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AN EXTENSION OF MOTOO'S THEOREM

Joseph Glover*

Let $\mathfrak{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a right process on a Lusin topological space $(E, \mathcal{E})$, and let $\mathcal{F}^* = \sigma\{f(X_s): s \geq 0$, f is universally measurable on E$\}$. Let A_t and B_t be $\mathcal{F}^*$-measurable continuous raw additive functionals. If A_t and B_t are $(\mathcal{F}_t)$-adapted and $dA_t << dB_t$ almost surely, then a very useful theorem (due first to Motoo and extended by Getoor) says that there is a positive function h so that $dA_t = h(X_t)\, dB_t$ almost surely. We prove an extension of this theorem by weakening the hypothesis of adaptedness.

Define $[B] = \{(t,\omega): B_t(\omega) < B_{t+e}(\omega)$ for all $e>0\}$. A process $C_t \in \mathcal{B}(R^+) \times \mathcal{F}^*$ is said to be [B]-intrinsically predictable if whenever $T \in \mathcal{F}^*$ is a positive random variable with $[T] \subset [B]$, then $C_t(k_{T(\omega)}\omega) = C_t(\omega)$ for all $t \leq T(\omega)$, for all $\omega \in \Omega$. Here, k_t is the killing operator of Azema [1].

(1) <u>Theorem</u>. <u>Let</u> A_t <u>and</u> B_t <u>be</u> σ-<u>integrable</u> $\mathcal{B}(R^+) \times \mathcal{F}^*$-<u>measurable continuous raw additive functionals</u>. <u>If</u> A_t <u>and</u> B_t <u>are</u> [B]-<u>intrinsically predictable and</u> $dA_t << dB_t$ <u>almost surely, then there is a positive universally measurable function</u> h <u>on</u> E <u>so that</u> $A_t = \int_0^t h(X_s)\, dB_s$ <u>almost surely</u>.

Examples of such raw additive functionals can be found in [2] and [3], where a theory of time change by the inverses of such additive functionals is discussed. The proof of the theorem given below is in much the same vein as those given in Section 1 of [3], but the objective and hypotheses are a bit different.

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Let $\mathcal{O}(\mathcal{F}_t)$ denote the collection of $(\mathcal{F}_t)$-optional processes. If C_t is an increasing process, let C_t° denote its dual optional projection. As usual, $\mathcal{F}^\circ = \sigma\{X_s: s \geq 0\}$, $\mathcal{E}^e = \sigma\{f: f$ is 1-excessive for $X\}$, $\mathcal{E}^* = \sigma\{f: f$ is universally measurable on $E\}$, $\mathcal{F}^e = \sigma\{f(X_s): s \geq 0, f \in \mathcal{E}^e\}$.

<u>Proof</u>. Let $T_t = \inf\{s: B_s > t\}$. Then $T_{t+s} = T_t + T_s \circ \theta_{T(t)}$. For each s, set $\mathcal{H}_s = \sigma\{H \in \mathcal{F}$: there exists $Z \in \mathcal{O}(\mathcal{F}_t)$ with $H = Z_{T(s)}$ on $\{T_s < \infty\}\}$.

(2) <u>Lemma</u>. <u>$(\mathcal{H}_s)$ is an increasing family of σ-algebras</u>.

<u>Proof</u>. If $B_t \in \mathcal{F}^e$ for each t, the proof is very simple and goes as follows. Fix $t > 0$ and $s > 0$ and define $V_r = \inf\{u \leq r: B_u \circ k_r > s\}$. It is simple to check that $V_r \in \mathcal{O}(\mathcal{F}_r)$ and $V_{T(t+s)} = \inf\{u \leq T_{t+s}: B_u \circ k_{T(t+s)} > s\} = \inf\{u \leq T_{t+s}: B_u > s\}$ on $\{T_{t+s} < \infty\}$ by the hypothesis of [B]-intrinsic predictability. Thus $V_{T(t+s)} = T_s$ on $\{T_{t+s} < \infty\}$, and it follows that $\mathcal{H}_s \subset \mathcal{H}_{t+s}$. Assuming that B_t is only $\mathcal{F}^*$-measurable complicates the proof in only technical ways: the full proof is given in (1.3) of [3]. ✠

(3) <u>Lemma</u>. <u>There are</u>

(i) <u>a kernel</u> K <u>from</u> $(E, \mathcal{E}^*)$ <u>to</u> $(\Omega, \mathcal{F}^*)$, <u>and</u>

(ii) <u>for each</u> $x \in E$, <u>a set</u> $M^x \subset R^+$ <u>of full Lebesgue measure</u>,

<u>so that</u> $E^x[G \circ \theta_{T(t)} | \mathcal{H}_t] = KG(X_{T(t)})$ <u>almost surely</u> (P^x) <u>for each</u> $t \in M^x$ for all G $b\mathcal{F}^{\circ +}$.

<u>Proof</u>. In assuming that A_t and B_t are σ-integrable, we mean there is a strictly positive optional process (R_t) so that $E^x \int R_t \, dA_t < \infty$ and $E^x \int R_t \, dB_t < \infty$ for all x. (If A_t and B_t are $\mathcal{F}^e$-measurable for each t, they are always σ-integrable: take $R_t = \exp(-A_t \circ k_t - B_t \circ k_t))$. Let $Z_t \in b\mathcal{O}(\mathcal{F}_t)^+$, and let $G \in b\mathcal{F}^{\circ +}$. Then

(4) $$E^x \int (RZ)_{T(t)} G \circ \theta_{T(t)} dt = E^x \int (RZ)_t G \circ \theta_t dB_t.$$

Set $D_t = \int_0^t G \circ \theta_s \, dB_s$. Since $dD_t^\circ << dB_t^\circ$ and both D_t° and B_t° are continuous additive functionals of X_t, there is a function $f^G \in \mathcal{E}^{e+}$ so that we may rewrite the right hand side of (4) as

$$E^x \int (RZ)_t f^G(X_t)\, dB_t = E^x \int (RZ)_{T(t)} f^G(X_{T(t)})\, dt.$$

Standard arguments yield existence of a kernel K from $(E, \mathcal{E}^*)$ to $(\Omega, \mathcal{F}^*)$ so that

$$E^x \int (RZ)_{T(t)} G\circ\theta_{T(t)}\, dt = E^x \int (RZ)_{T(t)} KG(X_{T(t)})\, dt.$$

Fix x in E. There is an $(\mathcal{F}_t)$-optional process (W_t^x) so that $W_{T(t)}^x = e^{-at}$ on $\{T_t<\infty\}$. (See Lemma (1.4) in [3]. If B_t is assumed to be $\mathcal{F}^e$-measurable, then $W_t^x = \exp(-aB_t\circ k_t)$). Replacing Z_t with $Z_tW_t^x$ and applying Fubini's theorem, we obtain

(5) $$\int e^{-at} E^x[(RZ)_{T(t)} G\circ\theta_{T(t)}]\, dt = \int e^{-at} E^x[(RZ)_{T(t)} KG(X_{T(t)})]\, dt.$$

There is a separable σ-algebra $\mathcal{O}^x \subset \mathcal{O}(\mathcal{F}_t)$ so that for each process $Y_t \in \mathcal{O}(\mathcal{F}_t)$ there is a process $Y_t^x \in \mathcal{O}^x$ so that Y_t and Y_t^x are P^x-indistinguishable ([4], p.366). Let $(Z_t^{x,n})_{n\geq 1}$ be an algebra of bounded processes generating $\mathcal{O}^x$, and let $(G^m)_{m\geq 1}$ be an algebra of bounded random variables generating $\mathcal{F}°$. Equation (5) implies that for each n and m, there is a set $M_{n,m}^x \subset R^+$ of full Lebesgue measure so that for each $t \in M_{n,m}^x$, $E^x[(RZ^{x,n})_{T(t)} G^m\circ\theta_{T(t)}] = E^x[(RZ^{x,n})_{T(t)} KG^m(X_{T(t)})]$. Thus there is <u>one</u> set $M^x \subset R^+$ of full Lebesgue measure so that for each $t \in M^x$, $E^x[Z_{T(t)} G\circ\theta_{T(t)}] = E^x[Z_{T(t)} KG(X_{T(t)})]$ for all $Z_t \in \mathcal{O}(\mathcal{F}_t)^+$ and for all $G \in \mathcal{F}^{\circ +}$. It follows that $E^x[G\circ\theta_{T(t)} | \mathcal{H}_t] = KG(X_{T(t)})$ almost surely (P^x) for each $t \in M^x$. ✝

Now let $C_t = A_{T(t)}$. Then $C_t \in \mathcal{F}^*$ for each t, $C_{t+s} = C_t + C_s\circ\theta_{T(t)}$, and $dC_t << dt$. If we set $Z_t = \liminf_{n\to\infty} n(C_{t+1/n} - C_t)$, then $Z_t \in \mathcal{F}^*$ for each t, and $Z_{t+s} = Z_t\circ\theta_{T(s)}$. By Lebesgue's differentiation theorem, $C_t = \int_0^t Z_s\, ds$. Let ν be the measure on $(\Omega, \mathcal{F}°)$ defined by setting $\nu(H) = E^x[H\circ k_{T(t)}]$ for all $H \in \mathcal{F}^{\circ +}$. Since $A_{T(t)} \in \mathcal{F}^*$, there is a random variable $Q \in \mathcal{F}°$ so that $Q = A_{T(t)}$ almost surely (ν). Let (Y_s) be the $(\mathcal{F}_s)$-predictable process $(Q\circ k_s)$. Then $Y_{T(t)} = Q\circ k_{T(t)} \in \mathcal{H}_t$. Since $Q\circ k_{T(t)} = A_{T(t)}\circ k_{T(t)} = A_{T(t)}$ almost surely (P^x), we conclude that for each s, Z_s differs from an element of $\mathcal{H}_{s+}$ by a P^x-null set. Set $g(x) = K(x,Z_0) \in \mathcal{E}^*$, and let μ be the measure on $(E, \mathcal{E})$ defined by setting $\mu(f) = E^x[f(X_{T(t)})]$. Since $g \in \mathcal{E}^*$, there is a function $h \in \mathcal{E}$ so that $\mu(|h-g|) = 0$. Thus $E^x[|h(X_{T(t)}) - g(X_{T(t)})|] = 0$ and $h(X_{T(t)}) \in \mathcal{H}_t$

since $h(X_t)$ is an optional process. Therefore, if $t \in M^x$, $g(X_{T(t)}) = E^x[g(X_{T(t)})|\mathcal{H}_t] = E^x[Z_0 \circ \theta_{T(t)}|\mathcal{H}_t] = E^x[Z_t|\mathcal{H}_t]$ almost surely (P^x). Recall that there is a set $N^x \subset R^+$ so that $R^+ - N^x$ is countable and $E^x[H|\mathcal{H}_t] = E^x[H|\mathcal{H}_{t+}]$ almost surely (P^x) for all $H \in b\mathcal{F}^{\circ +}$ for each $t \in N^x$. Thus if $t \in M^x \cap N^x$, $g(X_{T(t)}) = Z_t$ almost surely (P^x) since Z_t is in the P^x-completion of $\mathcal{H}_{t+}$. Since $M^x \cap N^x$ is of full Lebesgue measure, standard Fubini arguments yield that $C_t = \int_0^t g(X_{T(s)})\, ds$, and it follows that $A_t = \int_0^t g(X_s)\, dB_s$. This completes the proof of Theorem (1). ✛

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