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# Multiple stochastic integrals. A counter example

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# Multiple Stochastic Integrals -- A Counter Example

by

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In this note we give an example of a continuous square integrable martingale M such that $d\langle M,M\rangle_t \ll dt$ (in fact M is an Itô integral) but for which the multiple stochastic integral

$$\iint_{\{0<s<t<\infty\}} f_{st}\, dM_s\, dM_t$$

does not exist as an $L^0$-integrator on the space of bounded predictable integrands.

We work on a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, satisfying the usual conditions. Let

$$C_2 = \{(s,t)\,|\,0<s<t<\infty\}.$$

A simple predictable set is a set of the form

$$\{(s,t,\omega)\in C_2\times\Omega\,|\; S(\omega)<s\leqq T(\omega) \text{ and } U(\omega)<t\leqq V(\omega)\}$$

where S and T are stopping times, and U and V are non-negative $\mathcal{F}_S$-measurable random variables such that $T(\omega)\leqq U(\omega)$ for all $\omega$. A simple predictable process is a linear combination of indicator functions of simple predictable sets.

<u>Definition. The predictable $\sigma$-field, $\mathcal{P}$, on $C_2\times\Omega$ is the $\sigma$-field generated by the elementary predictable sets.</u>

Note that, according to these definitions, a simple predictable process is not the same as "un processus prévisible simple", as defined in (3). The definition of $\mathcal{P}$, however, does agree with that in (2,3), as the reader can easily check.

If M is a square integrable martingale, then $\iint_{C_2} f(s,t)\,dM_s\,dM_t$ may be defined for simple predictable processes in the obvious way. Meyer (2) showed that if $\langle M,M\rangle_t = t$ then this multiple stochastic integral extends uniquely as an $L^2$-integrator to

$$\{f(s,t,\omega)\,|\; f \text{ predictable}, \; E(\iint_{C_2} f^2(s,t,\omega)\,ds\,dt)<\infty\}.$$

This result was extended by Ruiz de Chavez (3) to the case when

(1) $$\langle M,M\rangle_t - \langle M,M\rangle_s \leq m(t) - m(s) \quad \text{for all } 0 \leq s < t$$

for some deterministic m.

At first glance it seems that, at least if M is continuous, there should be no problem in defining $\iint_{C_2} f(s,t)dM_s dM_t$ by the iterated integral

$$\int_0^\infty \left(\int_0^t f(s,t)dM_s\right) dM_t.$$

The above integrand, however, is only uniquely defined for each t up to a null set and one is therefore faced with the task of selecting a predictable version of this process. This done in (2) when $\langle M,M\rangle_t = t$. It is natural to ask if a condition like (1) is really needed to handle this measurability problem and extend the multiple stochastic integral as an $L^0$ integrator to the bounded predictable process. The following examples show that the answer is "yes" even if M is an Itô integral. Although we have tried to disguise it by working with a Brownian filtration, the discerning reader will note that this example is closely related to ( and was inspired by) an example of a martingale measure that is not a stochastic integrator due to Bakry (1).

Assume $B_t$ is an $\mathcal{F}_t$-Brownian motion. Choose $\alpha\varepsilon(0,1/2)$ and $\delta_p \downarrow 0$ such that $\sum_{p=1}^\infty \delta_p^2 < \infty$ but $\sum_{p=1}^\infty \delta_p p^{-\alpha} = \infty$. Define a sequence of stopping times $\{T_p\}$ by

$$T_0 = 0$$
$$T_p = \inf \{t > T_{p-1} \mid |B_t - B_{T_{p-1}}| = \delta_p\}.$$

Then $T_p \uparrow T_\infty$, and since $T_\infty = \Sigma_{p=1}^\infty \delta_p^2 S_p$ where $\{S_p\}$ are i.i.d. copies of $\inf\{t \mid |B_t| = 1\}$, it is easy to see that $T_\infty \varepsilon L^q, \forall q > 0$. Define a random variable, U, uniformly distributed on (0,1), by

$$U = \sum_{p=1}^\infty I(B(T_p) < B(T_{p-1}))2^{-p},$$

and a sequence of Bernoulli random variables by

$$e_p(U) = \begin{cases} 0 & \text{if } B(T_p) > B(T_{p-1}) \\ 1 & \text{if } B(T_p) < B(T_{p-1}) \end{cases}.$$

In addition let $U_n(U) = \Sigma_{p=1}^{n} e_p(U)2^{-p}$, $V_n = U_n+T_\infty$, $V = U+T_\infty$ and choose $f(t) \geq 0$ such that

$$\int_0^t f^2(s)ds = (\log \tfrac{1}{t})^{-\alpha} \equiv \Phi(t), \quad 0 \leq t \leq 1/2$$

Our continuous martingale is

$$M_t = \int_0^t \left(I_{(0,T_\infty)}(s) + I_{(V,V+1/2)} f(s-V)\right)dB_s.$$

Then $\langle M,M\rangle_\infty \varepsilon L^q \ \forall q > 0$. If

$$H_n = \bigcup_{p=1}^{n} \{(s,t,\omega) \mid T_{p-1}(\omega) < s \leq T_p(\omega),\ V_{p-1}(\omega) < t \leq V_{p-1}(\omega) + 2^{-p}\},$$

then $I_{H_n}$ is a simple predictable process and $I_{H_n} \uparrow I_H$ as $n \to \infty$, where $H \varepsilon \mathcal{P}$. We claim, however, that $\iint_{C_2} I_{H_n} dM_s dM_t$ does not converge in probability. Note that

$$M(V_{p-1}+2^{-p}) - M(V_{p-1}) = I(e_p = 0) \int_V^{V_{p-1}+2^{-p}} f(s-V)dB_s,$$

so that

$$\iint_{C_2} I_{H_n}(s,t)dM_s dM_t = \sum_{p=1}^{n} \left(M(T_p)-M(T_{p-1})\right)\left(M(V_{p-1}+2^{-p})-M(V_{p-1})\right)$$

(2)
$$= \sum_{p=1}^{n} \delta_p I\left(e_p(U)=0\right) \int_0^\infty I\left(s \leq U_{p-1}(U)+2^{-p}-U\right) f(s)d\tilde{B}_s,$$

where $\tilde{B}_s = B(V+s) - B(V)$ is a Brownian motion independent of $\mathcal{F}_V$. Conditional on $U = u$, (2) has a mean zero normal distribution with variance

$$\sigma_n^2(u) = \sum_{p=1}^{n} \delta_p^2 \, I\left(e_p(u) = 0\right) \, \Phi\left(U_{p-1}(u)+2^{-p}-u\right)$$

$$+2 \sum\sum_{1 \leq p<q \leq n} \delta_p\delta_q \, I\left(e_p(u) = e_q(u) = 0\right) \, \Phi\left(U_{q-1}(u)+2^{-q}-u\right).$$

Therefore

(3) $$E\left(\exp\{i\lambda \iint_{C_2} I_{H_n}(s,t)dM_s dM_t\}\right) = \int_0^1 \exp\{-\lambda^2 \sigma_n^2(u)/2\}du.$$

We claim that

(4) $$\lim_{n\to\infty} \sigma_n^2(u) = \infty \text{ for Lebesgue - a.a.u.}$$

Fix $p\in$ . Then

$$\sum_{p<q\leq n} \delta_q I(e_q(u) = 0)\left[\Phi(U_{q-1}(u)+2^{-q}-u) - 2^q \int_0^{2^{-q}} \Phi(s)ds\right]$$

$$= \sum_{p<q\leq n} \delta_q I(e_q(u) = 0)\left[\Phi(U_q(u)+2^{-q}-u) - 2^q \int_0^{2^{-q}} \Phi(s)ds\right]$$

$\xrightarrow{a.s.}$ as $n\to\infty$ (w.r.t. Lebesgue measure on $[0,1]$),

by the martingale convergence theorem, as the conditional distribution of $U_q(u) + 2^{-q}-u$ given $\sigma(U_r(u) \mid r\leq q)$ is uniform on $[0,2^{-q}]$. As $e_p(u)=0$ for infinitely many p a.s. $[du]$, (4) will follow if for each p

(5) $$\lim_{n\to\infty} \sum_{p<q\leq n} \delta_q I(e_q(u) = 0)2^q \int_0^{2^{-q}} \Phi(s)ds = \infty \quad \text{a.s.}[du].$$

The above expression is bounded below by

$$\lim_{n\to\infty} \sum_{p<q\leq n} \delta_q I(e_q(u) = 0)\, \Phi(2^{-q-1}) \quad (\Phi \text{ is concave})$$

$$\geq \lim_{n\to\infty} c \sum_{p<q\leq n} \delta_q q^{-\alpha} I(e_q(u) = 0) = \infty \quad \text{a.s.} \quad [du].$$

The last by the choice of $\{\delta_q\}$. This proves (5) and hence (4). (3) and (4) together show

$$\lim_{n\to\infty} E\left(\exp\{i\lambda\iint_{C_2} I_{H_n}(s,t)dM_s dM_t\}\right) = I(\lambda=0)$$

so that $\iint_{C_2} I_{H_n}(s,t)dM_s dM_t$ cannot converge in distribution as $n\to\infty$, as required.

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## References

(1) D. Bakry, Une remarque sur les semimartingales à deux indices, Séminaire de Probabilités XV, Lect. Notes 850, p.671-672 (1981).

(2) P.A. Meyer, Un cours sur les intégrales stochastiques, appendice au chapitre IV, Séminaire de Probabilités X, Lect. Notes 511, p. 321-331 (1976).

(3) J. Ruiz de Chavez, Sur les intégrales stochastiques multiples, ce volume, p. 248

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