SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

EDWIN A. PERKINS

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Séminaire de probabilités (Strasbourg), tome 17 (1983), p. 67-71

http://www.numdam.org/item?id=SPS_1983__17__67_0

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STOCHASTIC INTEGRALS AND PROGRESSIVE

MEASURABILITY -- AN EXAMPLE

by

Edwin Perkins

In this note we construct a measurable set $D \subset [0, \infty) \times \Omega$, a 3-dimensional Bessel process, X , and a filtration, $\{F_t^B\}$, containing the canonical filtration, $\{F_t^X\}$, of X satisfying the following properties:

- (i) X is an $\{F_t^B\}$ - semimartingale.
- (ii) D is an $\{F_t^X\}$ - progressively measurable set, i.e.,
 $D \cap [[0, t]] \in \text{Borel}([0, t]) \times F_t^X$ for all $t \geq 0$.
- (iii) $\int_0^t I_D dX = X(t)$, where the left side is interpreted with respect to $\{F_t^X\}$, and I_D denotes the indicator function of D .
- (iv) $\int_0^t I_D dX$ is an $\{F_t^B\}$ - Brownian motion when the stochastic integral is taken with respect to $\{F_t^B\}$.

As the local martingale part of X with respect to either filtration will be a Brownian motion (since $[X](t) = t$), $\int_0^t I_D dX$ may be defined in the obvious way even though D will not be predictable.

Let B be a 1-dimensional Brownian motion on a complete $(\Omega, \mathcal{F}, P)$. If $M(t) = \sup_{s \leq t} B(s)$, $Y = M - B$ and $X = 2M - B$, then Y is a reflecting Brownian motion, and X is a 3-dimensional Bessel process by a theorem of Pitman [4]. $\{F_t^X\}$, respectively $\{F_t^B\}$, will denote the smallest filtration, satisfying the usual conditions, that makes X , respectively B , adapted. $F_t^X \subseteq F_t^B$ is clear, and since $M(t) = \inf_{s \geq t} X(s)$, the inf being assumed at the next zero of Y , we must have $F_t^X \subsetneq F_t^B$ for $t > 0$, as $M(t)$ cannot be F_t^X - measurable. Finally, define

$$D = \{(t, \omega) \mid \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n I(X(t+2^{-k}) - X(t+2^{-k-1}) > 0) = 1/2\}.$$

Property (i) is immediate and for (ii), fix $t \geq 0$ and note that

$$D \cap [[0, t]] = (\{t\} \times D(t)) \cup \bigcup_{N=1}^{\infty} \{(s, \omega) \mid s \leq t - 2^{-N}\},$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=N}^{\infty} I(X(s+2^{-k}) - X(s+2^{-k-1}) > 0) = 1/2 \in \text{Borel}([0, t]) \times F_t^X.$$

Here $D(t)$ is the t -section of D . To show (iii) choose $t > 0$ and note that

$$X(t+2^{-k}) - X(t+2^{-k-1}) = B(t+2^{-k-1}) - B(t+2^{-k}) \text{ for large } k \text{ a.s.}$$

Therefore the law of large numbers implies that

$$(1) \quad P((t, \omega) \in D) = 1 \text{ for all } t > 0.$$

The canonical decomposition of X with respect to $\{F_t^X\}$ is (see McKean [3])

$$(2) \quad X(t) = W(t) + \int_0^t X(s)^{-1} ds,$$

where W is an $\{F_t^X\}$ -Brownian motion. Therefore with respect to $\{F_t^X\}$ we have

$$\int_0^t I_D dX = \int_0^t I_D dW + \int_0^t I_D X_s^{-1} ds = X(t) \text{ a.s. (by (1))}.$$

It remains only to prove (iv). If

$$T(t) = \inf\{s \mid M(s) > t\},$$

we claim that

$$(3) \quad P((T(t), \omega) \in D) = 0 \text{ for all } t \geq 0.$$

Choose $t \geq 0$ and assume $P((T(t), \omega) \in D) > 0$. Since $X(T(t) + \cdot) - X(T(t))$ is equal in law to $X(\cdot)$, the 0-1 law implies that $P((T(t), \omega) \in D) = 1$. The dominated convergence theorem and Brownian scaling imply

$$\begin{aligned}
1/2 &= n^{-1} \sum_{k=1}^n P(X(2^{-k}) - X(2^{-k-1}) > 0) \\
&= P(X(2) - X(1) > 0) \\
&= P(B(2) - B(1) < 2(M(2) - M(1))) \\
&> 1/2 .
\end{aligned}$$

Therefore (3) holds and, with respect to $\{F_t^B\}$, we have w.p.1

$$\begin{aligned}
\int_0^t I_D \, dX &= 2 \int_0^t I_D \, dM - \int_0^t I_D \, dB \\
&= 2 \int_0^t I_D(T(s), \omega) ds - B(t) \quad (\text{by (1)}) \\
&= -B(t) \quad (\text{by (3)})
\end{aligned}$$

This completes the proof.

It is not hard to see that the above result implies that the optional projections of I_D with respect to $\{F_t^X\}$ and $\{F_t^B\}$ are distinct. In particular D cannot be $\{F_t^X\}$ -optional. In fact, D is not $\{F_t^B\}$ -optional and both optional projections may be computed explicitly.

Proposition (a) The optional projection of I_D with respect to $\{F_t^X\}$ is $I_{(0, \infty) \times \Omega}$.

(b) The optional projection of I_D with respect to $\{F_t^B\}$ is I_{Z^c} where Z is the zero-set of Y .

(c) D is not $\{F_t^B\}$ -optional.

Proof (a) Let $\infty \geq T \geq \varepsilon > 0$ be an $\{F_t^X\}$ stopping time. The law of large numbers implies that

$$(4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n I(W(T+2^{-k}) - W(T+2^{-k-1}) > 0) = 1/2 \quad \text{a.s. on } \{T < \infty\},$$

where W is as in (2). Recall that $M(t) = \inf_{s \geq t} X(s)$. Therefore

$$\begin{aligned}
&E(|I(W(T+2^{-k}) - W(T+2^{-k-1}) > 0) - I(X(T+2^{-k}) - X(T+2^{-k-1}) > 0)| I(T < \infty)) \\
&\leq P(0 \geq W(T+2^{-k}) - W(T+2^{-k-1})) \geq \int_{T+2^{-k-1}}^{T+2^{-k}} X(s)^{-1} ds, \quad T < \infty \\
&\leq P(0 \geq (W(T+2^{-k}) - W(T+2^{-k-1})) 2^{(-k-1)/2} \geq -2^{(-k-1)/2} M(\varepsilon)^{-1}, \quad T < \infty)
\end{aligned}$$

$$\begin{aligned}
&\leq C E(\min(1, 2^{-(k-1)/2} M(\epsilon)^{-1})) \\
&\leq C(2^{-(k-1)/4} + P(M(\epsilon) < 2^{-(k-1)/4})) \\
&\leq C(\epsilon) 2^{-(k-1)/4} .
\end{aligned}$$

The Borel-Cantelli lemma implies that

$$\begin{aligned}
(5) \quad &W(T+2^{-k}) - W(T+2^{-k-1}) > 0 \Leftrightarrow X(T+2^{-k}) - X(T+2^{-k-1}) > 0 \\
&\text{for large } k \text{ a.s. on } \{T < \infty\} .
\end{aligned}$$

(4) and (5) imply that $(T, \omega) \in D$ a.s. Moreover by (3) with $t = 0$, $(0, \omega) \notin D$ a.s. Therefore if T is any $\{F_t^X\}$ - stopping time and

$$T' = \begin{cases} T & \text{if } T > 0 \\ \infty & \text{if } T = 0 \end{cases} ,$$

then

$$\begin{aligned}
E(I_D(T, \omega) I(T < \infty)) &= \lim_{\epsilon \rightarrow 0^+} E(I_D(T' \vee \epsilon, \omega) I(T' < \infty)) \\
&= P(T' < \infty) \quad (\text{since by the above } (T' \vee \epsilon, \omega) \in D \\
&\quad \text{a.s. on } \{T' < \infty\}) \\
&= P(0 < T < \infty) .
\end{aligned}$$

This proves (a) .

(b) Let $T \leq \infty$ be any $\{F_t^B\}$ - stopping time. Then just as in the derivation of (1) one has

$$(6) \quad (T, \omega) \in D \text{ a.s. on } \{Y(T) \neq 0, T < \infty\} .$$

Moreover just as in the derivation of (3) one has

$$(7) \quad (T, \omega) \notin D \text{ a.s. on } \{Y(T) = 0, T < \infty\} .$$

Therefore

$$E(I_D(T, \omega) I(T < \infty)) = P(Y(T) \neq 0, T < \infty) ,$$

and (b) is proved.

(c) If D is $\{F_t^B\}$ - optional then $D = Z^c$ (up to indistinguishability) by the above. Therefore Z is on $\{F_t^X\}$ - progressively measurable set. $M(t)$ is the local time of Z and hence can be constructed from Z as Lévy's mesure du voisinage [2, p.225]. It follows easily from this construction that $M(t)$ is $\{F_t^X\}$ - adapted. As $M(t)$ is the future minimum of X , this is absurd. $\square$

The above example was suggested by joint work with Michel Emery [1], in which the predictable set

$$\{(t, \omega) \mid \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n I((cM - B)(t-2^{-k}) - (cM - B)(t-2^{-k-1})) > 0\} = 1/2\}$$

was used to show $F^{cM-B} = F^B \iff c \neq 2$.

List of References

1. Emery, M. and Perkins, E. La filtration de $B+L$.
Z.f. Wahrscheinlichkeitstheorie 59, 383-390 (1982).
2. Lévy, P. Processus Stochastiques et Mouvement Brownien. Gauthier-Villars, Paris, 1948.
3. McKean, M.P. The Bessel motion and a singular integral equation.
Mem. Coll. Sci. Univ. Kyoto. Ser. A Math. 33, 317-322 (1960).
4. Pitman, J. One-dimensional Brownian motion and the three-dimensional Bessel process. Adv. Appl. Prob. 7, 511-526 (1975).

Edwin Perkins
Mathematics Department
U.B.C.
Vancouver, B.C.
Canada V6T 1Y4