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The first passage problem for generalized Ornstein-Uhlenbeck processes with non-positive jumps

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1. Introduction. Let \((\Omega,F,P)\) be a probability space. We consider a cadlag stationary random process \(S_t, t \geq 0\), with independent increments and non-positive jumps

\[ \Delta S_t = S_t - S_{t^-} = S_t - \lim_{s \uparrow t} S_s \leq 0, \]

that is defined on this space and satisfies \(S_0 = 0\).

It is well known (\cite{31}) that the characteristic function of \(S_t\) has the form

\[ \mathbb{E} \exp(iuS_t) = \exp t(ibu - cu^2 + \int_{(-\infty,0)} F(dx)(e^{iux} - iux.1_{x \geq 1})]], \]

where \(-\infty < b < c \leq 0\), and the Lévy measure \(F(.)\) satisfies

\[ \int_{(-\infty,0)} F(dx) 1_{x^2 < 0}. \]

Following Skorokhod (\cite{8}) one can use the analytical continuation of (1.1) to the half-plane \(\text{Re}(iu) > 0\) and obtain the Laplace transform of \(S_t\) by substituting \(u\) instead of \(iu\). Thus, we have

\[ \mathbb{E} \exp(uS_t) = \exp t\psi(u), \quad u \geq 0, \]

where

\[ \psi(u) = bu + cu^2 + \int_{(-\infty,0)} F(dx)(e^{ux} - iux.1_{x \geq 1}). \]

For arbitrary \(\lambda > 0\) and \(-\infty < x < \infty\) we define the random process \(X_t, t \geq 0\), by the formula

\[ X_t = e^{-\lambda t}(x + \int_0^t e^{\lambda v} dS_v), \]

the stochastic integral w.r.t. the semi-martingale \(S\) being understood in the usual sense.

Definition. The random process \(X\) will be called the starting at \(x\) generalized Ornstein-Uhlenbeck process with parameter \(\lambda > 0\).
Certainly, the process \( X \) is characterized by the triplet \((b,c,F(.))\) as well. With \( b = 0, \ c = \frac{1}{2} \) and \( F(.) = 0 \) our definition yields the standard Wiener process \( S \) and the usual Ornstein-Uhlenbeck process \( X \).

Given a real number \( \mu > x \), let us introduce the first passage time

\[
T_{\mu}(x) = \inf \{ t \geq 0 : X_t \geq \mu \}.
\]

As far as \( \Delta X_t = \Delta S_t \leq 0 \), if \( T_{\mu}(x) < \infty \) one gets immediately the equality

\[
X_{T_{\mu}(x)} = \mu.
\]

The purpose of this paper is to determine the distribution of \( T_{\mu}(x), \mu > x \), by means of Laplace transform

\[
\gamma_{\mu}(\theta,x) = E \exp \left( -\theta T_{\mu}(x) \right), \ \theta > 0.
\]

It should be noted that generally speaking, we have no equation for the transition density of \( X \) and the usual Darling-Siegert approach to the first passage problem of diffusion processes ([2]) is not applicable in our case. Our approach is based on martingale techniques and depends essentially on the existence of suitable martingales on the process \( X \) (see Theorem 1 below). Besides the new generality of the explicit representation for \( \gamma_{\mu}(\theta,x) \) (Section 4), this approach gives us in particular the possibility to obtain ones again and in a natural way the interesting result of Novikov ([6]) concerning the first passage times of a stable process \( S \) through one-sided non-linear boundaries. The basic tool in this special case is the suitable time-change (Section 6) that transfers the linear problems for \( X_t, t \geq 0 \), into some non-linear problems for \( S_t, t \geq 0 \), and conversely. We make use of the reconversion in order to give an example of optimal stopping problem that admits a solution in terms of \( T_{\mu}(x) \).

2. The process \( X \). For the next we need to calculate the conditional Laplace transforms of the process \( X \) that was defined in (1.5). Let us introduce the \( \sigma \)-algebras

\[
\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t); \ t \geq 0,
\]

and the functions \( L(u;t,s) = E(\exp(uX_t)|\mathcal{F}_s^X), s < t, u > 0 \).
Since the stochastic integral in (1.5) might be looked at as an integral taken in the sense of convergence in probability ([4]), a simple argument leads to the following result.

**Proposition 1.** For any $0 \leq s < t$ and $u \geq 0$ one has

\[(2.1) \quad L(u; t, s) = \exp(e^{-\lambda(t-s)}X_s,u + \int_s^t \psi(u,e^{-\lambda(t-v)}) \, dv),\]

**Proof.** With an arbitrary subdivision $s = t_0 < t_1 < \ldots < t_n = t$, $\varepsilon = \max_{i \leq n} |t_i - t_{i-1}|$ and $Y_t = \int_0^t e^{\lambda v} \, dS_v$, we get

\[
E(\exp(uY_t) | F_s) = \exp(uY_s) \cdot E(\exp(u \int_s^t e^{\lambda v} \, dS_v) | F_s)
\]

\[
= \exp(uY_s) \lim_{\varepsilon \downarrow 0} \prod_{i=1}^n E(\exp(u e^{\lambda(t_i-t_{i-1})} \cdot (S_{t_i} - S_{t_{i-1}})))
\]

\[
= \exp(uY_s) \lim_{\varepsilon \downarrow 0} \prod_{i=1}^n \exp(\psi(u,e^{\lambda(t_i-t_{i-1})})
\]

\[
= \exp(uY_s + \int_s^t \psi(u,e^{\lambda v}) \, dv)
\]

as a consequence of (1.3) and the independent increments property of $S$.

Now starting with (1.5) we have

\[
L(u; t, s) = \exp(e^{-\lambda t}X_s,u + \int_s^t \psi(u,e^{-\lambda(t-v)}) \, dv)
\]

and the latter obviously implies (2.1).

**Corollary 1.** The Laplace transform of $X_t$ has the form

\[
E \exp(uX_t) = \exp(e^{-\lambda t}X_s,u + \int_s^t \psi(u,e^{-\lambda(t-v)}) \, dv), \quad u \geq 0.
\]

**Corollary 2.** The process $X$ is a cadlag Markov process. (Certainly, $X$ has also the strong Markov property.)

3. The martingale $M$. We are going to introduce a martingale $M_t(\theta)$, $t \geq 0$, depending on the process $X$ trajectories. To this end, one observes that because of (1.2) the quantity $F[-1,-z]$ is finite for every $z$, $0 < z \leq 1$. Thus, the measure
\[ G(dz) = F[-1,-z] \, dz \]
on \((0, 1]\) is well defined. We need the following assumption.

**Hypothesis G.** Either \(c > 0\) or the measure \(G(.)\) satisfies the condition

\[ \lim_{z \to 0^+} z^\kappa G(z,1) = C > 0 \]
for some constant \(\kappa, \ 0 < \kappa < 1.\)

Next, one defines successively

\[ g(y) = -\frac{y}{\lambda} \psi(u) \, du, \quad y > 0, \]
and

\[ M_t(b) = e^{-\theta t} \int_{\lambda b}^{\lambda b-1} \exp\{X_t y + g(y)\} \, dy, \quad t \geq 0. \]

The next statement is crucial because it permits an essential use of the martingale theory later on.

**Theorem 1.** Under the hypothesis G for any positive \(\theta\) the random process \(M_t(b)\), \(t \geq 0\), is a martingale w.r.t. \(F^X_t\), \(t \geq 0\).

**Proof.** First, we observe that our hypothesis G implies the convergence of the integral in (3.3). In fact, we have

\[ g(y) = -\frac{b}{\lambda} (y - 1) - \frac{c}{2\lambda} (y^2 - 1) - \frac{1}{\lambda} g_1(y) - \frac{1}{\lambda} g_2(y), \]
where

\[ g_1(y) = \int_{\lambda}^{\lambda b} \psi_1(u) \, du, \quad g_2(y) = \int_{\lambda}^{\lambda b} \psi_2(u) \, du \]
and

\[ \psi_1(u) = \int_{-\infty}^{0} F(dx) (e^{ux} - 1), \psi_2(u) = \int_{[1,0]} F(dx) (e^{ux} - 1 - ux), \quad u \geq 0. \]

The convergence of the integral at \(y = 0\) is obvious, because \(\lim_{y \to 0} g(y) = -\infty.\)

Now let us denote \(d_1 = \int_{(-\infty,0]} F(dx) \geq 0, \ d_2 = \int_{[0,1]} F(dx) \, x^2 \geq 0.\) In consequence of (1.2) one gets \(0 \leq d_1 + d_2 < \infty.\) Our function \(\psi_1\) satisfies \(0 \leq \frac{\psi_1(u)}{u} + d_1\) and \(0 \geq \frac{\psi_1'(u)}{u} + 0\) as \(u \to \infty.\) This means that \(\frac{\sqrt{y}}{2} \leq \int_1^{y} \psi_2(u) \, du \leq d_1 \ln y.\) On the other hand \(0 < e^{ux} - 1 - ux \leq u \, x^2, \quad u > 0, -1 \leq x < 0,\) and in this way one obtains the inequalities \(0 \leq \frac{\psi_2(u)}{u} \leq \frac{u}{2}; \ d_2 < \infty; \) and \(0 \leq g_2(y) \leq d_2 (y^2 - 1).\)
If $c > 0$, the corresponding term $-\frac{c}{2\lambda} (y^2 - 1)$ in $g(y)$ ensures the convergence. If $c = 0$, by the equality $\frac{\psi_2(u)}{u} = \int_0^1 (1 - e^{-uz}) G(dz)$, where obviously $0 \leq \int_0^1 z G(dz) = \frac{\lambda}{2} < \infty$, the hypothesis (3.1) and the corollary of Theorem 4.15 in [1] one gets $\lim_{u \to \infty} u^{-\frac{1}{2}} \psi_2(u) \geq C \Gamma(1 - \kappa) > 0$. Consequently, $\frac{\psi_2(u)}{u} \geq C_u u^\kappa$ for any $C_u$ belonging to the interval $(0, C \Gamma(1 - \kappa))$ and $u \geq u_2(C_2) > 0$ (sufficiently large). This implies $g_2(y) \geq C_2 y^{1+\kappa} + C_1$, $y > u_2(C_2)$, and the convergence of our integral too.

Secondly, applying Fubini's lemma and (2.1) for $0 \leq s \leq t$ (and with $z = ye^{-\lambda(t-s)}$) we get

$$
E(M_t(\theta) \mid F_s) = e^{-\theta t} \int_0^t y^{-\frac{\lambda}{2}} -1 E(\exp(X_t.y + g(y)) \mid F_s) dy
$$

$$
= e^{-\theta s} \int_0^t y^{-\frac{\lambda}{2}} -1 \exp(g(y) - \theta(t-s) + e^{-\lambda(t-s)} y X_s + \psi(ye^{-\lambda(t-s)})) dy
$$

$$
= e^{-\theta s} \int_0^t z^{-\frac{\lambda}{2}} -1 \exp(z X_s + g(z e^{-\lambda(t-s)})) dy.
$$

But the function $f(u,z) = g(z e^{\lambda u}) + \int_0^u \psi(z e^{\lambda v}) dv$, $u \geq 0$, satisfies the condition

$$
\frac{\partial f(u,z)}{\partial u} = g'(z e^{\lambda u}) \cdot z e^{\lambda u} + \psi(z e^{\lambda u}) = g'(y) \cdot \lambda y + \psi(y) = 0
$$

with $y = z e^{\lambda u}$, in view of (3.2). Therefore,

$$
f(u,z) = \text{const} = f(0,z) = g(z)
$$

and we get $E(M_t(\theta) \mid F_s) = X_s$, that completes the proof.

**Remark 1.** We emphasize the fact that Theorem 1 is valid for every process $\{X_t\}$ with $S$ containing a Gaussian component ($c > 0$). If the process $S$ has no Gaussian component ($c = 0$), the condition (3.1) is nevertheless fulfilled for a class of measures $F(.)$ that includes the stable processes $S$ with parameter $\alpha$ satisfying $1 < \alpha < 2$. Because of its importance, we consider this special case in Section 5.

4. The Laplace transform of $T_\mu(x)$. Now we are in a position to derive an explicit expression for the Laplace transform $\gamma_\mu(\theta,x)$. Due to the particular structure of
the martingale $M(t)$ we have the following result.

**Theorem 2.** Under the hypothesis $G$ the next equality holds:

$$
\gamma_\mu (\theta, x) = \frac{\int_0^{y_\lambda} \exp(xy + g(y)) \, dy}{\int_0^{y_\lambda} \exp(\mu y + g(y)) \, dy}, \quad \theta > 0.
$$

**Proof.** We put $T_\mu (x)\Delta t$ instead of $t$ in (3.3) and we make use of the well known martingale property that

$$
E M_{T_\mu (x)\Delta t} (\theta) = E M_\theta (\theta) = \int_0^{y_\lambda} \exp(xy + g(y)) \, dy.
$$

Next, one observes that

$$
0 \leq M_{T_\mu (x)\Delta t} (\theta) \leq \int_0^{y_\lambda} \exp(\mu y + g(y)) \, dy
$$

and, moreover, when $T_\mu (x) = \infty$ then

$$
0 \leq M_{T_\mu (x)\Delta t} (\theta) = M_\theta (\theta) \leq e^{-\theta t} \int_0^{y_\lambda} \exp(\mu y + g(y)) \, dy
$$

as well. Therefore,

$$
\lim_{t \to \infty} E M_{T_\mu (x)\Delta t} (\theta) = E M_{T_\mu (x)} (\theta) \cdot 1_{\{T_\mu (x) < \infty\}} = \int_0^{y_\lambda} \exp(\mu y + g(y)) \, dy, \quad \gamma_\mu (\theta, x).
$$

The right-hand sides of our equalities give directly (4.1).

**Remark 2.** For the validity of Theorem 2 we need not (and we did not use) any fact about the finiteness of $T_\mu (x)$. It is well known that $T_\mu (x) < \infty$ P-a.s. if and only if

$$
\lim_{y \to 0} g(y) = \infty
$$

or when

$$
\int_{y \to 0} F(dx) \cdot |x| < \infty.
$$

5. The case of stable process $S$ with parameter $1 < \alpha < 2$. Now we turn to the particular case when the following hypothesis is satisfied.

**Hypothesis $H_\alpha$.** Either $F(.) = 0$ and $c > 0$ (we characterize this by posing $\alpha = 2$),

or $c = 0$ and $F(dx) = \frac{\sigma \cdot dx}{|x|^{\alpha+1}} \cdot 1_{\{x < 0\}}$ for some $\sigma > 0$ and $1 < \alpha < 2$.

Using standard arguments (see [8], §25, Theorem 4) one obtains the equivalent form of $H_\alpha$ in the terms of our function $\psi : H_\alpha, 1 < \alpha < 2$, means that

$$
(5.1) \quad \psi(u) = \frac{\overline{\psi}(u)}{u} = E \psi + \overline{\psi} u^\alpha
$$
with some $\delta$, $-\infty < \delta < \infty$, and $\sigma > 0$. In this situation by (3.2) we get

$$g(y) = \bar{g}(y) = -\frac{\delta}{\lambda} (y - 1) - \frac{\sigma}{a\lambda} (y^a - 1),$$

and the martingale $M(t)$ is well defined via (3.3).

Following Novikov we introduce the function

$$H(\nu, \alpha, x) = \frac{1}{\Gamma(-a\nu)} \int_0^\infty y^{-a\nu - 1} \exp(xy - \frac{1}{\alpha} y^\alpha) \, dy,$$

which turns to be analytic in the half-plane $\text{Re} \, \nu < 1$. All the essential properties of $H(\nu, \alpha, x)$ are collected in the supplement of [6].

Next we obtain a special case of Theorem 2.

**Proposition 2.** Under the hypothesis $H_\alpha$, $1 < \alpha \leq 2$, the following equality holds for $\theta > 0$:

$$g(\theta, x) = H(-\theta, \alpha, x) = H(-\theta \alpha, \alpha, x)$$

Moreover, this formula defines also an analytical continuation of the Laplace transform $g(\theta, x)$ to the half-plane $\text{Re} \, \theta > -\alpha \nu_a(\mu)$, where $\nu_a(\mu) = (\frac{1}{\alpha}) \nu_a(\mu - \frac{\delta}{\lambda})$ and $\nu_a(\mu)$ is the smallest positive zero of $H(\nu, \alpha, \mu)$ with $(\alpha, \mu)$ fixed.

**Proof.** Applying the change of variables $y = (\frac{1}{\alpha}) \mu z$ we see the formula (5.3) is another form of (4.1) for $\theta > 0$. As far as the right-hand side of (5.3) is analytic in $\theta$ in the half-plane $\text{Re} \, \theta > -\nu_a(\mu)$ (see [6]), the left-hand side can be analytically continued in $\theta$ to this half-plane.

**Corollary 3.** Since $\lim_{\nu \to 0} H(\nu, \alpha, x) = 1$, $-\infty < x < \infty$, under the hypothesis $H_\alpha$ we get $\lim_{\nu \to 0} g(\theta, x) = 1$ and, consequently, $T(\nu, x) < \infty$ P-a.s.

6. The time change - two applications. Throughout this section we suppose the hypothesis $H_\alpha$ holds with some $\alpha$, $1 < \alpha \leq 2$, and $\delta = 0$ (see (5.1)). As a consequence we have

$$\psi(u) = \psi(u) = \frac{\sigma}{\alpha} u^\alpha, 1 < \alpha \leq 2,$$

and the process $X$ is stationary too (see (2.1)).
Let us introduce the real (increasing and continuous) function
\[ \delta(t) = (a\lambda)^{-1}(e^{a\lambda t} - 1), \quad t \geq 0, \]
which determines an one-to-one mapping of \([0,\infty)\) onto \([0,\infty)\), and the converse function
\[ \rho(t) = (a\lambda)^{-1}\ln(1 + a\lambda t), \quad t \geq 0. \]

**Lemma 1.** The distributions of \( S_t, t \geq 0, \) and of \( \frac{\rho(t)}{\rho'(t)} \int_0^t e^{\lambda t} dS_t, \quad t \geq 0, \) coincide.

**Proof.** As in Proposition 1 one calculates
\[ E \exp(\rho(t)) = E \exp(\rho'(t)) = \exp(\rho(t)) = \exp(\mu \alpha \delta(t)), \quad u > 0. \]
But under the hypothesis stated \((H_a \text{ and } E = 0)\) the latter term is just \( E \exp(\mu \alpha \delta(t)) \).

The lemma is proved.

Now for any constants \( a, b \) and \( c \) such that \( b \geq 0 \) and \( ab + c > 0 \), define the stopping time \( \tau(a,b,c) \) w.r.t. \( F_t, t \geq 0, \) by the formula
\[ \tau(a,b,c) = \inf \{t > 0 : t \leq a(t + b)a + c\} \]
and pose
\[ \tau(\mu)(x) = \tau(\mu(\alpha a), (a\lambda)^{-1}, -x), \quad \mu > x. \]

The following simple fact is valid in our situation.

**Theorem 3.** The stopping time \( \tau(\mu)(x) \) has the same distribution as \( \rho(\tau(\mu)(x)) \) does.

**Proof.** We define similarly \( \tau(a,b,c) \) and \( \tau(\mu)(x) \) by replacing \( S_t \) by \( S^t \) in (6.1) and (6.2). Next, starting with (1.6), we calculate
\[ \tau(\mu)(x) = \inf \{t : x + y_t \geq e^{\lambda t}\} \]
\[ = \inf \{s : y_s \geq e^{\lambda s} - x\} \]
\[ = \inf \{s : S_s \geq (1 + \lambda s)x\} = \rho(\tau(\mu)(x)). \]

The statement of the theorem follows from Lemma 1 which says the distribution of \( \tau(\mu)(x) \) coincides with the distribution of \( \tau(\mu)(x) \).

From Theorem 3 and Proposition 2 we deduce the following result of A.Novikov
Theorem 4. For every $a, b, c$ with $b > 0$, $ab^2 + c > 0$, one has

\begin{equation}
(6.3) \quad E (\tau(a, b, c) + b)^\nu = \frac{b^\nu H(\nu, a, -cb^{-\frac{1}{\alpha}}d)}{H(\nu, a, ad)} , \quad \text{if } b > 0 \text{ and } \nu < \nu_a(ad),
\end{equation}

and

\begin{equation}
(6.4) \quad E (\tau(a, b, c)^\nu) = \begin{cases} 
\frac{\nu^\nu}{H(\nu, a, ad)} & \text{if } \nu < \nu_a(ad), \\
+\infty & \text{if } \nu \geq \nu_a(ad),
\end{cases}
\end{equation}

where $d = (\frac{1}{ad})^{\frac{1}{\alpha}}$.

Proof. Assume $b > 0$ and put $x = -c$, $\lambda = (ab)^{-1}$, $\mu = ab^{-\frac{1}{\alpha}}$. Then

\begin{equation}
\mu - x = ab^2 + c > 0, \quad \bar{\mu} = (\frac{1}{a})^{\frac{1}{\alpha}} = ad
\end{equation}

and by Proposition 2 with $\nu = -\frac{\mu}{\alpha \lambda}$ we get the equalities

\begin{align*}
E (\tau(a, b, c) + b)^\nu &= E (\tau(x) + \frac{1}{\alpha \lambda})^\nu \\
 &= b^\nu E (\alpha \lambda \tau(x) + 1)^\nu = b^\nu \exp\{\nu \ln(1 + \alpha \lambda \tau(x))\} \\
 &= b^\nu \exp(-\theta(\tau(x))) = \frac{b^\nu H(\nu, a, -cb^{-\frac{1}{\alpha}}d)}{H(\nu, a, ad)}
\end{align*}

provided that $\theta > -\alpha \lambda \nu_a(ad)$ (or $\nu < \nu_a(ad)$). The rest statements of the theorem follow from the properties of $H(\nu, a, x)$, the case $b = 0$ being taken into account by letting $b \to 0$ (or $\lambda \to +\infty$).

Remark 3. In the original theorem of Novikov (with $d = 1$, see [6]) one makes use of the fact that

\begin{equation}
(t + b)^\nu . H(\nu, a, \frac{S_t - b^2}{1} ), \quad t \geq 0, \quad b > 0,
\end{equation}

is a complex-valued martingale (w.r.t. $F^S_t$, $t \geq 0$) for every complex $\nu$ with $\text{Re} \nu < 1$.

This fact involves an analytical continuation in contrast to our Theorem 1.

As a second example we consider an optimal stopping problem originally treated in more general setting in [5], [7] and [9]. This problem admits a simple solution in terms of stopping times $T_{\mu}(x)$.
Under the hypothesis stated at the beginning of this section \( H_a \) and \( \bar{b} = 0 \) the quantity

\[
(6.5) \quad v(x,b,\tau) = E \frac{x + S_T}{b + T}, \quad b > 0, \quad -\infty < x < \infty,
\]
is to be maximized on stopping times \( \tau = \tau(\omega) \) w.r.t. \( F^S_t, t \geq 0 \).

By Lemma 1 we have

\[
v(x,b,\tau) = v(x,b,\tau) = E \frac{x + S_T}{b + T},
\]
using \( S_T = Y, t \geq 0, \) and \( \tau \) in the place of \( S_T, t \geq 0, \) and \( \tau \). Now taking

\[
\lambda = \frac{1}{ab} \quad \text{and} \quad t = \delta(s), \quad s \geq 0,
\]
we get

\[
\frac{x + S_T}{b + T} = \frac{x + Y_{\rho(t)}}{b + t} = \frac{e^{\lambda_0(t)}X_{\rho(t)}}{e^{\lambda t}e^{\alpha_0 s}} = \alpha e^{-(a - 1)\lambda s}X_s.
\]

Consequently, it is equivalent to consider the problem of maximizing the quantity

\[
(6.6) \quad V(x,b,T) = \frac{1}{b} E e^{-sT}X_T, \quad s = \frac{a - 1}{ab} > 0,
\]
on stopping times \( T = T(\omega) \) w.r.t. \( F^X_s, s \geq 0, \) provided that \( T = \rho(\tau) \), because

\[
V(x,b,T) = v(x,\tau).
\]

By [7] for \( a = 2 \) and [5] for \( 1 < a < 2 \) one knows the solution of the original problem of maximizing (6.5) is one of the stopping times \( \tau(a,b,-x) \) or the stopping time \( \tau_0 = 0 \).

Let us denote

\[
\bar{\psi}(\mu) = \begin{cases} \int_{-\infty}^{0} \frac{y^{a-2} \exp(\mu y - \bar{b}y^a)}{0} dy, \quad \mu < \infty, \\ \int_{0}^{\infty} \frac{y^{a-1} \exp(\mu y - \bar{b}y^a)}{0} dy, \quad \mu > \infty. \end{cases}
\]

As far as \( \bar{\psi}(\mu) \) is positive, decreasing and continuous and \( \bar{\psi}(0) = \Gamma \left( \frac{a-1}{a} \right) > 0 \), the equation \( \mu = \bar{\psi}(\mu) \) has a unique solution \( \bar{\mu} \) (moreover, \( 0 < \bar{\mu} < \bar{\psi}(0) \)). The corresponding result in our case is given below without proof because it can be justified as in [5] and [7] (see also [9], Example 2, for the case \( a = 2 \) and \( \lambda = 1 \)).

Theorem 5. For every real \( x \) and \( b > 0 \), either the stopping time \( T_x(x) \), or the stopping time \( T_x(x) = 0 \) maximizes the quantity (6.6). More precisely,
\[
\sup_T V(x,b,T) = V(x,b,T_\bar{u}) = \frac{\bar{u}}{b} \psi(\bar{u},x) \quad \text{if} \quad x \leq \bar{u},
\]

and
\[
\sup_T V(x,b,T) = V(x,b,0) = \frac{x}{b} \quad \text{if} \quad x > \bar{u}.
\]

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References