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GIRSANOV TYPE FORMULA FOR A LIE GROUP VALUED BROWNIAN MOTION

by

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Let G be a Lie group of $d \times d$ matrices and X^1 be a G -valued continuous semimartingale, P^1 be its distribution on $\Omega = \mathcal{C}([0,1], G)$. Let $X^2 = AX^1$ be the left translate of X^1 by a G -valued adapted continuous process with finite variation paths, and P^2 be the distribution of X^2 on Ω . The question analogous to the classical Girsanov theorem is : Under what conditions on A, X is $P^2 \ll P^1$, and what is the density $\frac{dP^2}{dP^1}$?

We denote by \mathfrak{G} the Lie algebra of G , and by W the sample space $\mathcal{C}([0,1], \mathfrak{G})$. Using the pathwise integration formula (see Karandikar [3]) for multiplicative stochastic integration, we may define an "exponential" mapping $\mathcal{E} : W \rightarrow \Omega$ and a "logarithm" $\mathcal{L} : \Omega \rightarrow W$, which are independent of the choice of the laws and, in a reasonable sense, inverse to each other. Then we may denote by Y^1, Y^2 the processes $\mathcal{L}(X^1), \mathcal{L}(X^2)$ and by Q^1, Q^2 the corresponding laws on W . Next, using the "integration by parts formula" for multiplicative stochastic integration (Karandikar [4]), we show that $Y^2 = Y^1 + B$, where the process B is expressible in terms of A . Therefore the ordinary Girsanov theorem in the additive set-up will give conditions for the absolute continuity $Q^2 \ll Q^1$, and explicit expressions for the density. Returning to Ω by the exponential mapping \mathcal{E} , we can in this way solve a multiplicative Girsanov problem.

We study in particular the case where X^1 is a G -valued (multiplicative) brownian motion.

I. GENERALITIES

We first introduce some notation. Let U, V be continuous semimartingales (on some fixed probability space Ω with a filtration (\mathcal{F}_t) , not necessarily the same Ω as above), taking values in the space $L(d)$ of all $d \times d$ matrices. We denote by $\langle U, V \rangle$ the $L(d)$ valued process defined by

$$\langle U, V \rangle_j^i = \sum_k \langle U_k^i, V_j^k \rangle$$

The paths of $\langle U, V \rangle$ are continuous with finite variation, equal to 0 for $t=0$. We denote by $V \cdot U$ and $V \circ U$ the Ito stochastic integral and the Stratonovich stochastic integral of V with respect to U :

$$(V \cdot U)_t = \int_0^t V_s dU_s \quad (\text{matrix product}), \quad (V \circ U)_t = \int_0^t V_s \circ dU_s$$

and we denote by $U:V$, $U \circ V$ the similar integrals, with matrix products on the right side $((U:V)_t = \int_0^t (dU_s)V_s \dots)$. As usual, we may express Stratonovich integrals in terms of Ito integrals

$$(1) \quad V \circ U = V \cdot U + \frac{1}{2} \langle V, U \rangle, \quad U \circ V = U:V + \frac{1}{2} \langle U, V \rangle.$$

These formulas can be verified by looking at the entries and using the 1-dimensional relation (see Ito-Watanabe [2]). In the Lie group - Lie algebra setting, the Stratonovich integrals arise naturally.

We now assume that $U_0=0$. The Ito exponential of U , denoted by $\varepsilon(U)$, is the only solution to the stochastic differential equation

$$(2) \quad V = I + V \cdot U.$$

More precisely, this is the left exponential (see Karandikar [3]. The right exponential will not be used here). It can be shown that $\varepsilon(U)=V$ is invertible (see Karandikar [3]) and we can recover U from V by the formula

$$(3) \quad U = \mathcal{L}(V) = V^{-1} \cdot V \quad (\text{hence } \varepsilon(U)=\varepsilon(U') \Rightarrow U=U').$$

Similarly, we define the Stratonovich (left) exponential $\varepsilon^*(U)$ as the solution to

$$(4) \quad V = I + V \circ U$$

It can be easily seen that if V is a solution to (4), then $\langle V, U \rangle = V \cdot \langle U, U \rangle$, and therefore $V = I + V \cdot (U + \frac{1}{2} \langle U, U \rangle)$, hence

$$(5) \quad \varepsilon^*(U) = \varepsilon(U + \frac{1}{2} \langle U, U \rangle)$$

and $\varepsilon^*(U)$ is invertible. Just as above, one can recover U from $V=\varepsilon^*(U)$ by the formula

$$(6) \quad U = \mathcal{L}^*(V) = V^{-1} \circ V \quad (\text{hence } \varepsilon^*(U)=\varepsilon^*(U') \Rightarrow U=U').$$

Let U and U' denote two continuous semimartingales, such that $U_0=U'_0=0$, and let W denote $\varepsilon(U')$. Then we have the integration by parts formula for multiplicative stochastic integrals

$$(7) \quad \varepsilon(U + U' + \langle U, U' \rangle) = \varepsilon(W \cdot U; W^{-1}) \varepsilon(U').$$

This is a direct consequence of Ito's formula (Karandikar [4]). The same arguments with Stratonovich integrals in place of Ito's integrals will give

$$(8) \quad \varepsilon^*(U + U') = \varepsilon^*(W \circ U \circ W^{-1}) \varepsilon^*(U') \text{ with } W = \varepsilon^*(U').$$

Also, (8) can be deduced from (7) and (5).

We are going to apply this formula in the situation described in the introduction. Let X be a continuous semimartingale such that $X_0 = I$, and let A be a continuous semimartingale with finite variation paths, such that $A_0 = I$. We assume that these two processes take their values in the set of invertible matrices, and define

$$(9) \quad Y_t = \int_0^t X_s^{-1} \circ dX_s$$

$$(10) \quad B_t = \int_0^t (A_s X_s)^{-1} \circ dA_s X_s \quad (\text{Stieltjes integral})$$

Then the paths of B have finite variation, and we have :

THEOREM 1. $AX = \varepsilon^*(Y+B)$.

Proof. We have $X = \varepsilon^*(Y)$ according to (9) and (6). Similarly, we set $\alpha = A^{-1} \circ A$, so that $A = \varepsilon^*(\alpha)$. Then $AX = \varepsilon^*(\alpha) \varepsilon^*(Y)$, which we try to identify with the right side of (8). We must have $\varepsilon^*(U') = \varepsilon^*(Y)$, hence $U' = Y$, $W = X$. Then we must have $W \circ U \circ W^{-1} = \alpha$, and therefore since $W = X$, $U = X^{-1} \circ \alpha \circ X = X^{-1} \circ (A^{-1} \circ A) \circ X = (AX)^{-1} \circ A \circ X = B$. Note that we didn't really use in this proof the fact that A has finite variation.

II. CONSTRUCTION OF THE MAPS ε AND \mathcal{L}

Let W be the set of all continuous mappings $w : [0,1] \rightarrow L(d)$ such that $w(0) = 0$. We denote by Y_t the coordinate mapping $w \mapsto w(t)$ on W , and by \mathcal{G}_t the σ -field $\sigma(Y_s, s \leq t)$.

Let Ω be the set of all continuous mappings $\omega : [0,1] \rightarrow L(d)$ such that $\omega(0) = I$, and $\omega(t)$ is invertible for every t . The coordinate mappings and fields are denoted here by X_t and \mathcal{F}_t .

If one is interested in a particular pair (\mathcal{G}, G) , the mappings in W will be restricted to be \mathcal{G} -valued, and those in Ω to be G -valued. This makes no essential difference, as we shall see.

We say that a probability law on W (Ω) is a semimartingale measure if the corresponding coordinate process is a semimartingale (w.r. to the corresponding filtration, made right-continuous and complete).

Our aim in this section consists in constructing Borel mappings $\varepsilon : W \rightarrow \Omega$, $\mathcal{L} : \Omega \rightarrow W$ such that, for any semimartingale measure on W , $X \circ \varepsilon$ is a version of the Stratonovich exponential $\varepsilon^*(Y)$, and for any semimartingale measure on Ω , $Y \circ \mathcal{L}$ is a version of the Stratonovich "logarithm" $\mathcal{L}^*(X)$. These mappings, however, do not depend on the choice of a measure on W or Ω .

For $n \geq 1$, $w \in W$, $\omega \in \Omega$, define $s_i^n(w)$ and $t_i^n(\omega)$ for $i \geq 0$ inductively by

$$s_0^n(w) = t_0^n(\omega) = 0 \quad \text{and for } i \geq 0$$

$$s_{i+1}^n(w) = \inf \{ s \geq s_i^n(w) : |Y(s, w) - Y(s_i^n(w), w)| \geq 2^{-n} \text{ or } s \geq 1 \}$$

$$t_{i+1}^n(\omega) = \inf \{ t \geq t_i^n(\omega) : |X(t, \omega) - X(t_i^n(\omega), \omega)| \geq 2^{-n} \text{ or } \\ |X^{-1}(t, \omega) - X^{-1}(t_i^n(\omega), \omega)| \geq 2^{-n} \text{ or } \\ |X^{-1}(t_i^n(\omega))X(t, \omega) - I| \geq 2^{-n} \text{ or } t \geq 1 \}.$$

Here the norm $|\cdot|$ is chosen so that the logarithm of a matrix (i.e. the inverse mapping of the usual matrix exponential \exp) is defined on the neighbourhood $|x - I| < 1$ of the identity. We now set for $s, t \in [0, 1]$

$$\mathcal{L}_n(t, \omega) = \sum_{i \geq 0} \log(X^{-1}(t \wedge t_i^n(\omega), \omega) X(t \wedge t_{i+1}^n(\omega), \omega))$$

$$\mathcal{E}_n(s, w) = \prod_{i \geq 0} \exp(Y(s \wedge s_{i+1}^n(w), w) - Y(s \wedge s_i^n(w), w))$$

It is easy to see that if ω is G -valued, then $\mathcal{L}_n(\cdot, \omega)$ is G -valued, and similarly in the other direction. Let

$$W_0 = \{ w \in W : \mathcal{E}_n(\cdot, w) \text{ converges uniformly} \}$$

$$\Omega_0 = \{ \omega \in \Omega : \mathcal{L}_n(\cdot, \omega) \text{ converges uniformly} \}$$

We denote the corresponding limits by $\mathcal{E}(w) = \mathcal{E}(\cdot, w)$ and $\mathcal{L}(\omega) = \mathcal{L}(\cdot, \omega)$; outside W_0 or Ω_0 we set $\mathcal{E}(t, w) = I$, $\mathcal{L}(t, \omega) = 0$ for all t . Of course, since the coordinate mappings are denoted by Y_t on W , X_t on Ω , we may write $Y_t(\mathcal{L}(\omega))$ instead of $\mathcal{L}(t, \omega)$, $X_t(\mathcal{E}(w))$ instead of $\mathcal{E}(t, w)$.

THEOREM 2. Let Z be a continuous $L(d)$ -valued semimartingale defined on some filtered probability space $(\theta, \mathcal{H}, (\mathcal{H}_t)_{t \leq 1}, \mu)$.

1) Assume that $Z_0 = 0$. Then for μ -a.e. $\theta \in \theta$ the path $Z_\cdot(\theta)$ belongs to W_0 and the path $\mathcal{E}^*(Z)(\cdot, \theta)$ of the Stratonovich exponential $\mathcal{E}^*(Z)$ is equal to $\mathcal{E}(Z_\cdot(\theta))$.

2) Assume that $Z_0 = I$ and Z takes its values in the set of invertible matrices. Then for μ -a.e. $\theta \in \theta$ the path $Z_\cdot(\theta)$ belongs to Ω_0 , and the path $\mathcal{L}^*(Z)(\cdot, \theta)$ of the Stratonovich logarithm $\mathcal{L}^*(Z)$ is equal to $\mathcal{L}(Z_\cdot(\theta))$.

COROLLARY. Let P be a semimartingale measure on Ω (Q be a semimartingale measure on W). Then the image measure $\mathcal{L}(P)$ is a semimartingale measure on W ($\mathcal{E}(Q)$ a semimartingale measure on Ω) and we have

$$(11) \quad \mathcal{E}(\mathcal{L}(\omega)) = \omega \text{ a.s. } P \quad (\mathcal{L}(\mathcal{E}(w)) = w \text{ a.s. } Q).$$

Proof. The proof relative to the exponential is outlined in Karandikar [3] (Sém. Prob. XVI), and fully given in Karandikar [5]. Keeping the notation $t_i^n(\theta)$ for $t_i^n(Z,(\theta))$, the statement amounts to the fact that

$$J_n(t, \theta) = \sum_{i \geq 0} \log(Z^{-1}(t \wedge t_i^n(\theta), \theta) Z(t \wedge t_{i+1}^n(\theta), \theta))$$

converges μ -a.s. to $J^*(Z)(t, \theta)$, uniformly in $t \in [0, 1]$. We rewrite $J_n(t, \theta)$ as

$$\begin{aligned} & \sum_{i=0}^{\infty} \log(I + Z^{-1}(t \wedge t_i^n) [Z(t \wedge t_{i+1}^n) - Z(t \wedge t_i^n)]) \\ &= \sum_{i=0}^{\infty} Z^{-1}(t \wedge t_i^n) [Z(t \wedge t_{i+1}^n) - Z(t \wedge t_i^n)] \\ & \quad - \frac{1}{2} \sum_{i=0}^{\infty} (Z^{-1}(t \wedge t_i^n) [Z(t \wedge t_{i+1}^n) - Z(t \wedge t_i^n)])^2 \\ & \quad + \text{higher order terms.} \end{aligned}$$

By the methods used in Karandikar [3], [5] it can be shown that a.s. J_n converges uniformly to $Z^{-1} \cdot Z - \frac{1}{2} \langle Z^{-1} \cdot Z, Z^{-1} \cdot Z \rangle$. Denoting this process by U , we have $\langle U, U \rangle = \langle Z^{-1} \cdot Z, Z^{-1} \cdot Z \rangle$ and

$$U + \frac{1}{2} \langle U, U \rangle = Z^{-1} \cdot Z.$$

Therefore, applying (5) and (3)

$$\varepsilon^*(U) = \varepsilon(U + \frac{1}{2} \langle U, U \rangle) = \varepsilon(Z^{-1} \cdot Z) = Z$$

Which implies that $U = J^*(Z)$ according to (6). The corollary is almost obvious, and left to the reader.

III. THE MAIN RESULT

We may now translate theorem 1 in the situation of path spaces, to get our main result. Let $P = P^1$ be a semimartingale measure on Ω , and let $A(t, \omega)$ be a G -valued, \mathcal{F}_t -adapted continuous process with finite variation paths, such that $A(0, \omega) = I$. Let φ be the mapping from Ω to Ω defined by

$$(12) \quad X(t, \varphi(\omega)) = A(t, \omega) X(t, \omega)$$

We denote by P^2 the image law of P^1 under φ .

We now denote by α the process on W (G -valued)

$$(13) \quad \alpha_t(w) = \int_0^t A_s^{-1}(\mathcal{E}(w)) dA_s(\mathcal{E}(w))$$

and by B_t the process on W (G -valued).

$$(14) \quad B_t(w) = \int_0^t X_s^{-1}(\mathcal{E}(w)) d\alpha_s(w) X_s(\mathcal{E}(w)).$$

Finally, let $Q = Q^1$ be the image of P^1 under φ , and Q^2 be the image of Q^1 under the mapping ψ from W to W defined by

$$(15) \quad Y(t, \psi(w)) = B(t, w) + Y(t, w).$$

Then theorem 1 gives at once the following result :

THEOREM 3 . 1) $\Psi(\mathfrak{L}(\omega)) = \mathfrak{L}(\Psi(\omega))$ a.s. P^1 . Hence Q^2 is the image of P^2 under \mathfrak{L} , and P^2 the image of Q^2 under \mathfrak{E} .

2) $Q^2 \ll Q^1$ if and only if $P^2 \ll P^1$. Further, if $Q^2 \ll Q^1$, then

$$(16) \quad \frac{dP^2}{dP^1}(\omega) = \frac{dQ^2}{dQ^1}(\mathfrak{L}\omega) \quad \text{a.e. } P^1.$$

Proof. The first statement follows from theorem 1 and theorem 2, 2), both processes being versions of the Stratonovich logarithm of AX under the law P^1 . The other statements follow from the fact that P^1, P^2, Q^1, Q^2 are semimartingale measures, and therefore \mathfrak{E} and \mathfrak{L} are almost inverse to each other under any of them (theorem 2, (11)).

Let us apply this to the case of brownian motions on G : we choose some euclidean norm on the Lie algebra \mathfrak{G} , and an orthonormal basis (D_1, \dots, D_m) relative to it (we denote by $\tilde{D}_1, \dots, \tilde{D}_m$ the corresponding left invariant vector fields on G). Let Q^1 be the probability law on W for which (Y_t) is a m -dimensional motion in the euclidean space \mathfrak{G} (that is, the components Y_t^i of Y_t in the basis D_i are independent real-valued standard brownian motions). Then, according to Ibéro [1], the Stratonovich exponential $\varepsilon^*(Y)$ is a G -valued brownian motion corresponding to the left invariant "laplacian" $L = \sum_1^m \tilde{D}_i^2$. Otherwise stated, the law P^1 of this G -valued brownian motion on Ω is the image of Q^1 under \mathfrak{E} , and Q^1 is the image of P^1 under \mathfrak{L} .

The classical Girsanov theorem asserts that the law Q^2 of $Y+B$ under Q^1 will be absolutely continuous with respect to Q^1 , if the \mathfrak{G} -valued process B on W is continuous, of finite variation, with a density b (progressively measurable) such that

$$(17) \quad E_{Q^1} \left[\int_0^1 b_s \cdot dY_s - \int_0^1 |b_s|^2 ds \right] = 1$$

Here $||$ is the euclidean norm in \mathfrak{G} and \cdot denotes the corresponding scalar product. Besides that, the function under the sign E is equal to the density dQ^2/dQ^1 .

In the multiplicative set up, this corresponds to the absolute continuity with respect to P^1 of the law P^2 of the process AX , where A is a G -valued process given as a (deterministic) multiplicative integral:

$$(18) \quad A(t, \omega) = I + \int_0^t A(u, \omega) d\alpha(u, \omega), \quad \alpha(t, \omega) = \int_0^t A^{-1}(u, \omega) dA(u, \omega)$$

where the \mathfrak{G} -valued process $\alpha(t, \omega)$ on Ω is absolutely continuous

with a progressively measurable density $a(t, \omega)$, and a, b are related by

$$(19) \quad b(t, \mathcal{F}(\omega)) = X_t^{-1}(\omega) a(t, \omega) X_t(\omega)$$

Therefore, the condition for absolute continuity is the uniform integrability of the local martingale $\varepsilon(M)$, where

$$(20) \quad M_t(\omega) = \int_0^t (X_s^{-1} a_s X_s) \cdot (X_s^{-1} dX_s)$$

and in that case, the density is equal to $\varepsilon(M)_1$. There is no obvious sufficient condition for absolute continuity, except the case where G is compact and the density a is bounded.

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