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by Martin L. Silverstein

0. Introduction

Let B_t, $0 \leq t \leq 1$ be standard one dimensional Brownian motion starting at 0. Fix a positive integer m and for $0 < t \leq 1$ let $H_m^t(x)$ be an orthogonal polynomial of degree m for B_t. That is, $H_m^t(x)$ is a polynomial of degree m in x and $E\, H_m^t(B_t)B_t^j = 0$ for $j = 0,1,\ldots m-1$. Then for some choice of constants a_t the process

$$a_t H_m^t(B_t), \quad 0 < t \leq 1$$

is a martingale. This well known fact can be found in Chapter 2 in McKean (1969) where $a_t H_m^t(x)$ is identified as the Hermite polynomial $H_m(t,x)$. Our main result is that this property of Brownian motion is shared by the following discrete time process defined on the unit sphere in n-dimensional Euclidean space.

For $n \geq 3$ let $x_1,\ldots,x_n$ be the usual Euclidean coordinates defined on the unit sphere $S^{n-1}(1)$ equipped with uniform measure normalized to have total mass one. The process of interest is the sum of squares process ss_k defined for $1 \leq k \leq n-1$ by

$$ss_k = x_1^2 + \ldots + x_k^2 .$$

Orthogonal polynomials $Q_m^{n,k}(s)$ can be defined in terms of certain Jacobi polynomials $P_m^{(\alpha,\beta)}$:

$$Q_m^{n,k}(s) = P_m^{(\alpha,\beta)}(2s-1) \tag{0.1}$$

with $\alpha = \frac{1}{2}(n-k)-1$ and $\beta = \frac{1}{2}k-1$. We will prove

<u>Theorem. For dimension</u> $n \geq 3$ and for $m = 1,2\ldots$ the process $\{M_m^n(k),\ 1 \leq k \leq n-1\}$ <u>defined by</u>

$$M_m^n(k) = \frac{\Gamma(\frac{1}{2}(n-k))}{\Gamma(\frac{1}{2}(n-k)+m)} \quad Q_m^{n,k}(ss_k)$$

<u>is a martingale</u>.

The proof will show that for the conditioning σ-algebra (past) at time k we can take the one generated by the coordinates $x_1,\ldots,x_k$.

A weak version of the theorem is true for the process of partial sums $s_k = x_1 +\dots+ x_k$. The orthogonal polynomials $P_m^{n,k}$ are certain Gegenbauer polynomials. If $k < \ell$ then we recover a constant times $P_m^{n,k}(s_k)$ if we condition $P_m^{n,\ell}(s_\ell)$ on the σ-algebra generated by s_k alone but in general not if we condition on the σ-algebra generated by all of $s_1,\dots,s_k$. Of course the latter σ-algebra is needed for the martingale property.

Preliminaries on Jacobi polynomials and integration on spheres are collected in Section 1. This theorem is proved in Section 2.

1. Preliminaries

Good references for integration on spheres are the beginning of Chapter IX in Vilenkin (1968) and of Chapter 1 in Müller (1961). Starting with formulae given in the references and making routine substitutions, we see that ss_k has the distribution with density

$$(1.1) \qquad \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}(n-k))\,\Gamma(\frac{1}{2}k)} (1-s)^{\frac{1}{2}(n-k)-1} s^{\frac{1}{2}k-1},$$

for $0 \le s \le 1$. The pair ss_k, ss_{k+1} has joint density

$$(1.2) \qquad \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}k)\Gamma(\frac{1}{2}(n-k-1))} (t-s)^{-\frac{1}{2}} s^{\frac{1}{2}k-1} (1-t)^{\frac{1}{2}(n-k-3)}$$

for $0 \le s \le t \le 1$, with s,t corresponding respectively to ss_k and ss_{k+1}. Of course $1 \le k \le n-1$ in (1.1) and $1 \le k \le n-2$ in (1.2). Also at one point we will need the joint density for $x_1,\dots,x_k$, ss_{k+1}:

$$(1.3) \qquad \pi^{-\frac{1}{2}(k-1)} (\Gamma(\tfrac{1}{2}n) / \Gamma(\tfrac{1}{2}(n-k-1))) (1-t)^{\frac{1}{2}(n-k-3)} (t-x_1^2-\dots-x_k^2)^{-\frac{1}{2}}$$

for $x_1^2 +\dots+ x_k^2 \le t$.

A very accessible reference for the Jacobi poynomials is Rainville (1960). For $\alpha,\beta > -1$ the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$, $m \ge 0$ are orthogonal polynomials for the density $(1-x)^\alpha (1+x)^\beta$, $-1 \le x \le 1$. We will use the explicit representation

$$(1.4) \qquad P_m^{(\alpha,\beta)}(x) = \sum_{j=0}^{m} \frac{(1+\alpha)_m (1+\alpha+\beta)_{m+j} (-1)^j 2^{-j} (1-x)^j}{(1+\alpha)_j (1+\alpha+\beta)_m j!\ (m-j)!}$$

with the notation $(a)_j = \Gamma(a+j)/\Gamma(a)$.

2. Proof of the theorem

Beginning with known orthogonality properties of the Jacobi polynomials mentioned in the last paragraph of Section 1, and substituting $x = 2s-1$ in (1.1), we conclude that (0.1) does indeed define orthogonal polynomials in ss_k. The theorem will be proved if we can show that

$$(2.1) \qquad E(Q_m^{n,k+1}(ss_{k+1}) \mid x_1,\dots,x_k) = \frac{(\frac{1}{2}(n-k-1))_m}{(\frac{1}{2}(n-k))_m} Q_m^{n,k}(ss_k)$$

From (1.3) we conclude that the conditional expectation in (2.1) is unchanged if we condition only on ss_k. Combining this observation with (1.2), we see that we need only verify

$$(2.2) \qquad \int_s^1 dt\ (t-s)^{-\frac{1}{2}}(1-t)^{\frac{1}{2}(n-k-3)}\ Q_m^{n,k+1}(t)$$
$$= \int_s^1 dt\ (t-s)^{-\frac{1}{2}}(1-t)^{\frac{1}{2}(n-k-3)}\ \frac{(\frac{1}{2}(n-k-1))_m}{(\frac{1}{2}(n-k))_m}\ Q_m^{n,k}(s)$$

By (1.4) and (0.1) we can write

$$(2.3) \qquad Q_m^{n,k}(s) = \sum_{j=0}^{m} \frac{(\frac{1}{2}(n-k))_m(\frac{1}{2}(n-2))_{m+j}(-1)^j\ (1-s)^j}{(\frac{1}{2}(n-k))_j(\frac{1}{2}(n-2))_m\ j!\ (m-j)!}$$

Comparing coefficients of $(1-t)^j$ and $(1-s)^j$ on the two sides of (2.2), we see that it is enough to show

$$(2.4) \qquad \int_s^1 dt\ (t-s)^{-\frac{1}{2}}(1-t)^{\frac{1}{2}(n-k-3)+j}$$
$$= \int_s^1 dt\ (t-s)^{-\frac{1}{2}}(1-t)^{\frac{1}{2}(n-k-3)}\ \frac{(\frac{1}{2}(n-k-1))_j}{(\frac{1}{2}(n-k))_j}\ (1-s)^j$$

Substituting $x = (1-t)/(1-s)$ and using the well known Beta function identity

$$\int_0^1 dx\ x^{p-1}(1-x)^{q-1} = \Gamma(p)\,\Gamma(q)/\Gamma(p+q)$$

we reduce (2.4) to

$$\frac{\Gamma(\frac{1}{2}(n-k-1)+j)}{\Gamma(\frac{1}{2}(n-k)+j)}\ (1-s)^j = \frac{\Gamma(\frac{1}{2}(n-k-1))}{\Gamma(\frac{1}{2}(n-k))}\ \frac{(\frac{1}{2}(n-k-1))_j}{(\frac{1}{2}(n-k))_j}\ (1-s)^j$$

which is certainly true. This finishes the proof.

Remark. The basic identity (2.1) generalizes to Jacobi polynomials with α, β general. Also the increment $\frac{1}{2}$ in the parameters α, β can be replaced by any positive number. The author is presently investigating these generalizations.

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