$(1.7) \quad G_{\mu r}(x) = \int G_r(y-x) \mu(dy).$$

1.3. We start from a probability density $q(z)$ on $\mathbb{R}^2$ such that

$$(1.8) \quad \begin{aligned} \int |\ln |x||^k q(x) dx < \infty & \quad \text{for all } k > 0, \\ \int e^{\beta |x|} q(x) dx < \infty & \quad \text{for some } \beta > 0. \end{aligned}$$

Put

$$(1.9) \quad q^\epsilon(x) = \epsilon^{-2} q(x/\epsilon)$$

and consider a sequence of functionals

$$(1.10) \quad T_k(\epsilon, \lambda, u) = \int_{D_k(u)} dt_1 \dots dt_k \rho(X_{t_1}) q^\epsilon(X_{t_2} - X_{t_1}) \dots q^\epsilon(X_{t_k} - X_{t_{k-1}}),$$

$$k=1, 2, \dots$$

Here $\lambda(dx) = \rho(x)dx$ and

$$(1.11) \quad D_k(u) = \{0 < t_1 < \dots, t_k < u\}.$$

Theorem 1.1. Let (μ, λ) be an admissible pair of measures and let q satisfy condition (1.8). There exist functionals $\mathcal{T}_k(\lambda, u)$

(independent of q) such that

$$(1.12) \quad \mathcal{T}_k(\epsilon, \lambda, u) \approx \mathcal{T}_k(\lambda, u).$$

Here

$$(1.13) \quad \mathcal{T}_k(\epsilon, \lambda, u) = \sum_{\ell=1}^k \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix} (\kappa - h_\epsilon)^{k-\ell} T_\ell(\epsilon, \lambda, u),$$

$$(1.13a) \quad \kappa = \frac{1}{\pi} \int [C + \ln \frac{|y|}{\sqrt{2}}] q(y) dy,$$

$C = 0.5772157 \dots$ is Euler's constant.

$$1.4. \text{ Putting } \{F(T)\} = \sum_1^n a_k T_k \text{ for every polynomial } F(T) = \sum_1^n a_k T^k, \text{ we}$$

rewrite formula (1.13) in a compact form

$$\mathcal{T}_k = \{T(T + \kappa - h_\epsilon)^{k-1}\}.$$

We note that

$$(1.14) \quad \phi_h(w)^\ell = \sum_{k=\ell}^{\infty} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix} w^k (-h) ^{k-\ell}$$

and therefore we get from (1.13) the following equation for generating functions

$$(1.15) \quad \sum_{k=1}^{\infty} \mathcal{T}_k(\epsilon, \lambda, u) w^k = \sum_{\ell=1}^{\infty} \phi_{h_\epsilon - \kappa}(w)^\ell T_\ell(\epsilon, \lambda, u)$$

or

$$(1.16) \quad \sum_{\ell=1}^{\infty} \mathcal{T}_\ell(\epsilon, \lambda, u) \phi_{\kappa - h_\epsilon}(v)^\ell = \sum_{k=1}^{\infty} v^k T_k(\epsilon, \lambda, u).$$

By comparing coefficients at v^k and then taking into account (1.12), we get

$$(1.17) \quad T_k(\epsilon, \lambda, u) = \sum_{\ell=1}^{\infty} \binom{k-1}{\ell-1} (h_\epsilon - \kappa)^{k-\ell} \tau_\ell(\epsilon, \lambda, u) \approx \sum_{k=1}^{\infty} \binom{k-1}{\ell-1} (h_\epsilon - \kappa)^{k-\ell} \tau_\ell(\lambda, u).$$

1.5. Let

$$(1.18) \quad T(\epsilon, z, u) = \int_0^u q^\epsilon(X_t - z) dt$$

We consider a sequence

$$(1.19) \quad T^k(\epsilon, \lambda, u) = \frac{1}{k!} \int \lambda(dz) T(\epsilon, z, u)^k \\ = \int \lambda(dz) \int_{D_k(u)} q^\epsilon(X_{t_1} - z) \dots q^\epsilon(X_{t_k} - z) dt_1 \dots dt_k, \quad k=1, 2, \dots$$

and we renormalize it by the formula

$$(1.20) \quad \tau^k(\epsilon, \lambda, u) = \sum_{\ell=1}^k L_{k\ell}(h_\epsilon) T^\ell(\epsilon, \lambda, u), \quad k=1, 2, \dots$$

where $L_{k\ell}$ is a polynomial with the leading term $h^{k-\ell}$.

Theorem 1.2. Suppose that (μ, λ) and q satisfy conditions of Theorem 1.1 and let $\tau_k(\lambda, u)$ be the functionals described there.

Polynomials $L_{k\ell}$ can be chosen in such a way that

$$(1.21) \quad \tau^k(\epsilon, \lambda, u) \approx \tau_k(\lambda, u).$$

Namely,

$$(1.22) \quad \mathcal{Y}[\phi_h(w)]^\ell = \sum_{k=\ell}^{\infty} L_{k\ell}(h) w^k.$$

To describe $\mathcal{Y}$ we consider independent random variables $Y_1, \dots, Y_n, \dots$ with the probability distribution $q(x)dx$ and we put

$$(1.23) \quad \varphi_j = -\frac{1}{\pi} [C + \ln(|Y_j - Y_{j+1}|/\sqrt{2})],$$

$$(1.24) \quad \mathcal{Q}(v) = v \left[1 + \sum_{n=1}^{\infty} v^n E(\varphi_1 \dots \varphi_n) \right].$$

The power series $\mathcal{Y}(w)$ is uniquely determined by either of two conditions

$$(1.25) \quad \mathcal{Y}[\mathcal{Q}(w)] = w \quad \text{or} \quad \mathcal{Q}[\mathcal{Y}(v)] = v.$$

1.6. The same argument as in subsection 1.4 shows that

$$(1.26) \quad \sum_{k=1}^{\infty} \tau^k(\epsilon, \lambda, u) w^k = \sum_{\ell=1}^{\infty} \mathcal{Y}[\phi_{h_\epsilon}(w)]^\ell T^\ell(\epsilon, \lambda, u)$$

or

$$(1.27) \quad \sum_{\ell=1}^{\infty} \mathcal{T}^{\ell}(\epsilon, \lambda, u) \phi_{-h_{\epsilon}}[\mathcal{Q}(v)]^{\ell} = \sum_{k=1}^{\infty} v^k T^k(\epsilon, \lambda, u).$$

We get from (1.27) the following asymptotic decomposition

$$(1.28) \quad T^k(\epsilon, \lambda, u) \approx \sum_{\ell=1}^k M_{k\ell}(h_{\epsilon}) \mathcal{T}_{\ell}(\lambda, u)$$

where $M_{k\ell}$ are polynomials defined by the formula

$$(1.29) \quad \sum_{k=\ell}^{\infty} M_{k\ell}(h) v^k = \phi_{-h}[\mathcal{Q}(v)]^{\ell}.$$

We note that for $n \geq k$, M_{nk} is a polynomial of degree $n-k$ with the leading term h^{n-k} (for $n < k$, $M_{nk} = 0$).

1.7. We denote by $\ell_1 = \ell_1(b)$ the number of elements equal to 1 in a route $b = (b_1, \dots, b_M)$ and we denote by $\mathfrak{S}_k$ the set of all routes for which $1 \leq \ell_i \leq k_i$, $i = 1, \dots, n$.

For every $n = 0, 1, 2, \dots$ there exists a unique polynomial $\mathcal{P}_n$ such that

$$(1.30) \quad \int \mathcal{P}_n(\log t) e^{-rt} dt = \left[-\frac{\ln r}{2\pi} \right]^n r^{-1}.$$

Theorem 1.3. For every $k_1, \dots, k_n \geq 1$

$$(1.31) \quad P_{\mu}[\mathcal{T}_{k_1}(\lambda_1, u_1) \dots \mathcal{T}_{k_n}(\lambda_n, u_n)] = m_k(\lambda, u)$$

where

$$(1.32) \quad m_k(\lambda, u) = \sum_{b \in \mathfrak{S}_k} a(k, b) \int \lambda(dz) \int_{D_M(u)} p_{\mu}(t_1, z_{b_1}; \dots; t_M, z_{b_M}) \mathcal{P}_{\nu}[\log(u - t_M)] dt$$

with

$$(1.33) \quad a(k, b) = \prod_{j=1}^n \binom{k_j - 1}{\ell_j - 1}, \quad \nu = \sum_{j=1}^n (k_j - \ell_j);$$

$$(1.34) \quad \lambda(dz) = \lambda_1(dz_1) \dots \lambda_M(dz_M), \quad dt = dt_1 \dots dt_M.$$

1.8. All the stated results follow from Theorem 1.4. In this theorem we deal simultaneously with several density functions q and, to avoid confusion, we write q as an extra argument for functions which depend on q .

Theorem 1.4. Suppose that densities $q_1, \dots, q_n$ satisfy condition (1.8) and (μ, λ_1) is an admissible pair of measures for $i=1, \dots, n$. Let $1 \leq m \leq n$. Put

$$\begin{aligned}
 (1.35) \quad \varphi_i(h, v) &= \phi_{\kappa(q_i)}(v) && \text{for } i=1, \dots, m, \\
 &= \phi_{-h}[\delta(q_i, v)] && \text{for } i=m+1, \dots, n; \\
 (1.36) \quad T(i, \epsilon_i, u) &= T_{k_i}(q_i, \epsilon_i, \lambda_i, u) && \text{for } i=1, \dots, m, \\
 &= T_{k_i}^1(q_i, \epsilon_i, \lambda_i, u) && \text{for } i=m+1, \dots, n.
 \end{aligned}$$

We have

$$\begin{aligned}
 (1.37) \quad & \int_0^\infty e^{-ru} du \, P_\mu \left[\bigcap_{i=1}^n \mathcal{T}(i, \epsilon_i, u) \right] \\
 & \approx r^{-1} \sum_{b \in \mathfrak{B}_k} a(k, b) \left[-\frac{\ln r}{2\pi} \right]^\nu g_{rb}(\mu, \lambda_1, \dots, \lambda_n)
 \end{aligned}$$

where $a(k, b)$ and ν are defined by (1.33).

1.9. Theorems 1.1 through 1.4 will be proved in Section 4 after we develop necessary tools in Sections 2 and 3. The relation of the paper to the previous work is discussed in Section 5.

We use the following notation: if a_j is a real-valued function on a finite set J , then a_J means the product of a_j over all $j \in J$.

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2. SOME PROPERTIES OF GREEN'S FUNCTION

2.1. In this section we get some estimates and asymptotic formulae for Green's function $G_r(X)$ defined by (1.3).

It is well-known (see e.g. [IM], p.233) that

$$(2.1) \quad G_r(x) = \frac{1}{\pi} K_0(\sqrt{2r}|x|)$$

where K_0 is a modified Bessel function which can be described (see [W], 3.71.14, and 3.7.2) by the formula

$$(2.2) \quad K_0(r) = -I_0(r) \ln \frac{r}{2} + B(r).$$

Here

$$(2.3) \quad I_0(r) = \sum_{m=0}^{\infty} a_m r^{2m} / (2m)!, \quad a_m = \left[\frac{2m}{m} \right] 2^{-2m};$$

$$(2.4) \quad B(r) = -C + \sum_{m=1}^{\infty} a_m \left(1 + \frac{1}{2} + \dots + \frac{1}{m} - C \right) r^{2m} / (2m!).$$

It follows from (2.2) that

$$(2.5) \quad \frac{1}{\pi} K_0(2\epsilon r) = h_{\epsilon} \varphi_{\epsilon}(r) + \psi_{\epsilon}(r),$$

with

$$\varphi_{\epsilon}(r) = I_0(2\epsilon r), \quad \psi_{\epsilon}(r) = \frac{1}{\pi} [B(2\epsilon r) - I_0(2\epsilon r) \ln r]$$

and h_{ϵ} given by (1.4).

Since $a_m \rightarrow 0$ and $a_m(1 + \frac{1}{2} + \dots + \frac{1}{m} - C) \rightarrow 0$ as $m \rightarrow \infty$, there exist constants $\gamma_1, \gamma_2, \gamma_3$ such that

$$(2.6) \quad \varphi_{\epsilon}(r) \leq \gamma_1 e^{2\epsilon r}, \quad |\psi_{\epsilon}(r)| \leq (\gamma_2 + \gamma_3 |\ln r|) e^{2\epsilon r} \text{ for all } r > 0.$$

2.2. Suppose that a random variable Y has a probability density q which satisfies condition (1.8) and put $N = |Y|/\sqrt{2}$. It follows from (2.1), (2.2) and (2.5) that

$$(2.7) \quad G(\epsilon Y) = h_{\epsilon} \varphi_{\epsilon}(N) + \psi_{\epsilon}(N) \leq (\gamma_1 h_{\epsilon} + \gamma_2 + \gamma_3 |\ln N|) e^{2\epsilon N}$$

and by (1.8) there exist constants β_k such that

$$(2.8) \quad E[G(\epsilon Y)^k] \leq \beta_k |\ln \epsilon|^k$$

for all sufficiently small ϵ .

We claim that

$$(2.9) \quad E \int [G(z) - G(z - \epsilon Y)]^2 dz \approx 0.$$

Indeed, the left side is equal to $2E[F(0) - F(\epsilon Y)]$ where

$$F(y) = \int G(z) G(z - y) dz = \int_0^{\infty} e^{-t} t p_t(y) dt$$

and (2.9) follows from an estimate

$$0 \leq F(0) - F(y) = (2\pi)^{-1} \int_0^\infty (1 - e^{-y^2/2t}) e^{-t} dt \leq \text{const. } y^2 \left(1 + \int_0^\infty dt e^{-t}/t\right) y^{2/2}.$$

By (2.7)

$$(2.10) \quad EG(\epsilon Y) = a(\epsilon) h_\epsilon + b(\epsilon), \quad a(\epsilon) = E\varphi_\epsilon(N), \quad b(\epsilon) = E\psi_\epsilon(N).$$

The functions $a(\epsilon)$ and $b(\epsilon)$ are even and analytic in a neighbourhood of 0. Since $a(0)=1, b(0)=-\kappa$ (cf. (1.13a)), we have $a(\epsilon)=1+O(\epsilon^2)$, $b(\epsilon)=-\kappa+O(\epsilon^2)$ and

$$(2.11) \quad EG(\epsilon Y) \approx h_\epsilon - \kappa$$

2.3. Now we investigate the functions

$$(2.12) \quad c_k(\epsilon) = Eg(\epsilon V_1, \dots, \epsilon V_k), \quad k=1, 2, \dots$$

where g is given by (1.3) (with $r=1$) and $V_1, \dots, V_k$ are i.i.d. random variables with a probability density q subject to the condition (1.8).

By (2.1)

$$(2.13) \quad c_k(\epsilon) = E \prod_{j \in J} \left[\frac{1}{\pi} K_0(2\epsilon R_j) \right]$$

where $J = \{1, 2, \dots, k-1\}$, $R_j = |V_j - V_{j+1}| / \sqrt{2}$.

By (2.1) and (2.5),

$$(2.14) \quad c_k(\epsilon) = \sum_\epsilon h_\epsilon^{|A|} f_{A\Gamma}(\epsilon).$$

Here $f_{A\Gamma}(\epsilon) = E(\varphi_{\epsilon, A} \varphi_{\epsilon, \Gamma})$ and the sum is taken over all partitions of J into disjoint sets r and A , $|A|$ meaning cardinality of A .

The functions $f_{A\Gamma}(\epsilon)$ have the same properties as $a(\epsilon)$ and $b(\epsilon)$, and $f_{A\Gamma}(0) = E\varphi_r$ where φ_j are defined by (1.23). Therefore $f_{A\Gamma}(\epsilon) = E\varphi_r + O(\epsilon^2)$. By (2.14)

$$c_k(\epsilon) \approx \sum_\epsilon h_\epsilon^{|A|} E\varphi_r.$$

2.4. Consider the set J as a linear graph with bonds $(1, 2), \dots, (k-2, k-1)$. Denote the connected components of r enumerated in the natural order by $r_1, \dots, r_m$. The sets r_j and r_{j+1} are separated by a connected component A_j of A . Besides $A_1, \dots, A_{m-1}$ the set A can have two extra components: A_0 - to the left of r_1 , and A_m - to the right of r_m . All numbers $k_j = |r_j|$ and $\ell_j = |A_j|$ are strictly positive except ℓ_0 and

ℓ_m which can vanish. The case $m=0$ is exceptional. In this case $A=J$.

Since $r_1, \dots, r_m$ are independent, $E r_i = a_1 \dots a_m$ where $a_i = E(r_1 \dots r_i)$. Therefore

$$(2.15) \quad c_k(\epsilon) = h_\epsilon^{k-1} \sum h_\epsilon^{\ell_0 + \ell_1 + \dots + \ell_m} a_{k_1} \dots a_{k_m},$$

the sum is taken over all $m \geq 1$ and all representations

$$(2.16) \quad k-1 = \ell_0 + k_1 + \ell_1 + \dots + \ell_{m-1} + k_m + \ell_m$$

such that $\ell_0, \ell_m \geq 0$ and the rest of terms are strictly positive.

It follows from (2.16) that

$$(2.17) \quad M(\epsilon, v) = \sum_{k=1}^{\infty} c_k(\epsilon) v^k \approx_{\phi_{-h_\epsilon}} [d(v)]$$

where d is defined by (1.24) and the equivalence relation $\approx$ for power series should be interpreted as an analogous relation between the corresponding coefficients.

3. RANDOM FIELDS ON DIRECTED TREES

3.1. A directed tree S is a finite collection of sites connected by arrows in such a way that:

- (a) every site is the end of at most one arrow;
- (b) there are no loops $s_1 \rightarrow s_2 \rightarrow \dots \rightarrow s_m \rightarrow s_1$.

We say that a site s is *initial* if no arrow enters it. Every connected component of S contains exactly one initial site.

We consider a family of independent random variables Z_s indexed by sites $s \in S$ and random variables $Y_{ss'}$, indexed by arrows ss' and we assume that, within every connected component S_b , all Z_s are identically distributed with a law λ_b , and all $Y_{ss'}$ are identically distributed with a density q_b .

Let $\epsilon = \epsilon_s$ be a positive function on S constant on each connected component. Obviously there exists a unique solution V_s of the equations:

$$(3.1) \quad \begin{aligned} V_{s'} - V_s &= \epsilon Y_{ss'}, \text{ for every arrow } ss', \\ V_s &= Z_s \text{ for every initial site } s. \end{aligned}$$

We call it a *random field over S with parameters* (ϵ, λ, q) .

3.2. Suppose that a directed tree is ordered and let $1, \dots, k$ be its sites enumerated according to the ordering. We consider only orderings with the property: all arrows have the form ij with $i < j$.

If a directed tree S is connected, then 1 is its only initial site. We note that the joint density for $V_1, \dots, V_k$ is equal to

$$q_S(x_1, \dots, x_k; \epsilon, \lambda) = \rho(x_1) \prod_{ij} q^\epsilon(x_j - x_i)$$

and the joint density for $V_2, \dots, V_k$ is

$$\tilde{q}_S(x_2, \dots, x_k; \epsilon, \lambda) = \int dx_1 q(x_1, \dots, x_k; \epsilon, \lambda)$$

where the product is taken over all arrows, $\lambda(dz) = \rho(z)dz$, and q^ϵ is defined by (1.9). Put

$$(3.2) \quad \begin{aligned} T_S(q, \epsilon, \lambda, u) &= \int_{D_k(u)} q_S(X_{t_1}, \dots, X_{t_k}; \epsilon, \lambda) dt_1 \dots dt_k, \\ \tilde{T}_S(q, \epsilon, \lambda, u) &= \int_{D_{k-1}(u)} \tilde{q}_S(X_{t_1}, \dots, X_{t_{k-1}}; \epsilon, \lambda) dt_1 \dots dt_{k-1}. \end{aligned}$$

(the domains $D_k(u)$ are defined by (1.11)). In particular, random variables T_{L_k} corresponding to the ordered tree

$$(3.3) \quad L_k: 1 \rightarrow 2 \rightarrow \dots \rightarrow k$$

coincide with T_k defined by (1.10), and the random variables $\tilde{T}_{L^k}$ corresponding to

$$(3.4) \quad L^k: \begin{array}{c} 3 \\ \uparrow \\ 2 \leftarrow 1 \rightarrow 4 \\ \downarrow \\ k+1 \end{array}$$

are identical to T^k given by (1.19).

Theorem 3.1. Consider a tree S with ordered connected components $S_1, \dots, S_n$ and put

$$\begin{aligned} T(b, u) &= T_{S_b}(q_b, \epsilon_b, \lambda_b, u) & \text{for } b=1, \dots, m; \\ T(b, u) &= \tilde{T}_{S_b}(q_b, \epsilon_b, \lambda_b, u) & \text{for } b=m+1, \dots, n. \end{aligned}$$

Let V be the random field over S with parameters (ϵ, λ, q) and let S^* be the set of all the sites in S except the initial sites of the components $S_{m+1}, \dots, S_n$. Consider all one-to-one mappings from the set $\{1, 2, \dots, N\}$ onto S and put $a \in A$ if the restriction of a to any component S_b is monotone increasing relative to the ordering of S_b .

We have

$$(3.5) \quad \int_0^\infty e^{-ru} du P_\mu \left[\bigcap_{b=1}^n T(b, u) \right] = r^{-1} \sum_{a \in A} E g_{\mu r}(V_{a_1 \epsilon}, \dots, V_{a_N \epsilon})$$

where V is the random field over S with parameters (ϵ, λ, q) and

$$g_{\mu r} = \int \mu(dx_0) g_r(x_0, x_1, \dots, x_N).$$

Proof. We note that

$$\begin{aligned} & P_\mu \left[\bigcap_{b=1}^n T(b, u) \right] \\ &= \sum_{a \in A} \int_{0 < t_{a_1} < \dots < t_{a_N} < u} P_\mu f_{\epsilon a}(X_{t_{a_1}}, \dots, X_{t_{a_N}}) dt_1 \dots dt_N \end{aligned}$$

where $f_{\epsilon a}(x_1, \dots, x_N)$ is the joint density for $V_{a_1 \epsilon}, \dots, V_{a_N \epsilon}$. Since $p_\mu(t_{a_1}, x_1; \dots; t_{a_N}, x_N)$ given by (1.1) is the joint density for $X_{t_{a_1}}, \dots, X_{t_{a_N}}$, we have

$$\begin{aligned} (3.6) \quad & P_\mu f_{\epsilon a}(X_{t_{a_1}}, \dots, X_{t_{a_N}}) \\ &= \int p_\mu(t_1, x_1; \dots; t_N, x_N) f_{\epsilon a}(x_1, \dots, x_N) dx_1 \dots dx_N \\ &= E p_\mu(V_{a_1 \epsilon}, \dots, V_{a_N \epsilon}). \end{aligned}$$

Formula (3.5) follows from (3.6) if we take into account that

$$(3.7) \quad \int_0^\infty e^{-ru} du \int_{D_N(u)} p_\mu(t_1, x_1; \dots; t_N, x_N) dt = r^{-1} g_{\mu r}(x_1, \dots, x_N).$$

3.3. Theorem 3.2. Consider a tree S with ordered connected components

$$(3.8) \quad \begin{aligned} S_b &= L_{k_b} \text{ for } b=1, \dots, m; \\ &= L_{k_b}^{k_b} \text{ for } b=m+1, \dots, n \end{aligned}$$

and let $\mathbf{a}=(a_1, \dots, a_N) \in \mathcal{A}$. Suppose that the first ℓ_1 elements in $(a_1, \dots, a_N)$ belong to S_{b_1} , the next ℓ_2 elements belong to S_{b_2} with $b_2 \neq b_1$ etc. Elements $b_1, b_2, \dots, b_M$ form a route b in the sense of Subsection 1.2.

If $(\mu, \lambda_1, \dots, \lambda_n)$ and $\mathbf{q}=(q_1, \dots, q_n)$ satisfy the conditions of Theorem 1.4, then

$$(3.9) \quad \begin{aligned} & \mathbb{E} g_{\mu r}(V_{a_1 \epsilon}, \dots, V_{a_N \epsilon}) \\ & \approx g_{rb}(\mu, \lambda) \prod_{j=1}^M c_{\ell_j b_j}(\epsilon_{b_j} \sqrt{r}) \end{aligned}$$

where $g_{rb}(\mu, \lambda)$ is given by (1.6) and

$$(3.10) \quad \begin{aligned} c_{\ell b}(\epsilon) &= \left[\int G(\epsilon y) q_b(y) dy \right]^{\ell-1} & \text{if } b \leq m, \\ &= \int g(\epsilon y_1, \dots, \epsilon y_\ell) \prod_{j=1}^{\ell} q_b(y_j) dy_j & \text{if } b > m. \end{aligned}$$

Proof. We have

$$(3.11) \quad g_{\mu r}(V_{a_1 \epsilon}, \dots, V_{a_N \epsilon}) = A_J$$

where $J = \{1, 2, \dots, N\}$,

$$(3.12) \quad \begin{aligned} A_1(\epsilon) &= G_{\mu r}(V_{a_1 \epsilon}), \\ A_j(\epsilon) &= G_r(V_{a_j \epsilon} - V_{a_{j-1} \epsilon}) \text{ for } j=2, \dots, N. \end{aligned}$$

Let σ_b be the initial site in S_b ,

$r = \{j: a_{j-1} \text{ and } a_j \text{ belong to different connected components of } S\}$,

$\Lambda = \{j: a_{j-1} \text{ and } a_j \text{ belong to the same connected component of } S\}$.

Note that $J = \{1\} \cup r \cup \Lambda$ and

$$(3.13) \quad \begin{aligned} A_1(\epsilon) &= G_{\mu r} Z(a_1) & \text{if } a_1 \in S_b, b \leq m, \\ &= G_{\mu r} [Z(\sigma_b) + \epsilon_{a_1} Y(\sigma_b, a_1)] & \text{if } a_1 \in S_b, b > m; \\ A_j(0) &= G_r [Z(a_j) - Z(a_{j-1})] & \text{if } j \in r; \\ A_j(\epsilon) &= G_r [\epsilon_{a_j} Y(a_{j-1}, a_j)] & \text{if } a_{j-1}, a_j \in S_b, b \leq m, \\ &= G_r \{ \epsilon_{a_j} [Y(\sigma_b, a_j) - Y(\sigma_b, a_{j-1})] \} & \text{if } a_{j-1}, a_j \in S_b, b > m, j > 1. \end{aligned}$$

By (2.9),

$$(3.14) \quad E[A_j(\epsilon) - A_j(0)]^2 \approx 0 \text{ for } j \in r.$$

Taking into account (2.8), we get

$$(3.15) \quad EA_j(\epsilon) \approx E[A_r(0)A_{\lambda}(\epsilon)A_1(\epsilon)].$$

Note that

$$(3.16) \quad A_r(0) = g_r(Z_{s_1}, \dots, Z_{s_M})$$

where $s_1 = a_1, s_2 = a_{\ell_1+1}, \dots, s_M = a_{\ell_{M-1}+1}$. Since $A_{\lambda}(\epsilon)$ is a function of the Y 's, it is independent of (3.16) and, by (3.15)

$$(3.17) \quad EA_j(\epsilon) \approx E[A_1(\epsilon)g_r(Z_{s_1}, \dots, Z_{s_M})] EA_{\lambda}(\epsilon).$$

We claim that

$$(3.18) \quad E\{[A_1(\epsilon) - A_1(0)]g_r(Z_{s_1}, \dots, Z_{s_M})\} \approx 0.$$

Indeed the function $F(x) = Eg_r(x, Z_{s_2}, \dots, Z_{s_M})$ is bounded and therefore it is sufficient to check that

$$(3.19) \quad E[A_1(\epsilon) - A_1(0)]^2 \approx 0.$$

Suppose that $a_1 \in S_b$. If $b < m$, then $A_1(\epsilon)$ does not depend on ϵ . If $b > m$, then

$$A_1(\epsilon) - A_1(0) = G_{\mu r}[Z(\sigma_b) + \epsilon_{a_1} Y(\sigma_b, a_1)] - G_{\mu r}[Z(\sigma_b)].$$

If μ is finite, then we get (3.19) from (2.9). If μ has a bounded Hölder continuous density, then $G_{\mu r}(x)$ and its gradient are bounded and, since λ is finite, we get (3.19) from the inequality

$$|A_1(\epsilon) - A_1(0)| \leq \text{const.} \epsilon_{a_1} |Y(\sigma_b, a_1)|.$$

The set λ is the union of $\lambda_1 = [2, \ell_1], \dots, \lambda_M = [\ell_{M-1} + 2, \ell_M]$. By (2.1)

$G_r(x) = G(\sqrt{r}x)$ and therefore

$$(3.20) \quad EA_{\lambda_j} = c_{\ell_j} b_j (\epsilon_{b_j} \sqrt{r}).$$

We note that $A_{\lambda_1}, \dots, A_{\lambda_M}$ are independent and formula (3.9) follows from (1.6), (3.17) (3.18) and (3.20).

4. PROOFS OF MAIN RESULTS

4.1. Proof of Theorem 1.4. Let

$$(4.1) \quad \tilde{m}_k(\epsilon, \lambda, r) = \int_0^\infty e^{-ru} du P_\mu \left[\prod_{i=1}^n T(i, \epsilon_i, u) \right].$$

It follows from Theorems 3.1 and 3.2 that

$$(4.2) \quad \tilde{m}_k(\epsilon, \lambda, r) \approx r^{-1} \sum_{b \in \mathfrak{B}^k} g_{rb}(\mu, \lambda) \prod_{j=1}^M c_{\ell_j b_j}(\epsilon_{b_j} \sqrt{r})$$

where $\mathfrak{B}^k$ is the set of all routes $b = (b_1, \dots, b_M)$ in $\{1, \dots, n\}$ which contain k_1 elements equal to 1, ..., k_n elements equal to n .

We introduce generating functions

$$(4.3) \quad M_b(\epsilon, v) = \sum_{\ell=1}^{\infty} c_{\ell b}(\epsilon) v^\ell.$$

By (3.10), (2.11), (2.17) and (1.35),

$$(4.4) \quad M_b(\epsilon \sqrt{r}, v) \approx c_b(h_{\epsilon \sqrt{r}}, v) = c_b(h_{\epsilon} - \rho, v) \quad \text{with } \rho = \frac{1}{\pi} \ln r.$$

Since $\ell_1 + \dots + \ell_M = k_1 + \dots + k_n$, we get from (4.3) and (4.2) that

$$(4.5) \quad \sum_{k_1, \dots, k_n \geq 1} \tilde{m}_k(\epsilon, \lambda; r) v_1^{k_1} \dots v_n^{k_n} \approx r^{-1} \sum_b g_{rb}(\mu, \lambda) \prod_{j=1}^M c_{\ell_j b_j}(h_{\epsilon_{b_j}} - \rho, v_{b_j}),$$

the sum is taken over all routes b in the space $\{1, \dots, n\}$ which pass through every point.

We note that, if $w = c_1(h, v)$, then $v = \mathcal{D}_1(h, w)$ where

$$(4.6) \quad \begin{aligned} \mathcal{D}_i(h, w) &= \phi_{h-\kappa(q_i)}(w) \quad \text{for } i \leq m, \\ &= \psi[\phi_h(w)] \quad \text{for } i > m. \end{aligned}$$

In both cases, for every ρ ,

$$(4.7) \quad c_1[h - \rho, \mathcal{D}_1(h, w)] = \phi_\rho(w)$$

We rewrite (4.5) in the form

$$(4.8) \quad \sum_{k_1, \dots, k_n \geq 1} \tilde{m}_k(\epsilon, \lambda; r) \mathcal{D}_1(w_1)^{k_1} \dots \mathcal{D}_n(w_n)^{k_n} \approx r^{-1} \sum_b g_{rb}(\mu, \lambda) \prod_{j=1}^M \phi_\rho(w_{b_j})$$

It follows from (1.15) (1.26) and (4.6) that

$$(4.9) \quad \sum_{k=1}^{\infty} \mathcal{T}(i, k, \epsilon_1, u) w^k = \sum_{\ell=1}^{\infty} \mathcal{D}_1(h_{\epsilon_1}, w)^{\ell} T(i, \ell, \epsilon_1, u).$$

By comparing (4.8) and (4.9), we see that the right side in (1.37) is equal to the coefficient at $w_1^{k_1} \dots w_n^{k_n}$ in the right side of (4.8).

If ℓ_1 is the number elements in $(b_1, \dots, b_M)$ which are equal to 1, then by (1.14)

$$(4.10) \quad \begin{aligned} \prod_{j=1}^M \phi_{\rho}(w_{b_j}) &= \prod_{i=1}^n \phi_{\rho}(w_i)^{\ell_i} \\ &= \prod_{i=1}^n \left[\begin{matrix} k_i - 1 \\ \ell_i - 1 \end{matrix} \right] w_i^{k_i} (-\rho)^{k_i - \ell_i} \end{aligned}$$

The coefficient at $w_1^{k_1} \dots w_n^{k_n}$ in (4.10) is $a(k, b) \rho^{\nu}$ with $a(k, b)$ and ν defined by (1.33). This implies (1.37).

4.2. Proof of Theorems 1.1 and 1.2. The integral in formula

(1.32) is the convolution of functions $1_{t>0} \int \mu(dz_0) p_t(z_{b_1} - z_0)$,

$p_t(z_{b_j}, z_{b_{j+1}}) 1_{t>0}$ for $j=1, \dots, M-1$ and $\mathcal{P}_{\nu}(\log t) 1_{t>0}$. Therefore

$$(4.11) \quad \int_0^{\infty} e^{-ru} du m_k(\lambda, u) = r^{-1} \sum_{b \in \mathfrak{B}_k} a(k, b) g_{rb}(\mu, \lambda) \left[-\frac{\ln r}{2\pi} \right]^{\nu}.$$

We compare this expression with (1.37) and we get

$$(4.12) \quad \int_0^{\infty} e^{-ru} du P_{\mu}[\mathcal{T}_{k_1}(\epsilon_1, \lambda_1, u) \dots \mathcal{T}_{k_n}(\epsilon_n, \lambda_n, u)] \approx \int_0^{\infty} e^{-ru} du m_k(\lambda, u).$$

To every $r>0$ there corresponds a measure $M_r(du, d\omega) = e^{-ru} du P(d\omega)$ on $\mathbb{R}_+ \times \Omega$. It follows from (4.12) that $\|\mathcal{T}_k(\epsilon, \lambda, u) - \mathcal{T}_k(\epsilon', \lambda, u)\|_{r, p} \approx 0$ where $\|\cdot\|_{r, p}$ means the $L^{2p}(M_r)$ -norm. Thus there exists an $L^{2p}(M_r)$ -limit

$$(4.13) \quad \mathcal{T}_k(\lambda, u) = \lim_{\epsilon \downarrow 0} \mathcal{T}_k(\epsilon, \lambda, u)$$

and

$$(4.14) \quad \mathcal{T}_k(\epsilon, \lambda, u) \approx \mathcal{T}_k(\lambda, u).$$

We conclude from (1.37) that $\mathcal{T}_k(q, \epsilon, \lambda, u) \approx \mathcal{T}_k(\tilde{q}, \epsilon, \lambda, u)$. Hence $\mathcal{T}_k(\lambda, u)$ does not depend on the choice of q . Theorem 1.1 is proved.

The same arguments prove Theorem 1.2.

4.3. Proof of Theorem 1.3. By (4.14), (1.37) and (4.11)

$$\begin{aligned} & \int_0^\infty e^{-ru} du P_\mu \left\{ \prod_{i=1}^n \gamma_{k_i}(\lambda_i, u) \right\} \\ &= \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-ru} du P \left\{ \prod_{i=1}^n \gamma_{k_i}(\epsilon_i, \lambda_i, u) \right\} \\ &= \int_0^\infty e^{-ru} du m_k(\lambda, u) \end{aligned}$$

which implies (1.31).

5. BIBLIOGRAPHICAL NOTES

5.1. Interest in the self-intersections of the Brownian motion has increased significantly in connection with Symanzik's ideas in quantum field theory. The functional $\gamma_2(m, 1)$ where m is the Lebesgue measure has been introduced in a pioneering work [V] by Varadhan which has appeared as an Appendix to Symanzik's memoir. For $k \geq 2$, the functionals $\gamma_k(\lambda)$ have appeared first in [D1] and [D2] as a tool for a probabilistic representation of $P(\gamma)_2$ fields.

In [D2] we considered polynomials of the field

$$(5.1) \quad T_{\epsilon Z}(\zeta) = \int_0^\zeta p_\epsilon(z, X_t) dt$$

where p is a symmetric transition density, X_t is the corresponding Markov process and ζ is an exponential killing time independent of X . Assuming that Green's function

$$(5.2) \quad G_r(x, y) = \int_0^\infty e^{-rt} p_t(x, y) dt$$

has singularity of the same kind as Green's function of the planar Brownian motion, we defined functions $B_{k\ell}(\epsilon, z)$ such that there exists an L^p -limit

$$(5.3) \quad \|T^k\|_\lambda = \lim_{\epsilon \downarrow 0} \int \lambda(dz) \sum_{\ell=0}^n B_{k\ell}(\epsilon, z) T_{\epsilon Z}^\ell(\zeta).$$

for all $p \geq 2$ and for a wide class of measures λ . In our present notations $\|T^k\|_\lambda = \gamma_k(\lambda, \zeta)$.

The random fields (5.3) are closely related to Wick's powers $: \varphi^{2n} :_{\lambda}$ of the free Gaussian field associated with X . In fact, we have arrived at our renormalization by using this relation.

The direct construction of the fields $\mathcal{T}_k$ given in the present paper for the case of the Brownian motion on $\mathbb{R}^2$ has a number of advantages:

- (i) Computations are much simpler than in [D2] and we get fields $\mathcal{T}_k(\lambda, u)$ defined for each u (not only $\mathcal{T}_k(\lambda, \zeta)$).
- (ii) We prove that $\mathcal{T}_k(\lambda, u)$ is the limit of fields $\mathcal{T}_k^{\epsilon}(\epsilon, \lambda, u)$ corresponding to a rather general density function q not just to the transition density p .
- (iii) We get an explicit expression for the coefficients $B_{k\ell}(\epsilon)$ as polynomials in $\ln \epsilon$ (because of translation invariance of the Brownian motion, $B_{k\ell}$ do not depend on z).
- (iv) We show that the functionals T_k given by (1.10) also can be renormalized to converge to $\mathcal{T}_k$. Moreover the renormalization is much simpler than in the case of T^k .

The case $k=2$ has been studied also in [D3] and [D4]. In [D3], the existence of L^p -limits

$$(5.4) \quad \varphi_{\lambda}(f) = \lim_{\epsilon \downarrow 0} \int_{\lambda}(dz) \int \int_{0 < s < t} ds dt f(s, t) \left[p_{\epsilon}(z, X_s) p_{\epsilon}(z, X_t) - \frac{p_{\epsilon}(z, X_s)}{2\pi(t-s) + 2\epsilon} \right]$$

has been proved for all sufficiently smooth functions f with compact support. In [D4] the functional $\varphi_{\lambda}(f)$ has been expressed in terms of stochastic integrals. The method is due to Rosen who used it in [R1] to get a simple proof of Varadhan's result.

5.2. Various results about the functional $\mathcal{T}_2(m, u)$ are contained in [Y1], [Y2], [Y3] and [R1], [R2] and [L1]. In particular in [L1], a relation between this functional and the measure of the Brownian sausage has been established. A renormalization for $T_3(m, u)$ is given

in [Y4] (it has been discovered independently by J.Rosen).

5.3. Recently Rosen [R3] proved that for every bounded Borel set $B \subset (0 < t_1 < \dots < t_k)$ there exists an L^2 -limit

$$I^k(B) = \lim_{\epsilon \downarrow 0} \int_B \{p_\epsilon(X_{t_1}, X_{t_2})\} \dots \{p_\epsilon(X_{t_{k-1}}, X_{t_k})\} dt_1 \dots dt_k$$

where $\{Y\} = Y - EY$. An interesting open problem is to express $I^k(D_k(u))$ through $\mathcal{T}_\epsilon(m, u)$. Such an expression is known only for $k \leq 3$.

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