SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 16 (1982), p. 392-399

<http://www.numdam.org/item?id=SPS_1982__16__392_0>

THERE EXISTS NO ULTIMATE SOLUTION TO SKOROKHOD'S PROBLEM

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Abstract

Let (X,Y) be a mean zero martingale pair, i.e., X and Y possess mean zero and $E(Y|X) = X$ a.s.. It has been proved in various ways that (1) there exist stopping times τ on Brownian motion $\{B(t);\ t\geq 0\}$ such that $B(\tau)$ is distributed like X and $\{B(t\wedge\tau);\ t\geq 0\}$ is uniformly integrable; and (2) for any such τ there exist stopping times τ' such that $\tau \leq \tau'$ a.s., $(B(\tau), B(\tau'))$ is distributed like (X,Y) and $\{B(t\wedge\tau');\ t\geq 0\}$ is uniformly integrable. In other words (to explain the role of uniform integrability), a martingale pair can be embedded in a piece of Brownian motion that is itself a martingale.

We will show that unless Y lives on one or two points, there can exist <u>no</u> stopping time τ' with $\{B(t\wedge\tau');\ t\geq 0\}$ uniformly integrable and $B(\tau')$ distributed as Y, such that whenever (X,Y) is a martingale pair there exist τ with $\tau\leq\tau'$ a.s. and $B(\tau)$ distributed as X.

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1. INTRODUCTION

Let F be a distribution with mean zero and finite variance. It has been shown (Skorokhod [12], Dubins [3], Root [9], Chacon and Walsh [2], Azema and Yor [1] and others) that for some stopping time (st) τ of finite expectation on Brownian Motion starting at zero (BM) $\{B(t);\ t \geq 0\}$, $B(\tau)$ is distributed ($\sim$) F. This embeddability of F has been extended to more general Markov processes (Rost [10], Meyer [7]).

Monroe [8] has shown that if F has mean zero (but not necessarily finite variance), then $B(\tau) \sim F$ for some τ that makes $\{B(t\wedge\tau);\ t\geq 0\}$ uniformly integrable. Monroe proves this property of τ to be equivalent to τ being <u>minimal</u>, ..e. $B(\tau)$ has mean zero and for no $\tau' \leq \tau$ (except τ itself) is $B(\tau')\sim F$. An interesting property of minimality is that if $B(\tau)$ has mean zero and finite variance, then τ is minimal if and only if it has finite mean. The mean of a minimal stopping time is always equal to the variance of the variable it embeds.

A pair of distributions (F,G) <u>admits a martingale</u> (or, <u>belongs to</u> M), if F and G have expectations zero and for some two random variables X and Y on some space, $X \sim F$, $Y \sim G$ and $E(Y|X) = X$ a.s.. Equivalently, (see Meyer [6], $(F,G) \in M$ if for all real x, $\phi_F(x) \geq \phi_G(x)$, where

$$(1) \qquad \phi_{\mathcal{L}(Z)}(x) = -E|Z-x|.$$

Equivalently, $EX = EY = 0$ and $E\psi(X) \leq E\psi(Y)$ for all nonnegative non-decreasing convex functions ψ. For more on ϕ, see Chacon and Walsh [2]. Further details on convex inequalities can be found in Meilijson and Nádas [5] and in Rost [11], and their relation to extremal martingales in Dubins and Gilat [4] and in Azema and Yor [1].

It is clear from the proofs of embeddability that if $(F,G) \in M$ then for every minimal st τ with $B(\tau) \sim F$ there exists a minimal st τ' with $B(\tau') \sim G$ and $\tau \leq \tau'$ a.s.. A minimal st τ is said to be ultimate if for every distribution F with $(F, \mathcal{L}(B(\tau))) \in M$ there exists a st τ' with $B(\tau') \sim F$ and $\tau' \leq \tau$ a.s.

Let

(2) $\tau_A = \inf \{t \geq 0 \mid B(t) \in A\}$.

THEOREM. A stopping time τ is ultimate if and only if for some $a \leq 0 \leq b$, $\tau = \tau_{\{a,b\}}$ a.s..

2. PROOF OF THE THEOREM.

A distribution F has atomic ends if for some $a < b$, (the "ends"),

(3) $0 = F(a-) < F(a) \leq F(b-) < F(b) = 1$.

If strict inequality occurs throughout (3), we will say that F has non exclusive atomic ends. We will prove in lemma 2 that if τ is an ultimate st and $B(\tau)$ is not supported by one or two points, then there exists an ultimate st τ' such that $\mathcal{L}(B(\tau'))$ has non exclusive atomic ends. Lemma 4 will show this to be an impossibility.

LEMMA 1. For a distribution function G with mean zero for which there exist real numbers a and b with ess $\inf(G) < a < 0 < b <$ ess $\sup(G)$, let $0 < u < v < 1$ be defined by

(4) $E(G^{-1}(U) \mid U \leq u) = a, \; E(G^{-1}(U) \mid U \geq v) = b,$

where $U \sim U[0,1]$, and let $G_{(a,b)}$ be the distribution of a random variable X with

$$(5) \qquad X = \begin{cases} a & U \leqslant u \\ b & U \geqslant v \\ G^{-1}(U) & \text{otherwise} \end{cases}$$

$$= E(G^{-1}(U) \mid 1_{\{U \le u\}},\ 1_{\{U \le v\}},\ \max(u, \min(U,v))).$$

Then $(G_{(a,b)}, G) \varepsilon M$ and whenever F is supported by [a,b] and $(F,G) \varepsilon M$, then $(F, G_{(a,b)}) \varepsilon M$.

Proof. As mentioned in the introduction, the test for belonging to M is a pointwise inequality of the functions ϕ. This function ϕ (see Chacon and Walsh [2] is concave and is asymptotic to $-|x|$ as $|x| \to \infty$. It agrees with $-|x|$ outside any interval supporting the distribution. $\phi_{G_{(a,b)}}$ agrees with ϕ_G on $[G^{-1}(u), G^{-1}(v)]$ and is linear on $[a, G^{-1}(u)]$ and on $[G^{-1}(v), b]$. These two linear pieces are tangents to ϕ_G, and thus $\phi_{G_{(a,b)}}$ is the minimal concave function that exceeds ϕ_G and agrees with $-|x|$ outside [a,b].

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LEMMA 2.

(i) For every $a \le 0 \le b, \tau_{\{a,b\}}$ is ultimate.

(ii) If τ is ultimate, so is $\min(\tau, \tau_{\{a,b\}})$.

(iii) If τ is ultimate and $B(\tau)$ is not supported by one or two points, then there exists an ultimate τ' such that $B(\tau)$ has non exclusive atomic ends.

Proof. (i) If $a.b = 0$, the result is clear.
If $a.b > 0$ then every F with $(F,\mathcal{L}(B(\tau_{\{a,b\}}))) \in M$ is supported by $[a,b]$ and every minimal st τ for which $B(\tau) \sim F$ satisfies $\tau \leq \tau_{\{a,b\}}$ a.s. (ii) Let τ be ultimate. Denote $G = \mathcal{L}(B(\tau))$. Then every F witn $(F,G) \in M$ that is supported by $[a,b]$ must be embeddable before time τ and also before time $\tau_{\{a,b\}}$. In particular, this is the case for $F = G_{(a,b)}$. This implies that

$$(6) \qquad (G_{(a,b)},\mathcal{L}(B(\min(\tau,\tau_{\{a,b\}}))) \in M.$$

But, by lemma 1, (for, if its conditions are not met, the statement of (ii) is trivial),

$$(7) \qquad (\mathcal{L}(B(\min(\tau,\tau_{\{a,b\}}))), G_{(a,b)}) \in M.$$

Combine (6) and (7) to obtain that $\mathcal{L}(B(\min(\tau,\tau_{\{a,b\}}))) = G_{(a,b)}$ and that $\min(\tau,\tau_{\{a,b\}})$ is ultimate.

(iii) The conditions of (iii) imply the conditions of lemma 1. Take a proper pair (a,b) and apply (ii).

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LEMMA 3. Let $a < x < 0 < y < b$ and let

$$(8) \qquad \tau^{(1)} = \min(\tau_{\{a,b\}}, \max(\tau_{\{x\}}, \tau_{\{y\}})).$$

Let X_a have values a and y and mean zero, let X_b have values x and b and mean zero. Then

(9) $E(\tau^{(1)}) = xy - ay - xb =$

$$= \frac{a^2P(X_a=a)}{1-P(X_a=a)} + \frac{b^2P(X_b=b)}{1-P(X_b=b)} + \frac{abP(X_a=a)P(X_b=b)}{(1-P(X_a=a))(1-P(X_b=b))}.$$

(10) $P(\tau^{(1)} = \tau_{\{a\}}) = y/(v-a) = P(X_a = a)$

(11) $P(\tau^{(1)} = \tau_{\{b\}}) = -x/(b-x) = P(X_b = b).$

Proof. Express $\tau^{(1)}$ as the sum of two terms. The first is the hitting time $\min(\tau_{\{x\}}, \tau_{\{y\}})$. If $\tau_{\{x\}} \le \tau_{\{y\}}$ ($\tau_{\{y\}} < \tau_{\{x\}}$), the second is the hitting time $\min(\tau_{\{y\}}, \tau_{\{a\}})$ ($\min(\tau_{\{x\}}, \tau_{\{b\}})$). Apply repeatedly the well known fact that if $u < 0 < v$, then $E(\tau_{\{u,v\}}) = -uv$ and $P(\tau_{\{u\}} < \tau_{\{v\}}) = v/(v-u)$.

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LEMMA 4. If F has non exclusive atomic ends, then there is no ultimate st to embed F.

Proof. For $X \sim F$, $a = \text{ess inf}(F)$, $b = \text{ess sup}(F)$:

Let $X_a = E(X \mid 1_{\{X=a\}})$, $X_b = E(X \mid 1_{\{X=b\}})$.

Let $y = E(X \mid X > a)$, $x = E(X \mid X < b)$.

If τ is ultimate and $B_\tau \sim F$, then $\{a < B_\tau < b\} = \{\tau < \tau_{\{a,b\}}\}$ and on $\{a < B_\tau < b\}$, BM must have visited both x and y up to time τ, since X_a and X_b must have been embedded. Hence, $\tau \ge \tau^{(1)}$ of lemma 3.

In view of (10) and (11), on $\{\tau^{(1)} < \tau_{\{a,b\}}\}$ (which equals a.s.

$\{\tau < \tau_{\{a,b\}}\}$, the st $\tau - \tau^{(1)}$, defined on $B \circ \tau^{(1)}$, must embed the conditional distribution of X given that $a < X < b$. Following Monroe [8], if τ is minimal and $B(\tau)$ is bounded, then the expectation of τ is the variance of $B(\tau)$, which is finite. We thus obtain (upon substituting P_a for $P(X = a)$ and P_b for $P(X = b)$) that

$$(12) \qquad Var(X) = (1 - P_a - P_b)Var(X \mid a < X < b) + E(\tau^{(1)}).$$

But, on the other hand,

$$(13) \qquad Var(X) = E(X^2) = (1 - P_a - P_b)E(X^2 \mid a < X < b) + a^2P_a + b^2P_b =$$

$$= (1 - P_a - P_b)Var(X \mid a < X < b) + \frac{(aP_a+bP_b)^2}{1-P_a-P_b} + a^2P_a + b^2P_b.$$

Compare (12) and (13):

$$(14) \qquad E(\tau^{(1)}) = a^2P_a + b^2P_b + \frac{(aP_a+bP_b)^2}{1-P_a-P_b} =$$

$$= \frac{a^2P_a}{1-P_a} + \frac{b^2P_b}{1-P_b} + \frac{abP_aP_b}{(1-P_a)(1-P_b)}$$

$$- \frac{P_aP_b(b-a)^2(\frac{b}{b-a} - P_a)(\frac{-a}{b-a} - P_b)}{(1-P_a)(1-P_b)(1-P_a-P_b)}$$

But (14) conflicts with (9), since $b/(b-a)$ and $-a/(b-a)$ are strictly bigger than P_a and P_b respectively.

Hence, τ is not ultimate.

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