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ON THE AZEMA-YOR STOPPING TIME

by

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Azema and Yor [1] presented a stopping time that embeds in Brownian motion distributions with mean zero. Various other methods have been suggested in the literature (Dubins, Root, Rost, Chacon and Walsh, and others), starting with the original randomized method of Skorokhod. All of these methods are defined in terms of decompositions of distributions, limiting processes or existence proofs. The Azéma-Yor stopping time is explicit and extremely simple to describe:

Let $\{B_t \mid t \geq 0\}$ be standard Brownian motion and let $S_t = \sup \{B_s | s \leq t\}$. Let $F = L(X)$ with $E(X) = 0$ and let $\psi(x) = E(X|X \geq x)$. Let $\tau = \inf \{t | S_t \geq \psi(B_t)\}$. Then (i) $\tau < \infty$ a.s., (ii) $B_\tau \sim F$ and (iii) $E(\tau) = Var(X)$.

Property (i) is trivial. If $P(X = 0) = 1$ then $\tau = 0$. Otherwise take any $x < 0 < y = E(X|X \geq x)$ and let τ_x be the first visit to x following the first visit to y. Then $\tau \leq \tau_x < \infty$ a.s.

The proof of property (ii) given in [1] is very hard. By continuity and the oscillatory nature of Brownian motion, it is clearly enough to prove the statement for random variables X with finite support. The purpose of this note is to remark that for this case the Azéma-Yor stopping time is a special case of the Chacon-Walsh [2] family of stopping times. Property (ii) becomes then clear. Property (iii) can be obtained as in [2] or [3] by a monotone convergence argument on stopping times.

A Chacon-Walsh stopping time (of which Dubins' [3] is a special case) is built as follows:

Express X as the limit of a martingale with almost surely dichotomous transitions. Embed this martingale in Brownian motion successively by first hitting times of one of the two points. To accomplish this, express the potential function $U(x) = - E \mid X - x \mid$ of the distribution of X as the (decreasing) limit of a sequence $U_n(x)$ of functions defined as $U_0(x) = - |x|$, $U_{n+1}(x) = \min(U_n(x), a_n x + b_n)$, where $a_n x + b_n$ is tangent to U(x). The endpoints of the newly added linear piece are the support of the abovementioned dichotomous transition.

If X lives on a finite set, then U is a broken line that breaks at the atoms of X. If the Chacon-Walsh construction is done by taking as straight lines the ones through these broken pieces, one at a time, from left to right, the result is the Azéma-Yor stopping time.

For the sake of completeness, here is a direct description of the procedure. Consider $P(X = x_i) = P_i > 0$, $1 \leq i \leq n$, $x_1 \leq x_2 < \ldots < x_n$.
Now let $Y_i = E(X \mid 1_{\{X=x_1\}}, 1_{\{X=x_2\}}, \ldots, 1_{\{X=x_i\}})$.
Then $0 = Y_0, Y_1, Y_2, \ldots, Y_{n-2}, Y_{n-1} = X$ is a martingale with dichotomous transition distributions, which becomes constant (and equal to X) as soon as it decreases. If Y is embedded in Brownian motion by first hitting times, then, clearly, the eventual stopping time embeds L(X). But, what we have done is exactly the Azéma-Yor prescription.

REFERENCES

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[3] Dubins, L.E. On a theorem of Skorokhod. Ann.Math.Statist., 39 (1968), 2094-2097.

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