

SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 19 (1985), p. 496-503

http://www.numdam.org/item?id=SPS_1985__19__496_0

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THE GAUGE AND CONDITIONAL GAUGE THEOREM

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Let $\{X_t, t \geq 0\}$ be the Brownian motion in R^d , $d \geq 1$. Let D be a bounded domain in R^d , \bar{D} its closure, ∂D its boundary; and let q be a Borel function defined in R^d and satisfying the following condition:

$$(1) \quad \lim_{t \downarrow 0} \sup_{x \in D} E^x \left\{ \int_0^t 1_D |q|(X_s) ds \right\} = 0$$

where 1_D is the indicator of D . Such a function is said to belong to the Kato class K_d . The equivalent condition (1) is given by Aizenman and Simon [1].

The *gauge* for (D, q) is defined to be the function u on \bar{D} below:

$$(2) \quad u(x) = E^x \left\{ \exp \left(\int_0^{\tau_D} q(X_s) ds \right) \right\} .$$

From here on we write for abbreviation:

$$(3) \quad e_q(t) = \exp \left(\int_0^t q(X_s) ds \right) .$$

For a domain D with $m(D) < \infty$ (where m denotes the Lebesgue measure), without any regularity hypothesis on ∂D , and a bounded q , we proved the following theorem in [3].

The Gauge Theorem. *If $u(\cdot) \neq \infty$ in D , then u is bounded in \bar{D} .*

*Research supported in part by NSF grant MCS 83-01072 at Stanford University.

Actually, if ∂D is regular in the sense of the Dirichlet problem, then u is continuous and strictly positive in \bar{D} . However, in this note we shall concentrate on the main thing, as stated above. Zhao [6] extended the theorem to $q \in K_d$ for a bounded domain in R^d , $d \geq 3$; he also did the case $d = 2$ in yet unpublished notes. For $d = 1$ and D a half-line, see [2]. Prior to Zhao's work, Falkner extended the theorem in another direction by considering the *conditional gauge* for (D, q) as follows:

$$(4) \quad u(x, z) = E_z^x \{ e_q(\tau_D) \}, \quad (x, z) \in D \times \partial D;$$

where E_z^x is the expectation associated with the Brownian motion killed outside D , starting from x and conditioned to converge to z (at its life-time τ_D). For a class of bounded domains including those with C^2 boundary, and bounded q , he proved the following theorem in [5].

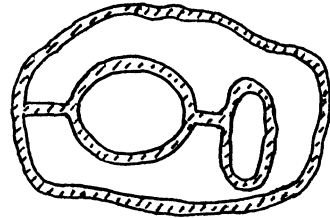
Conditional Gauge Theorem. *If $u(\cdot, \cdot) \not\equiv \infty$ in $D \times \partial D$, then it is bounded there. This is the case if and only if $u(\cdot) \not\equiv \infty$ in D , as in the gauge theorem.*

I gave a simpler proof of Falkner's theorem in [4]. Subsequently Zhao [7] proved that if $u(\cdot) \not\equiv \infty$, then $u(\cdot, \cdot) \not\equiv \infty$, for bounded C^2 domains. He has since proved the conditional gauge theorem as stated above for bounded $C^{1,1}$ domains. In this note I shall show that the conditional gauge theorem actually follows in a general way and rather quickly from the gauge theorem.

The basic probabilistic argument turns out to be an old one in [2] (see the proof of Theorem 1 there), easily adapted to the multi-dimensional case. The sole difficulty encountered in extending the class of bounded q to the class K_d is contained in Lemma 1 below.

We begin by setting up the framework of the probabilistic argument involving a sequence of hitting times. Let D_1 and D_2 be subdomains of D such that $\bar{D}_1 \subset D_2$, $\bar{D}_2 \subset D$, and $C \triangleq D - \bar{D}_1$ is connected and $m(C) < \varepsilon$ for an arbitrary $\varepsilon > 0$. This is possible if ∂D is Lipschitzian for instance. For then each connected component of $R^d - \bar{D}$ must contain a ball of fixed size, hence there are at most a finite number of

"holes" inside the outer boundary of D . Since D is connected, it is easy to see how to construct D_1 and D_2 as desired. A picture illustrates the result. I am indebted to Falkner for alerting me to the necessity of making C connected.



(The shaded portion represents C .)

Lemma 1. *If ∂D is sufficiently smooth, then for any given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that if the C described above has $m(C) \leq \delta(\varepsilon)$, then*

$$(5) \quad \sup_{\substack{x \in C \\ z \in \partial D}} E_z^x \left\{ \int_0^{\tau_C} |q|(X_t) dt \right\} \leq \varepsilon ;$$

$$(6) \quad \sup_{\substack{x \in C \\ z \in \partial D}} E_z^x \{ e_q(\tau_C) \} \leq \frac{1}{1-\varepsilon} .$$

In [7], Zhao proved that C^2 boundary is sufficient for the lemma to hold; more recently he has improved this result to require only $C^{1,1}$ boundary. In this connection it should be mentioned that the gauge theorem for an arbitrary bounded domain D , and $q \in K_D$, follows quickly from an easier analogue of (5) for a small ball B , as follows:

$$(7) \quad \sup_{\substack{x \in B \\ z \in \partial B}} E_z^x \left\{ \int_0^{\tau_B} |q|(X_t) dt \right\} \leq \varepsilon .$$

This was proved in Zhao [6]. The deduction of (6) from (5) is standard Markovian calculation.

Lemma 2 is a strengthened form of an argument I have indicated elsewhere (see [5], Remark 2.13). The constants a_1, a_2, \dots below are strictly positive, depending only on D_1, D_2 and D . We assume ∂D to be Lipschitzian below.

Lemma 2. For all $y \in \partial D_2$ and $z \in \partial D$, we have

$$(8) \quad a_1 \leq E_z^y \{ \tau_C = \tau_D; e_q(\tau_D) \} \leq a_2 .$$

Proof: Recall that

$$(9) \quad P_z^Y\{\tau_C < \tau_D\} = \frac{f(y, z)}{K(y, z)}$$

where K is the Poisson kernel for D , and

$$f(y, z) = E^Y\{\tau_C < \tau_D; K(X(\tau_C), z)\} .$$

For each $y \in \partial D_2$, $f(y, \cdot)$ is continuous on ∂D , because on $\{\tau_C < \tau_D\}$ we have $X(\tau_C) \in \partial D_1$ almost surely, and K is bounded continuous in $\partial D_1 \times \partial D$. For each $z \in \partial D$, $f(\cdot, z)$ is harmonic in C . Hence f is continuous on $\partial D_2 \times \partial D$ since ∂D_2 and ∂D are disjoint closed sets. It follows that the function of (y, z) in (9) is continuous and positive on $\partial D_2 \times \partial D$. The function $K(\cdot, z) - f(\cdot, z)$ is harmonic in C and unbounded in the neighborhood of z , because K is unbounded while f is bounded. Hence it is strictly positive in C by harmonicity, because C is connected and $z \in \partial C$. Therefore we have by continuity

$$(10) \quad b \triangleq \inf_{\substack{y \in \partial D_2 \\ z \in \partial D}} P_z^Y\{\tau_C = \tau_D\} > 0 .$$

Now it follows by Jensen's inequality and (15) that for $(y, z) \in \partial D_2 \times \partial D$:

$$(11) \quad E_z^Y\{e_q(\tau_D) \mid \tau_C = \tau_D\} \geq E_z^Y\{e_{-|q|}(\tau_D) \mid \tau_C = \tau_D\} \\ \geq \exp\{-E_z^Y[\int_0^{\tau_D} |q|(X_t) dt \mid \tau_C = \tau_D]\} \\ \geq \exp\{-\frac{1}{b} \int_0^{\tau_C} |q|(X_t) dt\} \geq e^{-\varepsilon/b} .$$

Combining (10), (11) and (16), we have proved (8) with $a_1 = b e^{-\varepsilon/b}$,
 $a_2 = \frac{1}{1-\varepsilon}$.

We are ready to prove the conditional gauge theorem for a bounded Lipschitzian domain for which the conclusions of Lemma 1 hold true, thus at least when ∂D belongs to $C^{1,1}$. Put $T_0 \equiv 0$, and for $n \geq 1$:

$$T_{2n-1} = T_{2n-2} + \tau_{D_2} \circ \theta_{T_{2n-2}},$$

$$T_{2n} = T_{2n-1} + \tau_C \circ \theta_{T_{2n-1}}.$$

For any $(x, z) \in D \times \partial D$, we have $P_z^X\{\tau_D < \infty\} = 1$. This nontrivial result has recently been proved by M. Cranston for a bounded Lipschitzian domain; for a bounded C^1 -domain D it follows from the fact that $K(\cdot, z)$ is integrable over D , by a remark communicated to me by Kenig. It follows that for some $n \geq 1$, $X(T_{2n}) \in \partial D$. Therefore we have by the strong Markov property of the conditioned process:

$$\begin{aligned} (12) \quad E_z^X\{e_q(\tau_D)\} &= \sum_{n=1}^{\infty} E_z^X\{T_{2n} = \tau_D; e_q(\tau_D)\} \\ &= \sum_{n=1}^{\infty} E_z^X\{T_{2n-2} < \tau_D; e_q(T_{2n-1}) E_z^{X(T_{2n-1})}[\tau_C = \tau_D; e_q(\tau_D)]\}. \end{aligned}$$

Observe that $\partial C = \partial D_1 \cup \partial D$. On the set $\{T_{2n-2} < \tau_D\}$, $X(T_{2n-1}) \in \partial D_2$. Hence by Lemma 2

$$\begin{aligned} (13) \quad a_1 \sum_{n=1}^{\infty} E_z^X\{T_{2n-2} < \tau_D; e_q(T_{2n-1})\} &\leq E_z^X\{e_q(\tau_D)\} \\ &\leq a_2 \sum_{n=1}^{\infty} E_z^X\{T_{2n-2} < \tau_D; e_q(T_{2n-1})\}. \end{aligned}$$

The general term in the series above is explicitly:

$$(14) \quad \frac{1}{K(x,z)} E^x \{ T_{2n-2} < \tau_D; e_q^{(T_{2n-1})K(X(T_{2n-1}),z)} \} .$$

Since K is continuous and strictly positive on $\bar{D}_2 \times \partial D$, we have for (x,z) and (x',z') in $\bar{D}_2 \times \partial D$, almost surely

$$(15) \quad a_3 \frac{K(X(T_{2n-1}),z')}{K(x',z')} \leq \frac{K(X(T_{2n-1}),z)}{K(x,z)} \leq a_4 \frac{K(X(T_{2n-1}),z')}{K(x',z')}$$

where a_3 and a_4 depend only on \bar{D}_2 and D . It follows from (13), (14) and (15) that

$$(16) \quad \sup_{x \in \bar{D}_2} \sup_{z \in \partial D} u(x,z) \leq \frac{a_2 a_4}{a_1 a_3} \inf_{x \in \bar{D}_2} \inf_{z \in \partial D} u(x,z) .$$

Since $u(x)$ is a probability average of $u(x,z)$ over $z \in \partial D$, we have

$$(17) \quad \inf_{z \in \partial D} u(x,z) \leq u(x) \leq \sup_{z \in \partial D} u(x,z) .$$

Now by hypothesis of the theorem, there exists $(x_0, z_0) \in D \times \partial D$ such that $u(x_0, z_0) < \infty$. Without loss of generality we may suppose $x_0 \in D_2$. Hence by (16)

$$(18) \quad \sup_{z \in \partial D} u(x_0, z) \leq \frac{a_2 a_4}{a_1 a_3} u(x_0, z_0) < \infty .$$

Next by (17), $u(x_0) < \infty$. Hence by the gauge theorem, $\sup_{x \in \bar{D}} u(x) < \infty$.

It follows then by (16) and (17) that

$$(19) \quad \sup_{x \in \bar{D}_2} \sup_{z \in \partial D} u(x,z) < \infty .$$

For $x \in D - \bar{D}_2$, we use the old argument in [3] adapted to the conditioned process, as follows:

$$\begin{aligned} u(x, z) &= E_z^x \{ \tau_C = \tau_D; e_q(\tau_C) \} + E_z^x \{ \tau_C < \tau_D; u(X(\tau_C), z) \} \\ &\leq \frac{1}{1-\epsilon} + \sup_{x \in \bar{D}_1} \sup_{z \in \partial D} u(x, z) < \infty . \end{aligned}$$

This establishes the first assertion of the conditional gauge theorem. The second assertion has also been proved between the lines above.

Remark : Conditional gauge theorem is also true for a bounded C^1 domain, and bounded q , using a hard inequality of Kenig's to prove lemma 1.

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