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A remark on the paper " Une martingale
d'opérateurs bornés, non représentable en intégrale
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by

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Recently, J.L. Journé has constructed a remarkable example of a bounded operator valued quantum martingale X in the usual filtration of the boson Fock space $\mathfrak{F}_+(L_2(0,\infty))$ which does not admit the representation

$$dX = E dA^\dagger + F d\Lambda + G dA \quad (1)$$

in the sense of [1] over the domain $\mathcal{E}$, where E, F, G are adapted operator processes, $A^\dagger, \Lambda, A$ are respectively the creation, conservation (gauge) and annihilation martingales, and $\mathcal{E}$ is the linear manifold generated by exponential vectors. In [3], a necessary and sufficient regularity condition was found for a bounded operator valued martingale X to satisfy (1) with E, F, G being bounded operator valued processes satisfying the condition

$$\int_0^t (\|E(s)\|^2 + \|G(s)\|^2) ds < \infty \quad \text{for all } t.$$

The purpose of this note is to indicate the possibility of achieving the representation (1) for the Journé example provided $\mathcal{E}$ is suitably restricted and the strictness of the definition of an adapted process is relaxed.

We begin with a class of examples of quantum martingales determined by second quantization of integral operators. Let $K(.,.)$ be a complex valued continuous function on $[0,\infty) \times [0,\infty)$. For every $t > 0$ define the bounded operator K_t on $L_2([0,\infty))$ by

$$(K_t f)(s) = \chi_{[0,t]}(s) \int_0^t K(s,\tau) f(\tau) d\tau + \chi_{(t,\infty)}(s) f(s) \quad (2)$$

where χ_C denotes the indicator function of the set C . Let K_∞ denote the integral operator defined by

$$(K_\infty f)(s) = \int_0^\infty K(s,\tau) f(\tau) d\tau. \quad (3)$$

If K_∞ is a bounded operator on $L_2[0,\infty)$ then $\|K_t\| \leq \max(1, \|K_\infty\|)$ for all t . Let $X_K(t)$ denote the operator on Fock space defined by

the relations

$$X_K(t)\psi(f) = \psi(K_t f) \quad \text{for } f \in L_2[0, \infty) \quad (4)$$

where $\psi(f)$ is the exponential vector corresponding to f . Then $X_K(t)$, the second quantization of K_t , is defined on the domain $\mathcal{E}$ and an easy computation shows that

$$\langle \psi(fx_{[0,a]}), X_K(t)\psi(gx_{[0,a]}) \rangle = \langle \psi(fx_{[0,a]}), X_K(a)\psi(gx_{[0,a]}) \rangle$$

for all $t \geq a$. In other words, $X_K = \{X_K(t), t \geq 0\}$ is a quantum martingale with domain $\mathcal{E}$, and $X_K(0) = \text{identity}$. If K_∞ is a contraction then K_t is a contraction for every t and hence $X_K(t)$ can be extended to a contraction on Fock space. In other words, X_K becomes a contraction valued operator martingale.

By straightforward differentiation we obtain the relation

$$\begin{aligned} \frac{d}{dt} \langle \psi(f), X_K(t)\psi(g) \rangle = \\ \langle \psi(f), X_K(t)\psi(g) \rangle \{-\bar{f}(t)g(t) + \bar{f}(t) \int_0^t K(t,s)g(s)ds + g(t) \int_0^t K(s,t)\bar{f}(s)ds\} \end{aligned} \quad (5)$$

in the generalized sense of absolute continuity. In the language of [1], (5) is equivalent to saying that X_K obeys the quantum stochastic differential equation

$$dX_K = X_K d\Lambda^\dagger - X_K d\Lambda + MX_K dA \quad (6)$$

where L and M are adapted processes of operators defined on the domain $\mathcal{E}$ by the relations

$$\begin{aligned} L(t) &= a(x_{[0,t]} \overline{K(t, \cdot)}) \\ M(t) &= a^\dagger(x_{[0,t]} \overline{K(\cdot, t)}) \end{aligned} \quad (7)$$

and $a, a^\dagger$ are the usual annihilation and creation fields over $L_2[0, \infty)$. Since $\mathcal{E}$ is left invariant by the operators $X_K(t)$, $a(h)$ for all $t \geq 0$, $h \in L_2[0, \infty)$, it follows that the coefficients $X_K L$, $-X_K$, $M X_K$ of $d\Lambda^\dagger$, $d\Lambda$ and dA respectively in (6) are all well defined adapted processes on the domain $\mathcal{E}$ satisfying the inequalities

$$\int_0^t \{ \|X_K(s)L(s)\psi(f)\|^2 + |f(s)|^2 \|X_K(s)\psi(f)\|^2 + \|M(s)X_K(s)\psi(f)\|^2 \} ds < \infty \quad (8)$$

for all t . We remark that the finiteness of the third integral in (8) follows from the canonical commutation rules.

We now try to relax the conditions on the kernel K . As long as K_t is a bounded operator for each t and

$$\int_0^t (|K(s,t)|^2 + |K(t,s)|^2) ds < \infty \quad \text{for each } t \quad (9)$$

it is clear that $L(t)$ and $M(t)$ are well defined on $\mathcal{E}$ by (7) and

condition (8) obtains. This implies (6). In the example of Journé, $K(s,t) = (s-t)^{-1}$ and (9) breaks down. Then the definition of $L(t)$ and $M(t)$ by (7) does not make any sense. To face this situation we have to interpret equation (6) in a weak sense and we proceed as follows.

Let $\mathcal{L} \subset L_2[0, \infty)$ denote the linear manifold of functions f which satisfy the local Lipschitz's condition

$$\|f\|_{\mathcal{L},t} = \sup_{0 \leq x, y \leq t} \left| \frac{f(x) - f(y)}{x - y} \right| < \infty \quad \text{for all } t.$$

Denote by $\mathcal{E}(\mathcal{L})$ the linear manifold generated by the set $\{\psi(f), f\mathcal{E}\mathcal{L}\}$. For the kernel $K(s,t) = (s-t)^{-1}$ we define the martingale X_K as before and observe that for $f, g \in \mathcal{L}$

$$\begin{aligned} \frac{d}{dt} < \psi(f), X_K(t)\psi(g) > = \\ < \psi(f), X_K(t)\psi(g) > + \left\{ -\bar{F}(t)g(t) + \int_0^t \frac{\bar{F}(t)g(s) - \bar{F}(s)g(t)}{t-s} ds \right\} \end{aligned} \quad (10)$$

The last integral in the above equation can be written as

$$\bar{F}(t) \int_0^t \frac{g(s) - g(t)}{t-s} ds + g(t) \int_0^t \frac{\bar{F}(t) - \bar{F}(s)}{t-s} ds.$$

This suggests the introduction of the Schwartz-Fock space $\mathcal{F}_S$ of sequences of distributions in the following sense. Any element of $\mathcal{F}_S$ is of the form $\lambda = (c, \lambda_1, \lambda_2, \dots)$ where λ_n is a symmetric distribution (in the sense of Schwartz) in the space $\mathbb{R}^n$ for $n=1, 2, \dots$, and $c \in \mathbb{C}$.

Let $\mathcal{M}$ be an arbitrary dense linear manifold in $L_2[0, \infty)$ and let $\mathcal{E}(\mathcal{M})$ be the linear manifold in Fock space generated by $\{\psi(f), f\mathcal{E}\mathcal{M}\}$. Let $E = \{E(t), t \geq 0\}$ be a family of linear maps $E(t) : \mathcal{E}(\mathcal{M}) \rightarrow \mathcal{F}_S$ satisfying the following conditions:

1) For any C_c^∞ function f on $[0, \infty)$, the series

$$< \psi(f), E(t)\psi(g) > = 1 + \sum_{n=1}^{\infty} n!^{-1/2} (E(t)\psi(g))_n (\bar{F}^{\otimes n}) \quad (11)$$

converges absolutely, where $(E(t)\psi(g))_n$ denotes the n -th term of the sequence $E(t)\psi(g)$ (a symmetric distribution on $\mathbb{R}^n$).

2) The scalar quantity

$$< \psi(f), E(t)\psi(g) > \exp\left(-\int_t^\infty \bar{F}(s)g(s)ds\right)$$

depends only on the values of f and g on the interval $[0, t]$ and is a Borel function of t .

Then we say that E is a generalized adapted process with domain $\mathcal{E}(\mathcal{M})$. If X, E, F, G, H are five generalized adapted processes over $\mathcal{E}(\mathcal{M})$ such that

$$\begin{aligned} < \psi(f), X(t_2)\psi(g) - X(t_1)\psi(g) > = \int_{t_1}^{t_2} < \psi(f), \bar{F}(s)E(s)\psi(g) + \\ + \bar{F}(s)g(s)F(s)\psi(g) + H(s)\psi(g) > ds \quad \text{for all } f \in C_c^\infty, g \in \mathcal{M}, t_1 < t_2 \end{aligned}$$

we say that

$$dX = EdA^\dagger + Fd\Lambda + GdA + Hdt$$

on the domain $\mathcal{E}(\mathcal{M})$. With these conventions it is straightforward to verify that the Journé martingale X_K obeys the generalized quantum stochastic differential equation

$$dX_K = EdA^\dagger + Fd\Lambda + GdA$$

where E, F, G are generalized adapted processes over the domain $\mathcal{E}(\mathcal{I})$, defined by

$$E(t)\psi(g) = \left(\int_0^t \frac{g(s) - g(t)}{t - s} ds \right) \psi(K_t g),$$

$$F(t)\psi(g) = -\psi(K_t g),$$

$$G(t)\psi(g) = (1, \lambda_1, \lambda_2, \dots) e^{\mathcal{F}_S},$$

where for any test function ϕ on $\mathbb{R}^n$

$$\begin{aligned} \lambda_n(\phi) = n!^{-1/2} \sum_{j=1}^n \int_{\mathbb{R}^n} \chi_{[0, t]}(x_j) h(x_1) \dots \hat{h}(x_j) \dots h(x_n) \\ \times \frac{\phi(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n) - \phi(x_1, \dots, x_n)}{t - x_j} dx_1 \dots dx_n, \end{aligned}$$

here $h = K_t g$, and the symbol $\hat{}$ over a term implies its omission.

Just as the derivative of a bounded function on the line could be a distribution, it seems that the 'partial derivatives' of a bounded operator valued quantum martingale X in Fock space, with respect to the fundamental creation, conservation and annihilation martingales could be generalized adapted processes determining X .

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Note. The norm conventions on Fock space are those of [1], and are slightly different from those used elsewhere in this volume (these would require a factor $n!^{-1}$ instead of $n!^{-1/2}$ in formula (11)).