SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

RAJEEVA L. KARANDIKAR

Girsanov type formula for a Lie group valued brownian motion

Séminaire de probabilités (Strasbourg), tome 17 (1983), p. 198-204

<http://www.numdam.org/item?id=SPS_1983__17__198_0>

GIRSANOV TYPE FORMULA FOR A LIE GROUP VALUED BROWNIAN MOTION

by

R.L. KARANDIKAR

Let G be a Lie group of $d\times d$ matrices and X^1 be a G-valued continuous semimartingale, P^1 be its distribution on $\Omega = \mathcal{C}([0,1],G)$. Let $X^2 = AX^1$ be the left translate of X^1 by a G-valued adapted continuous process with finite variation paths, and P^2 be the distribution of X^2 on Ω. The question analogous to the classical Girsanov theorem is : Under what conditions on A,X is $P^2 \ll P^1$, and what is the density $\frac{dP^2}{dP^1}$?

We denote by $\mathcal{G}$ the Lie algebra of G, and by W the sample space $\mathcal{C}([0,1],\mathcal{G})$. Using the pathwise integration formula (see Karandikar [3]) for multiplicative stochastic integration, we may define an " exponential" mapping $\mathcal{E} : W \to \Omega$ and a "logarithm" $\mathcal{L} : \Omega \to W$, which are independent of the chcice of the laws and, in a reasonable sense, inverse to each other. Then we may denote by Y^1, Y^2 the processes $\mathcal{L}(X^1)$, $\mathcal{L}(X^2)$ and by Q^1,Q^2 the corresponding laws on W. Next, using the " integration by parts formula " for multiplicative stochastic integration (Karandikar [4]), we show that $Y^2 = Y^1 + B$, where the process B is expressible in terms of A. Therefore the ordinary Girsanov theorem in the additive set-up will give conditions for the absolute continuity $Q^2 \ll Q^1$, and explicit expressions for the density. Returning to Ω by the exponential mapping $\mathcal{E}$, we can in this way solve a multiplicative Girsanov problem.

We study in particular the case where X^1 is a G-valued (multiplicative) brownian motion.

I. GENERALITIES

We first introduce some notation. Let U,V be continuous semimartingales (on some fixed probability space Ω with a filtration $(\mathcal{F}_t)$, not necessarily the same Ω as above), taking values in the space $L(d)$ of all $d\times d$ matrices. We denote by $\langle U,V\rangle$ the $L(d)$ valued process defined by

$$\langle U,V\rangle^i_j = \Sigma_k \langle U^i_k, V^k_j\rangle$$

The paths of $<U,V>$ are continuous with finite variation, equal to 0 for t=0 . We denote by $V \cdot U$ and $V \circ U$ the Ito stochastic integral and the Stratonovich stochastic integral of V with respect to U :

$$(V\cdot U)_t = \int_0^t V_s dU_s \quad (\text{ matrix product }) , \quad (V\circ U)_t = \int_0^t V_s \circ dU_s$$

and we denote by $U:V$, $U\overset{\circ}{:}V$ the similar integrals, with matrix products on the right side $((U:V)_t = \int_0^t (dU_s)V_s \ldots)$. As usual, we may express Stratonovich integrals in terms of Ito integrals

(1) $\quad V\circ U = V\cdot U + \frac{1}{2}< V,U > \quad , \quad U\overset{\circ}{:}V = U:V + \frac{1}{2}< U,V >$.

These formulas can be verified by looking at the entries and using the 1-dimensional relation (see Ito-Watanabe [2]). In the Lie group - Lie algebra setting, the Stratonovich integrals arise naturally.

We now assume that $U_0=0$. The Ito exponential of U , denoted by $\varepsilon(U)$, is the only solution to the stochastic differential equation

(2) $\quad V = I + V\cdot U$.

More precisely, this is the <u>left</u> exponential (see Karandikar [3]. The right exponential will not be used here). It can be shown that $\varepsilon(U)=V$ is invertible (see Karandikar [3]) and we can recover U from V by the formula

(3) $\quad U = \mathcal{L}(V) = V^{-1}\cdot V \quad$ (hence $\varepsilon(U)=\varepsilon(U') \Rightarrow U=U'$) .

Similarly, we define the Stratonovich (left) exponential $\varepsilon^*(U)$ as the solution to

(4) $\quad V = I + V\circ U$

It can be easily seen that if V is a solution to (4), then $< V,U > = V\cdot<U,U>$, and therefore $V = I+V\cdot(U + \frac{1}{2}<U,U>)$, hence

(5) $\quad \varepsilon^*(U) = \varepsilon(U + \frac{1}{2}<U,U>)$

and $\varepsilon^*(U)$ is invertible. Just as above, one can recover U from $V=\varepsilon^*(U)$ by the formula

(6) $\quad U = \mathcal{L}^*(V) = V^{-1}\circ V \quad$ (hence $\varepsilon^*(U)=\varepsilon^*(U') \Rightarrow U=U'$) .

Let U and U' denote two continuous semimartingales, such that $U_0=U'_0=0$, and let W denote $\varepsilon(U')$. Then we have the <u>integration by parts</u> formula for multiplicative stochastic integrals

(7) $\quad \varepsilon(U + U' + < U,U'>) = \varepsilon(W\cdot U:W^{-1})\varepsilon(U')$.

This is a direct consequence of Ito's formula (Karandikar [4]). The same arguments with Stratonovich integrals in place of Ito's integrals will give

$$\varepsilon^{*}(U + U') = \varepsilon^{*}(W \circ U \overset{\circ}{} W^{-1})\varepsilon^{*}(U') \text{ with } W=\varepsilon^{*}(U') . \tag{8}$$

Also, (8) can be deduced from (7) and (5).

We are going to apply this formula in the situation described in the introduction. Let X be a continuous semimartingale such that $X_0=I$, and let A be a continuous semimartingale with finite variation paths, such that $A_0=I$. We assume that these two processes take their values in the set of invertible matrices, and define

$$Y_t = \int_0^t X_s^{-1} \circ dX_s \tag{9}$$

$$B_t = \int_0^t (A_s X_s)^{-1} dA_s X_s \quad \text{(Stieltjes integral)} \tag{10}$$

Then the paths of B have finite variation, and we have :

THEOREM 1. $AX = \varepsilon^{*}(Y+B)$.

Proof. We have $X=\varepsilon^{*}(Y)$ according to (9) and (6). Similarly, we set $\alpha = A^{-1} \circ A$, so that $A=\varepsilon^{*}(\alpha)$. Then $AX = \varepsilon^{*}(\alpha)\varepsilon^{*}(Y)$, which we try to identify with the right side of (8). We must have $\varepsilon^{*}(U')=\varepsilon^{*}(Y)$, hence $U'=Y$, $W=X$. Then we must have $W \circ U \overset{\circ}{} W^{-1} = \alpha$, and therefore since $W=X$, $U=X^{-1} \circ \alpha \overset{\circ}{} X = X^{-1} \circ (A^{-1} \circ A) \overset{\circ}{} X = (AX)^{-1} \circ A \overset{\circ}{} X = B$. Note that we didn't really use in this proof the fact that A has finite variation.

II. CONSTRUCTION OF THE MAPS $\mathcal{E}$ AND $\mathcal{L}$

Let W be the set of all continuous mappings $w : [0,1] \longmapsto L(d)$ such that $w(0)=0$. We denote by Y_t the coordinate mapping $w \mapsto w(t)$ on W , and by $\mathcal{G}_t$ the σ-field $\sigma(Y_s,\ s \leq t)$.

Let Ω be the set of all continuous mappings $\omega : [0,1] \longmapsto L(d)$ such that $\omega(0)=I$, and $\omega(t)$ is invertible for every t . The coordinate mappings and fields are denoted here by X_t and $\mathcal{F}_t$.

If one is interested in a particular pair $(\mathcal{G},G)$, the mappings in W will be restricted to be $\mathcal{G}$-valued, and those in Ω to be G-valued. This makes no essential difference, as we shall see.

We say that a probability law on W (Ω) is a semimartingale measure if the corresponding coordinate process is a semimartingale (w.r. to the corresponding filtration, made right-continuous and complete).

Our aim in this section consists in constructing Borel mappings $\mathcal{E} : W \to \Omega$, $\mathcal{L} : \Omega \to W$ such that, for any semimartingale measure on W, $X \circ \mathcal{E}$ is a version of the Stratonovich exponential $\varepsilon^{*}(Y)$, and for any semimartingale measure on Ω , $Y \circ \mathcal{L}$ is a version of the Stratonovich " logarithm" $\ell^{*}(X)$. These mappings, however, do not depend on the choice of a measure on W or Ω .

For $n\geqq 1$, $w\in W$, $\omega\in\Omega$, define $s_i^n(w)$ and $t_i^n(\omega)$ for $i\geqq 0$ inductively by

$$s_0^n(w) = t_0^n(w) = 0 \quad \text{and for } i\geqq 0$$

$$s_{i+1}^n(w) = \inf\{ s\geqq s_i^n(w) : |Y(s,w)-Y(s_i^n(w),w)| \geqq 2^{-n} \text{ or } s\geqq 1\}$$

$$t_{i+1}^n(\omega) = \inf\{ t\geqq t_i^n(\omega) : |X(t,\omega)-X(t_i^n(\omega),\omega)| \geqq 2^{-n} \text{ or } |X^{-1}(t,\omega)-X^{-1}(t_i^n(\omega),\omega)| \geqq 2^{-n} \text{ or } |X^{-1}(t_i^n,\omega)X(t(\omega),\omega) - I| \geqq 2^{-n} \text{ or } t\geqq 1 \}.$$

Here the norm $|\ |$ is chosen so that the logarithm of a matrix (i.e. the inverse mapping of the usual matrix exponential exp) is defined on the neighbourhood $|x-I|<1$ of the identity. We now set for $s,t\in[0,1]$

$$\mathcal{L}_n(t,\omega) = \sum_{i\geqq 0} \log(X^{-1}(t\wedge t_i^n(\omega),\omega)\ X(t\wedge t_{i+1}^n(\omega),\omega))$$

$$\mathcal{E}_n(s,w) = \prod_{i\geqq 0} \exp(Y(s\wedge s_{i+1}^n(w),w) - Y(s\wedge s_i^n(w),w))$$

It is easy to see that if ω is G-valued , then $\mathcal{L}_n(\cdot,\omega)$ is $\mathcal{G}$-valued, and similarly in the other direction. Let

$$W_0 = \{ w\in W : \mathcal{E}_n(\cdot,w) \text{ converges uniformly } \}$$

$$\Omega_0 = \{ \omega\in\Omega : \mathcal{L}_n(\cdot,\omega) \text{ converges uniformly } \}$$

We denote the corresponding limits by $\mathcal{E}(w)=\mathcal{E}(\cdot,w)$ and $\mathcal{L}(\omega)=\mathcal{L}(\cdot,\omega)$; outside W_0 or Ω_0 we set $\mathcal{E}(t,w)=I$, $\mathcal{L}(t,\omega)=0$ for all t . Of course, since the coordinate mappings are denoted by Y_t on W , X_t on Ω , we may write $Y_t(\mathcal{L}(\omega))$ instead of $\mathcal{L}(t,\omega)$, $X_t(\mathcal{E}(w))$ instead of $\mathcal{E}(t,w)$.

THEOREM 2. Let Z be a continuous $L(d)$-valued semimartingale defined on some filtered probability space $(\Theta, \mathcal{H}, (\mathcal{H}_t)_{t\leq 1} , \mu)$.

1) Assume that $Z_0=0$. Then for μ-a.e. $\theta\in\Theta$ the path $Z_\cdot(\theta)$ belongs to W_0 and the path $\varepsilon^*(Z)(\cdot,\theta)$ of the Stratonovich exponential $\varepsilon^*(Z)$ is equal to $\mathcal{E}(Z_\cdot(\theta))$.

2) Assume that $Z_0=I$ and Z takes its values in the set of invertible matrices. Then for μ-a.e. $\theta\in\Theta$ the path $Z_\cdot(\theta)$ belongs to Ω_0 , and the path $\ell^*(Z)(\cdot,\theta)$ of the Stratonovich logarithm $\ell^*(Z)$ is equal to $\mathcal{L}(Z_\cdot(\theta))$.

COROLLARY . Let P be a semimartingale measure on Ω (Q be a semimartingale measure on W). Then the image measure $\mathcal{L}(P)$ is a semimartingale measure on W ($\mathcal{E}(Q)$ a semimartingale measure on Ω) and we have

(11) $$\mathcal{E}(\mathcal{L}(\omega)) = \omega \ \text{ a.s. } P \qquad (\mathcal{L}(\mathcal{E}(w)) = w \ \text{ a.s. } Q).$$

Proof. The proof relative to the exponential is outlined in Karandikar [3] (Sém. Prob. XVI), and fully given in Karandikar [5]. Keeping the notation $t_i^n(\theta)$ for $t_i^n(Z_\cdot(\theta))$, the statement amounts to the fact that

$$J_n(t,\theta) = \sum_{i\geq 0} \log(Z^{-1}(t\wedge t_i^n(\theta),\theta)Z(t\wedge t_{i+1}^n(\theta),\theta))$$

converges μ-a.s. to $\mathcal{L}^*(Z)(t,\theta)$, uniformly in $t\in[0,1]$. We rewrite $J_n(t,\theta)$ as

$$\sum_{i=0}^{\infty} \log(I + Z^{-1}(t\wedge t_i^n)[Z(t\wedge t_{i+1}^n)-Z(t\wedge t_i^n)])$$

$$= \sum_{i=0}^{\infty} Z^{-1}(t\wedge t_i^n)[Z(t\wedge t_{i+1}^n)-Z(t\wedge t_i^n)]$$

$$- \frac{1}{2}\sum_{i=0}^{\infty} (Z^{-1}(t\wedge t_i^n)[Z(t\wedge t_{i+1}^n)-Z(t\wedge t_i^n)])^2$$

+ higher order terms.

By the methods used in Karandikar [3], [5] it can be shown that a.s. J_n converges uniformly to $Z^{-1}\cdot Z - \frac{1}{2}< Z^{-1}\cdot Z, Z^{-1}\cdot Z >$. Denoting this process by U , we have $< U,U > = < Z^{-1}\cdot Z,Z^{-1}\cdot Z >$ and

$$U + \frac{1}{2}<U,U> = Z^{-1}\cdot Z .$$

Therefore, applying (5) and (3)

$$\varepsilon^*(U) = \varepsilon(U + \frac{1}{2}<U,U>) = \varepsilon(Z^{-1}\cdot Z) = Z$$

Which implies that $U=\mathcal{L}^*(Z)$ according to (6). The corollary is almost obvious, and left to the reader.

III. THE MAIN RESULT

We may now translate theorem 1 in the situation of path spaces, to get our main result. Let $P=P^1$ be a semimartingale measure on Ω , and let $A(t,\omega)$ be a G-valued, $\mathcal{F}_t$-adapted continuous process with finite variation paths, such that $A(0,\omega)=I$. Let φ be the mapping from Ω to Ω defined by

(12) $$X(t,\varphi(\omega)) = A(t,\omega)X(t,\omega)$$

We denote by P^2 the image law of P^1 under φ .

We now denote by α the process on W ($\mathcal{G}$-valued)

(13) $$\alpha_t(w) = \int_0^t A_s^{-1}(\varepsilon(w))dA_s(\varepsilon(w))$$

and by B_t the process on W ($\mathcal{G}$-valued) .

(14) $$B_t(w) = \int_0^t X_s^{-1}(\varepsilon(w))d\alpha_s(w)X_s(\varepsilon(w)) .$$

Finally, let $Q=Q^1$ be the image of P^1 under $\mathcal{L}$, and Q^2 be the image of Q^1 under the mapping ψ from W to W defined by

(15) $$Y(t,\psi(w)) = B(t,w) + Y(t,w) .$$

Then theorem 1 gives at once the following result :

THEOREM 3 . 1) $\psi(\mathcal{L}(\omega)) = \mathcal{L}(\varphi(\omega))$ a.s. P^1. Hence Q^2 is the image of P^2 under $\mathcal{L}$, and P^2 the image of Q^2 under $\mathcal{E}$.

2) $Q^2 \ll Q^1$ if and only if $P^2 \ll P^1$. Further, if $Q^2 \ll Q^1$, then

(16) $$\frac{dP^2}{dP^1}(\omega) = \frac{dQ^2}{dQ^1}(\mathcal{L}\omega) \quad \text{a.e. } P^1 .$$

Proof. The first statement follows from theorem 1 and theorem 2, 2), both processes being versions of the Stratonovich logarithm of AX under the law P^1. The other statements follow from the fact that P^1, P^2, Q^1, Q^2 are semimartingale measures, and therefore $\mathcal{E}$ and $\mathcal{L}$ are almost inverse to each other under any of them (theorem 2, (11)).

Let us apply this to the case of brownian motions on G : we choose some euclidean norm on the Lie algebra $\mathcal{G}$, and an orthonormal basis $(D_1,\dots,D_m)$ relative to it (we denote by $\tilde{D}_1,\dots\tilde{D}_m$ the corresponding left invariant vector fields on G). Let Q^1 be the probability law on W for which (Y_t) is a m-dimensional motion in the euclidean space $\mathcal{G}$ (that is, the components Y_t^i of Y_t in the basis D_i are independent real-valued standard brownian motions). Then, according to Ibéro [1], the Stratonovich exponential $\varepsilon^*(Y)$ is a G-valued brownian motion corresponding to the left invariant " laplacian" $L = \Sigma_1^m \tilde{D}_i^2$. Otherwise stated, the law P^1 of this G-valued brownian motion on Ω is the image of Q^1 under $\mathcal{E}$, and Q^1 is the image of P^1 under $\mathcal{L}$.

The classical Girsanov theorem asserts that the law Q^2 of Y+B under Q^1 will be absolutely continuous with respect to Q^1 , if the $\mathcal{G}$-valued process B on W is continuous, of finite variation, with a density b (progressively measurable) such that

(17) $$E_{Q^1}[\int_0^1 b_s \cdot dY_s - \int_0^1 |b_s|^2 ds] = 1$$

Here $|\ |$ is the euclidean norm in $\mathcal{G}$ and $\cdot$ denotes the corresponding scalar product. Besides that, the function under the sign E is equal to the density dQ^2/dQ^1 .

In the multiplicative set up, this corresponds to the absolute continuity with respect to P^1 of the law P^2 of the process AX , where A is a G-valued process given as a (deterministic) multiplicative integral :

(18) $$A(t,\omega) = I + \int_0^t A(u,\omega)d\alpha(u,\omega) \quad , \quad \alpha(t,\omega) = \int_0^t A^{-1}(u,\omega)dA(u,\omega)$$

where the $\mathcal{G}$-valued process $\alpha(t,\omega)$ on Ω is absolutely continuous with a progressively measurable density $a(t,\omega)$, and a,b are related by

(19) $$b(t, \mathcal{E}(\omega)) = X_t^{-1}(\omega)a(t,\omega)X_t(\omega)$$

Therefore, the condition for absolute continuity is the uniform integrability of the local martingale $\varepsilon(M)$, where

(20) $$M_t(\omega) = \int_0^t (X_s^{-1}a_sX_s)\cdot(X_s^{-1}dX_s)$$

and in that case, the density is equal to $\varepsilon(M)_1$. There is no obvious sufficient condition for absolute continuity, except the case where G is compact and the density a is bounded.

REFERENCES

[1] IBERO (M.). Intégrales stochastiques multiplicatives ét construction de diffusions sur un groupe de Lie. Bull. Soc. M. France, 100, 1976, p. 175-191

[2] ITO (K.) and WATANABE (S.). Introduction to stochastic differential equations. Proc. Intern. Symp. S.D.E. Kyoto, 1976. Kinokuniya publ. Tokyo.

[3] KARANDIKAR (R.L.). A.s. approximation results for multiplicative stochastic integrals. Sém. Prob. XVI, p. 384-391. Lecture Notes in M. n° 920, Springer-Verlag 1982.

[4] KARANDIKAR (R.L.). Multiplicative decomposition of non singular matrix valued continuous semimartingales. Ann. Prob., to appear.

[5] KARANDIKAR (R.L.). Pathwise stochastic calculus of continuous semimartingales. Ph.D. Thesis, Indian Statistical Institute, 1981.

Rajeeva L. Karandikar
Indian Statistical Institute
Stat-Math Division
Calcutta, India

in 1983 : Center for Stochastic Processes
University of North Carolina
Department of Statistics
Chapel Hill 27514, USA

The author wishes to thank Prof. K.R. Parthasarathy for repeatedly raising the above problem and Prof. P.A. Meyer for pointing out an error in an earlier version of the article.