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# AN EQUATION INVOLVING LOCAL TIME

by

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## 1. Introduction.

We show there is only one solution  $X$ , the obvious one, to the equation

$$X_t + \alpha L(X)_t = B_t + C_t \quad (|\alpha| > 1)$$

where  $L(X)$  is the symmetrized local time at 0 of the semimartingale  $X$ ;  $B$  is a given Wiener process; and  $C$  is any continuous finite variation process, adapted, whose support is contained in the zero set of  $B$ . More precisely:  $X$  must be  $B$ , and  $C$  must be  $\alpha L(B)$ .

HARRISON and SHEPP [3] have considered the equation  $X_t + \beta L(X)_t = B_t$ , and they showed that a unique solution  $X$  exists if  $|\beta| \leq 1$  and that no solution exists if  $|\beta| > 1$ . In addition, the problem of solving an equation where the solution involves finding a semimartingale together with its local time has recently been receiving attention.

Problems of this type seem to be related to questions of filtering with singular cumulative signals (cf [1]), as well as to questions concerning the equality of filtrations. In particular, it would be interesting to learn what happens when  $|\alpha| \leq 1$ , which seems to us to be tied to problems such as the equality of the filtrations of  $B + \alpha L$  and  $B$  (cf EMERY-PERKINS [2], and [1]).

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## 2. Results.

For all unexplained terminology and notations we refer the reader to JACOD [4]. In particular, we are using the symmetrized local time of [4, p.184], which is also the one HARRISON-SHEPP used. For a semimartingale  $X$ , we let  $L(X)$  denote its local time, which is known to exist always. We assume we are given a filtered probability space  $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$  supporting a standard Brownian motion  $B$  and verifying the usual conditions:  $\mathfrak{F}_0$  is  $P$ -complete and  $\mathfrak{F}_t = \bigcap_{s>t} \mathfrak{F}_s$ , all  $t \geq 0$ .

THEOREM. Let  $C$  be an adapted process with continuous paths of finite variation on compacts, and  $C_0 = 0$ . Suppose

$$(1) \quad C_t = \int_0^t 1_{(B_s = 0)} dC_s$$

Let  $X$  be a continuous semimartingale,  $X_0 = 0$ , verifying

$$(2) \quad X_t + \alpha L(X)_t = B_t + C_t$$

where  $|\alpha| > 1$ . Then  $(X.) = (B.)$ .

COMMENT. An immediate consequence of the theorem is that equation (2) has a solution  $(X, L(X))$  only if  $C_t = \alpha L(B)_t$ .

PROOF. Fix  $s > 0$ . We define:

$$S = \inf\{t \geq s: X_t = 0\}$$

$$T = \inf\{t \geq s: B_t = 0\}.$$

Step 1: We show  $P\{S \geq T\} = 1$ . Let  $\Lambda = \{S < T\}$  and suppose  $P(\Lambda) > 0$ .

Since  $X_S = 0$  on  $\Lambda$ , we have for all  $h > 0$  on  $\Lambda$ :

$$\begin{aligned}
 (3) \quad X_{(S+h) \wedge T} + \alpha [L(X)_{(S+h) \wedge T} - L(X)_S] \\
 = B_{(S+h) \wedge T} - B_S + C_{(S+h) \wedge T} - C_S \\
 = B_{(S+h) \wedge T} - B_S \quad (\text{from (1)}).
 \end{aligned}$$

Define  $\Omega' = \Omega \cap \Lambda$ ,  $\mathcal{T}'_h = \mathcal{T}_{S+h} \cap \Lambda$ , and  $P'$  by  $P'(A) = P(A \cap \Lambda)/P(\Lambda)$ . On  $(\Omega', \mathcal{T}', P')$  we have  $T' = T - S$  is an  $\mathcal{T}'_h$ -stopping time. Letting  $B'_h = B_{S+h} - B_S$  one easily checks that  $B'$  is an  $\mathcal{T}'_h$  Brownian motion; moreover  $X'_h = X_{S+h}$  is an  $\mathcal{T}'_h$  semimartingale ( $S < \infty$  a.s.). Thus equation (3) yields:

$$(4) \quad X'_{h \wedge T'} + \alpha L(X')_{h \wedge T'} = B'_{h \wedge T'}.$$

Using a technique due to HARRISON-SHEPP, we will show (4) is impossible. By Tanaka's formulas [4, p.184] and (4) we have:

$$(5) \quad (X')^-_{h \wedge T'} = -\int_0^{h \wedge T'} 1_{(X'_u < 0)} + \frac{1}{2} 1_{(X'_u = 0)} dB'_u + \left(\frac{1+\alpha}{2}\right) L(X')_{h \wedge T'}$$

and

$$(6) \quad (X')^+_{h \wedge T'} = \int_0^{h \wedge T'} 1_{(X'_u > 0)} + \frac{1}{2} 1_{(X'_u = 0)} dB'_u + \left(\frac{1-\alpha}{2}\right) L(X')_{h \wedge T'}.$$

Both  $(X')^+$  and  $(X')^-$  are nonnegative processes, zero at zero. Moreover since  $|\alpha| > 1$ , equations (5) and (6) imply that always one of  $(X')^-$  and  $(X')^+$  is a nonnegative supermartingale, and hence identically zero, since  $X_0^- = X_0^+ = 0$ . This implies (again from (5) and (6)) that  $L(X')_{h \wedge T'}$  is identically zero, and hence  $X'_{h \wedge T'} = B'_{h \wedge T'}$  from (5); thus  $B'_{h \wedge T'}$  never changes sign. Since  $B'_0 = 0$  and  $T' > 0$  a.s., we have a contradiction. We conclude that  $P(\Lambda) = 0$ ; that is,  $P(S \geq T) = 1$ .

Step 2: Recall  $s > 0$  is fixed. We will show that  $P(\{|B_s| \leq |X_s|\} \cap \{X_s B_s \geq 0\}) = 1$ .

Define:

$$\Delta_1 = \{0 < X_s < B_s\}$$

$$\Delta_2 = \{0 > X_s > B_s\}$$

$$\Delta_3 = \{-B_s < X_s < 0 < B_s\}$$

$$\Delta_4 = \{B_s < 0 < X_s < -B_s\}$$

We first show  $P(\Delta_i) = 0$ ,  $1 \leq i \leq 4$ . Note that on  $[s, T(\omega)[$ , we have

$B_u - B_s = X_u - X_s$ , so on  $\Delta_1$  and  $\Delta_2$  we have  $S < T$ ; thus step 1

gives us  $P(\Delta_1) = P(\Delta_2) = 0$ . If  $P(\Delta_3) > 0$ , we have  $P\{\exists u \in ]s, T(\cdot):$

$B_u = B_s - X_s | \Delta_3\} > 0$ , which contradicts the definition of  $T$

(since then  $X_u = 0$ ). Analogously,  $P(\Delta_4) = 0$ . Therefore  $P\{|B_s| \leq |X_s|\} = 1$ .

Define:

$$\Sigma_1 = \{X_s < -B_s < 0 < B_s\}$$

$$\Sigma_2 = \{X_s > -B_s > 0 > B_s\}.$$

Then  $P(\exists u \in [s, T(\cdot)[ : B_u - B_s = -B_s \text{ before } B_u - B_s = -X_s | \Sigma_1) > 0$ ,

since  $B_u - B_s = X_u - X_s$  on  $]s, T(\cdot)[$ . This would contradict that

$P(S \geq T) = 1$ , which we showed in step 1. Thus  $P(\Sigma_1) = 0$ . Analogously

$P(\Sigma_2) = 0$ , hence  $P\{X_s B_s \geq 0\} = 1$ . Thus step 2 is complete.

Step 3: By using step 2 for all  $s$  rational and then using the continuity of the paths of  $B$  and  $X$  we have that a.s., for all  $s > 0$ ,  $|B_s| \leq |X_s|$ , and  $X_s B_s \geq 0$ .

Step 4:  $X_s = B_s$ , all  $s > 0$ . Define

$$\Gamma_1 = \{X_s > B_s > 0\}$$

$$\Gamma_2 = \{X_s < B_s < 0\}.$$

Given step (3), it suffices to show  $P(\Gamma_1) = P(\Gamma_2) = 0$ . For fixed  $s$ ,

we have  $\tau_1 \leq \{T < S\}$ , since for any  $u \in ]s, T(\cdot)[$  we have

$X_u - B_u = X_s - B_s > 0$ . Thus by continuity we have  $X_T = X_s - B_s > 0$ .

Since  $B'_h = B_{T+h} - B_T = B_{T+h}$  is a new Brownian motion, we have

$$P\{\exists u \in ]T(\omega), S(\omega)[ B_u < 0 | \tau_1\} = 1,$$

which contradicts that  $B_u X_u > 0$ , since  $X_u > 0$  in  $]T(\omega), S(\omega)[$ . Thus

$P(\tau_1) = 0$ . Analogously,  $P(\tau_2) = 0$ . This completes step 4 and the

proof of the theorem.

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Ed PERKINS has written us that he and Martin BARLOW have established the non-uniqueness of solutions of  $X_t + \alpha L(X)_t = B_t + \alpha L(B)_t$  for  $0 < |\alpha| \leq 1$ .

Note de la rédaction : Voir l'article précédent dans ce volume.