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ROLLING WITH 'SLIPPING' : I

by

Gareth C. Price and David Williams

Throughout this note, ∂ will denote the Stratonovich differential, and d will denote the Itô differential.

Let B be a $BM(\mathbb{R}^3)$, a Brownian motion on $\mathbb{R}^3$. Let Z be a process on $\mathbb{R}^3$ with $|Z(0)| = 1$ and

$$\partial Z = Z \times \partial B, \tag{1}$$

where $\times$ is the vector product. Then Z is a $BM(S^2)$, a Brownian motion on the unit sphere S^2 in $\mathbb{R}^3$.

Though the representation (1) of a $BM(S^2)$ is very simple, it appears to suffer from the disadvantage that "there is too much freedom in B". Only the component

$$\partial Y = \partial B - (Z.\partial B)Z$$

of ∂B 'tangential' to S^2 really matters. Now, of course, we have

$$\partial Y = (\partial Z) \times Z$$

and the proper driving equation for Z:

$$\partial Z = Z \times \partial Y.$$

The apparent defect of redundancy in (1) is illusory. We have the very satisfying situation that the equation

$$\partial b = Z.\partial B, \qquad b(0) = 0,$$

defines a $BM(\mathbb{R}^1)$ process b <u>independent</u> of the process Z. We think of b as providing the information about B which is missing from Z.

Suppose now that $\tilde{Z}$ is a $BM(S^2)$ adapted to the filtration of Z. Let

$$\partial\tilde{Y} = (\partial\tilde{Z}) \times \tilde{Z}.$$

THEOREM. <u>We have</u>

$$d\tilde{Y} = HdY \qquad \text{(an } \underline{\text{Itô}} \text{ equation)},$$

where

(i) for each t, H_t is an orthogonal transformation such that

$$H_t Z_t = \tilde{Z}_t,$$

(ii) the process $H = \{H_t\}$ is Z previsible.

Proof that Z is a $BM(S^2)$. We have $\partial(Z.Z) = 2Z.\partial Z = 0$, so that Z stays on S^2. It is clear that Z is invariant under $O(3)$, whence Dynkin's formula shows that the infinitesimal generator of Z is a constant multiple of the spherical Laplacian. Of course, the usual argument for Stratonovich equations shows that if G is the generator of Z, then, with $\partial_k = \partial/\partial Z_k$,

$$2G = (Z_3\partial_2 - Z_2\partial_3)^2 + (Z_1\partial_3 - Z_3\partial_1)^2 + (Z_2\partial_1 - Z_1\partial_2)^2,$$

and this 'squared angular momentum' operator is known to be the Laplacian on S^2. □

Proof that b is a $BM(\mathbb{R}^1)$ independent of Z. The generator of the 4-dimensional process (b,Z) is

$$\tfrac{1}{2}(Z_1\partial_b + Z_3\partial_2 - Z_2\partial_3)^2 + \text{two cyclic permutations},$$

where $\partial_b = \partial/\partial b$. This operator splits as

$$\tfrac{1}{2}\partial_b^2 + G.$$ □

Martingale characterization of $BM(S^2)$. A process U on $\mathbb{R}^3$ with $|U(0)| = 1$ is a $BM(S^2)$ if and only if U is a continuous semimartingale such that

(i) $dU + Udt = dM$, where M is a martingale,

(ii) $d\langle U_m, U_n\rangle = (\delta_{mn} - U_m U_n)dt$.

A 'trick' way to prove this is to apply stereographic projection of S^2 to $\mathbb{R}^2 \cup \{\infty\}$, and then apply the Stroock-Varadhan result to the resulting process on $\mathbb{R}^2$.

<u>Proof of Theorem</u>. By Jacod's Theorem that "martingale characterization implies martingale representation" (see [1], and especially result (11) there), we have

$$d\tilde{Z}_m + \tilde{Z}_m dt = A^{(m)}.dN \tag{2}$$

where $A^{(m)}$ is a Z previsible process with values in $\mathbb{R}^3$, and

$$dN = dZ + Zdt = Z \times dB. \tag{3}$$

Thus,

$$d\tilde{Z}_m + \tilde{Z}_m dt = K^{(m)}.dB,$$

where

$$K^{(m)} = A^{(m)} \times Z.$$

We have

$$d\left\langle \tilde{Z}_m, \tilde{Z}_n \right\rangle /dt = \delta_{mn} - \tilde{Z}_m \tilde{Z}_n = K^{(m)}.K^{(n)},$$

and, also, the vectors $K^{(m)}$ are perpendicular to Z. But, the vectors

$$J^{(1)} = (0, -\tilde{Z}_3, \tilde{Z}_2),\ J^{(2)} = (\tilde{Z}_3, 0, -\tilde{Z}_1),\ J^{(3)} = (-\tilde{Z}_2, \tilde{Z}_1, 0) \tag{4}$$

satisfy

$$J^{(m)}.J^{(n)} = \delta_{mn} - \tilde{Z}_m \tilde{Z}_n,$$

and the vectors $J^{(m)}$ are perpendicular to $\tilde{Z}$. Hence, for some (in fact, unique) orthogonal matrix H_t with $H_t Z_t = \tilde{Z}_t$,

$$K_t^{(m)} = J_t^{(m)} H_t.$$

The process H is Z previsible, and

$$d\tilde{B} = HdB$$

defines a B Brownian motion. On combining (2), (3), and (4), we find that

$$d\tilde{Z} + \tilde{Z}dt = \tilde{Z} \times d\tilde{B}, \quad \partial\tilde{Z} = \tilde{Z} \times \partial\tilde{B}.$$

Of course, since H is orthogonal,

$$\tilde{Z}.d\tilde{B} = Z.dB,$$

so that, with an obvious notation, we have the satisfying relation

$$\tilde{b} = b.$$

Next,

$$\partial\tilde{Y} = (\partial\tilde{Z}) \times \tilde{Z}, \qquad d\tilde{Y} = (d\tilde{Z}) \times \tilde{Z},$$

so that

$$d\tilde{Y} = (\tilde{Z} \times d\tilde{B}) \times \tilde{Z} = d\tilde{B} - (\tilde{Z}.d\tilde{B})\tilde{Z}$$

$$= d\tilde{B} - (Z.dB)\tilde{Z} = HdY,$$

and our theorem is proved. □

The title indicates the 'frame-bundle' significance, something we hope to explain in a wider context. There seem to be some very nice - and potentially important - applications.

Acknowledgement. We wish to thank Professor A. Truman for helpful discussions on these topics. A.T. and D.W. intend to publish follow-up work. [The representation

$$dX = n(X) \times dB$$

for Brownian motion on a surface $(n(\cdot)$ is the unit normal) allows some neat formulae.]

REFERENCE

[1] J. JACOD, A general theorem of representation for martingales, Probability (ed. J.L. Doob), Proc. Symp. Pure Math. Amer. Math. Soc. XXXI (1977), 37-54.

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