SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

PHILIP PROTTER

ALAIN-SOL SZNITMAN

An equation involving local time

Séminaire de probabilités (Strasbourg), tome 17 (1983), p. 62-66

<http://www.numdam.org/item?id=SPS_1983__17__62_0>

© Springer-Verlag, Berlin Heidelberg New York, 1983, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

AN EQUATION INVOLVING LOCAL TIME

by

Philip PROTTER¹ and Alain-Sol SZNITMAN²

1. Introduction.

We show there is only one solution X , the obvious one, to the equation

$$X_t + \alpha L(X)_t = B_t + C_t \quad (|\alpha| > 1)$$

where $L(X)$ is the symmetrized local time at 0 of the semimartingale X ; B is a given Wiener process; and C is any continuous finite variation process, adapted, whose support is contained in the zero set of B . More precisely: X must be B , and C must be $\alpha L(B)$.

HARRISON and SHEPP [3] have considered the equation $X_t + \beta L(X)_t = B_t$, and they showed that a unique solution X exists if $|\beta| \leq 1$ and that no solution exists if $|\beta| > 1$. In addition, the problem of solving an equation where the solution involves finding a semimartingale together with its local time has recently been receiving attention.

Problems of this type seem to be related to questions of filtering with singular cumulative signals (cf [1]), as well as to questions concerning the equality of filtrations. In particular, it would be interesting to learn what happens when $|\alpha| \leq 1$, which seems to us to be tied to problems such as the equality of the filtrations of $B+cL$ and B (cf EMERY-PERKINS [2], and [1]).

---

¹Departments of Mathematics and Statistics; Purdue University, West Lafayette, IN 47907 USA. Supported in part by NSF Grant #0464-50-13955; Visitor at Université de Rennes, 1981-1982.

²Université de Paris VI, Tour 56, 4 place Jussieu 75230 Paris Cedex 05. Membre du Laboratoire de Probabilité associé au CNRS LA 224.

2. Results.

For all unexplained terminology and notations we refer the reader to JACOD [4]. In particular, we are using the symmetrized local time of [4, p.184], which is also the one HARRISON-SHEPP used. For a semimartingale X , we let $L(X)$ denote its local time, which is known to exist always. We assume we are given a filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ supporting a standard Brownian motion B and verifying the usual conditions: $\mathfrak{F}_0$ is P -complete and $\mathfrak{F}_t = \bigcap_{s>t} \mathfrak{F}_s$, all $t \geq 0$.

THEOREM. Let C be an adapted process with continuous paths of finite variation on compacts, and $C_0 = 0$. Suppose

$$(1) \quad C_t = \int_0^t 1_{(B_s = 0)} dC_s$$

Let X be a continuous semimartingale, $X_0 = 0$, verifying

$$(2) \quad X_t + \alpha L(X)_t = B_t + C_t$$

where $|\alpha| > 1$. Then $(X.) = (B.)$.

COMMENT. An immediate consequence of the theorem is that equation (2) has a solution $(X, L(X))$ only if $C_t = \alpha L(B)_t$.

PROOF. Fix $s > 0$. We define:

$$S = \inf\{t \geq s: X_t = 0\}$$

$$T = \inf\{t \geq s: B_t = 0\}.$$

Step 1: We show $P\{S \geq T\} = 1$. Let $\Lambda = \{S < T\}$ and suppose $P(\Lambda) > 0$.

Since $X_S = 0$ on Λ , we have for all $h > 0$ on Λ :

$$\begin{aligned}
(3) \quad X_{(S+h) \wedge T} + \alpha [L(X)_{(S+h) \wedge T} - L(X)_S] \\
= B_{(S+h) \wedge T} - B_S + C_{(S+h) \wedge T} - C_S \\
= B_{(S+h) \wedge T} - B_S \quad (\text{from (1)}).
\end{aligned}$$

Define $\Omega' = \Omega \cap \Lambda$, $\mathcal{F}'_h = \mathcal{F}_{S+h} \cap \Lambda$, and P' by $P'(A) = P(A \cap \Lambda)/P(\Lambda)$. On $(\Omega', \mathcal{F}', P')$ we have $T' = T - S$ is an $\mathcal{F}'_h$ -stopping time. Letting $B'_h = B_{S+h} - B_S$ one easily checks that B' is an $\mathcal{F}'_h$ Brownian motion; moreover $X'_h = X_{S+h}$ is an $\mathcal{F}'_h$ semimartingale ($S < \infty$ a.s.). Thus equation (3) yields:

$$(4) \quad X'_{h \wedge T'} + \alpha L(X')_{h \wedge T'} = B'_{h \wedge T'}.$$

Using a technique due to HARRISON-SHEPP, we will show (4) is impossible. By Tanaka's formulas [4, p.184] and (4) we have:

$$(5) \quad (X')^-_{h \wedge T'} = -\int_0^{h \wedge T'} 1_{(X'_u < 0)} + \frac{1}{2} 1_{(X'_u = 0)} dB'_u + \left(\frac{1+\alpha}{2}\right) L(X')_{h \wedge T'}$$

and

$$(6) \quad (X')^+_{h \wedge T'} = \int_0^{h \wedge T'} 1_{(X'_u > 0)} + \frac{1}{2} 1_{(X'_u = 0)} dB'_u + \left(\frac{1-\alpha}{2}\right) L(X')_{h \wedge T'}.$$

Both $(X')^+$ and $(X')^-$ are nonnegative processes, zero at zero. Moreover since $|\alpha| > 1$, equations (5) and (6) imply that always one of $(X')^-$ and $(X')^+$ is a nonnegative supermartingale, and hence identically zero, since $X'_0 = X_0^+ = 0$. This implies (again from (5) and (6)) that $L(X')_{h \wedge T'}$ is identically zero, and hence $X'_{h \wedge T'} = B'_{h \wedge T'}$ from (5); thus $B'_{h \wedge T'}$ never changes sign. Since $B'_0 = 0$ and $T' > 0$ a.s., we have a contradiction. We conclude that $P(\Lambda) = 0$; that is, $P(S \geq T) = 1$.

Step 2: Recall $s > 0$ is fixed. We will show that $P(\{|B_s| \leq |X_s|\} \cap \{X_s B_s \geq 0\}) = 1$.

Define:

$$\Delta_1 = \{0 < X_S < B_S\}$$

$$\Delta_2 = \{0 > X_S > B_S\}$$

$$\Delta_3 = \{-B_S < X_S < 0 < B_S\}$$

$$\Delta_4 = \{B_S < 0 < X_S < -B_S\}$$

We first show $P(\Delta_i) = 0$, $1 \leq i \leq 4$. Note that on $[s, T(\omega)[$, we have

$B_u - B_s = X_u - X_s$, so on Δ_1 and Δ_2 we have $S < T$; thus step 1

gives us $P(\Delta_1) = P(\Delta_2) = 0$. If $P(\Delta_3) > 0$, we have $P\{\exists u \in]s, T(\cdot):$

$B_u = B_s - X_s|_{\Delta_3}\} > 0$, which contradicts the definition of T

(since then $X_u = 0$). Analogously, $P(\Delta_4) = 0$. Therefore $P\{|B_S| \leq |X_S|\} = 1$.

Define:

$$\Sigma_1 = \{X_S < -B_S < 0 < B_S\}$$

$$\Sigma_2 = \{X_S > -B_S > 0 > B_S\}.$$

Then $P(\exists u \in [s, T(\cdot)[: B_u - B_s = -B_S \text{ before } B_u - B_s = -X_s|_{\Sigma_1}) > 0$,

since $B_u - B_s = X_u - X_s$ on $]s, T(\cdot)[$. This would contradict that

$P(S \geq T) = 1$, which we showed in step 1. Thus $P(\Sigma_1) = 0$. Analogously

$P(\Sigma_2) = 0$, hence $P\{X_S B_S \geq 0\} = 1$. Thus step 2 is complete.

Step 3: By using step 2 for all s rational and then using the

continuity of the paths of B and X we have that a.s., for all $s > 0$,

$|B_s| \leq |X_s|$, and $X_s B_s \geq 0$.

Step 4: $X_s = B_s$, all $s > 0$. Define

$$\Gamma_1 = \{X_s > B_s > 0\}$$

$$\Gamma_2 = \{X_s < B_s < 0\}.$$

Given step (3), it suffices to show $P(\Gamma_1) = P(\Gamma_2) = 0$. For fixed s ,

we have $\tau_1 \leq \{T < S\}$, since for any $u \in]s, T(\cdot)[$ we have

$X_u - B_u = X_s - B_s > 0$. Thus by continuity we have $X_T = X_s - B_s > 0$.

Since $B'_h = B_{T+h} - B_T = B_{T+h}$ is a new Brownian motion, we have

$$P\{\exists u \in]T(\omega), S(\omega)[B_u < 0 | \tau_1\} = 1,$$

which contradicts that $B_u X_u > 0$, since $X_u > 0$ in $]T(\omega), S(\omega)[$. Thus

$P(\tau_1) = 0$. Analogously, $P(\tau_2) = 0$. This completes step 4 and the proof of the theorem.

REFERENCES

1. Davis, B.; Protter, P.: Filtering with Singular Cumulative Signals, Purdue Mimeo, Series #81-8, April 1981 (unpublished).
2. Emery, M.; Perkins, E.: La Filtration de B+L; Z. Wahrscheinlichkeitstheorie und verw. Geb. 59, 383-390 (1982).
3. Harrison, J.; Shepp, L.: On skew Brownian Motion; Annals of Probability 9, 309-313 (1981).
4. Jacod, J.: Calcul Stochastique et Problèmes de Martingales. Springer Lecture Notes in Math. 714 (1979).

Ed PERKINS has written us that he and Martin BARLOW have established the non-uniqueness of solutions of $X_t + \alpha L(X)_t = B_t + \alpha L(B)_t$ for $0 < |\alpha| \leq 1$.

Note de la rédaction : Voir l'article précédent dans ce volume.