

SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 19 (1985), p. 258-262

[<http://www.numdam.org/item?id=SPS_1985__19__258_0>](http://www.numdam.org/item?id=SPS_1985__19__258_0)

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Multiple Stochastic Integrals -- A Counter Example

by

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In this note we give an example of a continuous square integrable martingale M such that $d\langle M, M \rangle_t \ll dt$ (in fact M is an Itô integral) but for which the multiple stochastic integral

$$\iint_{\{0 < s < t < \infty\}} f_{st} dM_s dM_t$$

does not exist as an L^0 -integrator on the space of bounded predictable integrands.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, satisfying the usual conditions. Let

$$C_2 = \{(s, t) | 0 < s < t < \infty\}.$$

A simple predictable set is a set of the form

$$\{(s, t, \omega) \in C_2 \times \Omega | S(\omega) < s \leq T(\omega) \text{ and } U(\omega) < t \leq V(\omega)\}$$

where S and T are stopping times, and U and V are non-negative \mathcal{F}_s -measurable random variables such that $T(\omega) \leq U(\omega)$ for all ω . A simple predictable process is a linear combination of indicator functions of simple predictable sets.

Definition. The predictable σ -field, \mathcal{P} , on $C_2 \times \Omega$ is the σ -field generated by the elementary predictable sets.

Note that, according to these definitions, a simple predictable process is not the same as "un processus prévisible simple", as defined in (3). The definition of \mathcal{P} , however, does agree with that in (2,3), as the reader can easily check.

If M is a square integrable martingale, then $\iint_{C_2} f(s, t) dM_s dM_t$

may be defined for simple predictable processes in the obvious way.

Meyer (2) showed that if $\langle M, M \rangle_t = t$ then this multiple stochastic integral extends uniquely as an L^2 -integrator to

$$\{f(s, t, \omega) | f \text{ predictable, } E(\iint_{C_2} f^2(s, t, \omega) ds dt) < \infty\}.$$

This result was extended by Ruiz de Chavez (3) to the case when

$$(1) \quad \langle M, M \rangle_t - \langle M, M \rangle_s \leq m(t) - m(s) \quad \text{for all } 0 \leq s < t$$

for some deterministic m .

At first glance it seems that, at least if M is continuous, there should be no problem in defining $\int_0^t \int_0^s f(s, t) dM_s dM_t$ by the iterated integral

$$\int_0^t \left(\int_0^s f(s, t) dM_s \right) dM_t.$$

The above integrand, however, is only uniquely defined for each t up to a null set and one is therefore faced with the task of selecting a predictable version of this process. This done in (2) when $\langle M, M \rangle_t = t$. It is natural to ask if a condition like (1) is really needed to handle this measurability problem and extend the multiple stochastic integral as an L^0 integrator to the bounded predictable process. The following examples show that the answer is "yes" even if M is an Itô integral. Although we have tried to disguise it by working with a Brownian filtration, the discerning reader will note that this example is closely related to (and was inspired by) an example of a martingale measure that is not a stochastic integrator due to Bakry (1).

Assume B_t is an \mathcal{F}_t -Brownian motion. Choose $\alpha \in (0, 1/2)$ and $\delta_p \downarrow 0$

such that $\sum_{p=1}^{\infty} \delta_p^2 < \infty$ but $\sum_{p=1}^{\infty} \delta_p p^{-\alpha} = \infty$. Define a sequence of stopping times $\{T_p\}$ by

$$\begin{aligned} T_0 &= 0 \\ T_p &= \inf \{t > T_{p-1} \mid |B_t - B_{T_{p-1}}| = \delta_p\}. \end{aligned}$$

Then $T_p \uparrow T_{\infty}$, and since $T_{\infty} = \sum_{p=1}^{\infty} \delta_p^2 S_p$ where $\{S_p\}$ are i.i.d. copies of $\inf\{t \mid |B_t| = 1\}$, it is easy to see that $T_{\infty} \in L^q, \forall q > 0$. Define a random variable, U , uniformly distributed on $(0, 1)$, by

$$U = \sum_{p=1}^{\infty} I(B(T_p) < B(T_{p-1})) 2^{-p},$$

and a sequence of Bernoulli random variables by

$$e_p(U) = \begin{cases} 0 & \text{if } B(T_p) > B(T_{p-1}) \\ 1 & \text{if } B(T_p) < B(T_{p-1}) \end{cases}.$$

In addition let $U_n(U) = \sum_{p=1}^n e_p(U) 2^{-p}$, $V_n = U_n + T_\infty$, $V = U + T_\infty$ and choose $f(t) \geq 0$ such that

$$\int_0^t f^2(s) ds = (\log \frac{1}{t})^{-\alpha} \equiv \phi(t), \quad 0 \leq t \leq 1/2$$

Our continuous martingale is

$$M_t = \int_0^t (I_{(0, T_\infty)}(s) + I_{(V, V+1/2)} f(s-V)) dB_s.$$

Then $\langle M, M \rangle_\infty \in L^q \quad \forall q > 0$. If

$$H_n = \bigcup_{p=1}^n \{(s, t, \omega) \mid T_{p-1}(\omega) < s \leq T_p(\omega), V_{p-1}(\omega) < t \leq V_{p-1}(\omega) + 2^{-p}\},$$

then I_{H_n} is a simple predictable process and $I_{H_n} \uparrow I_H$ as $n \rightarrow \infty$, where $H \in \mathcal{P}$. We claim, however, that $\iint_{C_2} I_{H_n} dM_s dM_t$ does not converge in probability. Note that

$$M(V_{p-1} + 2^{-p}) - M(V_{p-1}) = I(e_p = 0) \int_{V_{p-1}}^{V_{p-1} + 2^{-p}} f(s-V) dB_s,$$

so that

$$\begin{aligned} \iint_{C_2} I_{H_n}(s, t) dM_s dM_t &= \sum_{p=1}^n (M(T_p) - M(T_{p-1})) (M(V_{p-1} + 2^{-p}) - M(V_{p-1})) \\ (2) \quad &= \sum_{p=1}^n \delta_p I(e_p(U) = 0) \int_0^\infty I(s \leq U_{p-1}(U) + 2^{-p} - U) f(s) d\tilde{B}_s, \end{aligned}$$

where $\tilde{B}_s = B(V+s) - B(V)$ is a Brownian motion independent of \mathcal{F}_V . Conditional on $U = u$, (2) has a mean zero normal distribution with variance

$$\begin{aligned} \sigma_n^2(u) &= \sum_{p=1}^n \delta_p^2 I(e_p(u) = 0) \phi(U_{p-1}(u) + 2^{-p} - u) \\ &\quad + 2 \sum_{1 \leq p < q \leq n} \delta_p \delta_q I(e_p(u) = e_q(u) = 0) \phi(U_{q-1}(u) + 2^{-q} - u). \end{aligned}$$

Therefore

$$(3) \quad E \left\{ \exp \left\{ i \lambda \int_0^2 \int_{H_n} I_{H_n}(s, t) dM_s dM_t \right\} \right\} = \int_0^1 \exp \left\{ -\lambda^2 \sigma_n^2(u)/2 \right\} du.$$

We claim that

$$(4) \quad \lim_{n \rightarrow \infty} \sigma_n^2(u) = \infty \text{ for Lebesgue - a.a.u.}$$

Fix $p \in \mathbb{N}$. Then

$$\begin{aligned} & \sum_{p < q \leq n} \delta_q I(e_q(u) = 0) \left[\Phi(U_{q-1}(u) + 2^{-q} - u) - 2^q \int_0^{2^{-q}} \phi(s) ds \right] \\ &= \sum_{p < q \leq n} \delta_q I(e_q(u) = 0) \left[\Phi(U_q(u) + 2^{-q} - u) - 2^q \int_0^{2^{-q}} \phi(s) ds \right] \end{aligned}$$

$$\xrightarrow{\text{a.s.}} \text{ as } n \rightarrow \infty \quad (\text{w.r.t. Lebesgue measure on } [0, 1]),$$

by the martingale convergence theorem, as the conditional distribution of $U_q(u) + 2^{-q} - u$ given $\sigma(U_r(u) | r \leq q)$ is uniform on $[0, 2^{-q}]$. As $e_p(u) = 0$ for infinitely many p a.s. [du], (4) will follow if for each p

$$(5) \quad \lim_{n \rightarrow \infty} \sum_{p < q \leq n} \delta_q I(e_q(u) = 0) 2^q \int_0^{2^{-q}} \phi(s) ds = \infty \quad \text{a.s. [du]}.$$

The above expression is bounded below by

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{p < q \leq n} \delta_q I(e_q(u) = 0) \Phi(2^{-q-1}) \quad (\Phi \text{ is concave}) \\ & \geq \lim_{n \rightarrow \infty} c \sum_{p < q \leq n} \delta_q q^{-\alpha} I(e_q(u) = 0) = \infty \quad \text{a.s. [du]}. \end{aligned}$$

The last by the choice of $\{\delta_q\}$. This proves (5) and hence (4). (3) and (4) together show

$$\lim_{n \rightarrow \infty} E \left\{ \exp \left\{ i \lambda \int_0^2 \int_{H_n} I_{H_n}(s, t) dM_s dM_t \right\} \right\} = I(\lambda=0)$$

so that $\int_0^2 \int_{H_n} I_{H_n}(s, t) dM_s dM_t$ cannot converge in distribution as $n \rightarrow \infty$, as required.

Acknowledgement. I would like to thank P.A. Meyer for posing this problem and D. Bakry for pointing out his related work (1).

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