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The total continuity of natural filtrations

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THE TOTAL CONTINUITY OF NATURAL FILTRATIONS AND THE STRONG PROPERTY OF PREDICTABLE REPRESENTATION FOR JUMP PROCESSES AND PROCESSES WITH INDEPENDENT INCREMENTS

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1. INTRODUCTION. Let $(\Omega,\underline{\underline{F}},P)$ be a complete probability space, and let $\underline{F}=(\underline{\underline{F}}_t)_{t\geq 0}$ be a filtration which satisfies the usual conditions. We shall consider here the case of the filtration generated by a r.c.l.l. (cadlag) real valued process X, which will be either a jump process, or a Lévy process without fixed discontinuities. We want to find necessary and sufficient conditions on X, expressing the following properties of the filtration $\underline{F}$:

Total continuity. This idea was introduced in [3] by Yan, and means that $\underline{\underline{F}}_{T-}=\underline{\underline{F}}_T$ at every stopping time T (not necessarily predictable).

Strong predictable representation. This is usually called << predictable representation property >> in martingale theory, and means that every $\underline{F}$-local martingale N (with $N_0=0$) can be represented as a predictable stochastic integral H.M with respect to some fixed local martingale M. In the context of jump processes and Lévy processes, another representation using random measures turns out to be useful, and we find it convenient to distinguish them by the adjectives strong and weak.

There is some overlap of our discussion of predictable representation with that in [7] (however, some details in theorem 2.4 of [7] and its corollary need to be corrected).

NOTATION. We denote by X either a real valued Lévy process without fixed discontinuities (not necessarily homogeneous in time), or a real valued jump process. In both cases it is assumed that $X_0=0$, and that the sample functions are r.c.l.l.. It is shown in [4] and [5] that the family of σ-fields

$$\underline{\underline{F}}_t = \sigma\{X_s,\ s\leq t\ ,\ \underline{\underline{N}}\ \} \qquad (\underline{\underline{N}} \text{ is the family of P-null sets })$$

is automatically right continuous, and hence is the natural filtration of X as described above. Also, that we have for any stopping time T

$$(1) \qquad \underline{\underline{F}}_{T-} = \sigma\{T,X^{T-},\underline{\underline{N}}\}\ , \quad \underline{\underline{F}}_T = \sigma\{T,\ X^T,\ \underline{\underline{N}}\}$$

In the case of jump processes, we denote by T_n the successive jump times of X, by Δ_n the successive jump sizes (by convention $\Delta_n=0$ on $\{T_n=\infty\}$)

In the case of Lévy processes, we denote by X_t^c the Gaussian component of X_t (it is a << non homogeneous Brownian motion >>) and by Y_t the compensated sum of jumps of size between -1 and 1 . It is well known that Y is a martingale, even a square integrable martingale. The difference $X-X^c-Y$ is the sum of a **deterministic** continuous function $\alpha(t)$, and of the jump process

$$J_t = \Sigma_{s\leqq t} \Delta X_s I_{\{\Delta X_s>1 \text{ or } \Delta X_s<-1\}}$$

Set
$$J'_t = \Sigma_{s\leqq t} \varphi(\Delta X_s) I_{\{\Delta X_s>1 \text{ or } \Delta X_s<-1\}}$$

where φ induces a 1-1 mapping of $[1,\infty[$ on $[1,2[$ and $]-\infty,-1]$ on $]-2,-1]$. Then the two Lévy processes

$$X = X^c+Y+\alpha+J \qquad \text{and} \qquad X' = X^c+Y+J'-E[J']$$

generate the same filtration, and the second one has a Lévy measure carried by $]-2,2[$, and is a martingale (even a square integrable martingale). Therefore <u>it is no restriction to generality to assume</u>, if necessary, <u>that X is at the same time a Lévy process and a martingale</u>.

2. TOTAL CONTINUITY OF $\underline{F}$

A necessary condition for total continuity is quasi-left-continuity, which is automatically satisfied in the case of Lévy processes . In the case of jump processes, some conditions are needed : see [4] and [6] ; the most usual <u>sufficient</u> condition for quasi-left-continuity will imply that the T_n's are totally inaccessible - namely, the fact that T_{n+1} has a diffuse conditional distribution w.r. to $\underline{\underline{F}}_{T_n}$, except possibly for an atom at $+\infty$.

THEOREM 2.1. (Jump processes). Assume $\underline{F}$ is quasi-left-continuous. Then $\underline{F}$ is totally continuous if and only if, for every n, there exists a Borel function f_n on $\mathbb{R}^n$ such that

$$(2) \qquad \Delta_n = f_n(T_1,\dots,T_n) \text{ on } \{T_n<\infty\} .$$

PROOF. Necessity. Since the filtration is totally continuous, $\underline{\underline{F}}_{T_n} = \underline{\underline{F}}_{T_n-}$, hence Δ_n is $\underline{\underline{F}}_{T_{n-1}}$-measurable. According to (1), since the σ-field generated by X^{T_n-} is also generated by $(T_1,\Delta_1,\dots T_{n-1},\Delta_{n-1})$, Δ_n is a Borel function of $(T_1,\Delta_1,\dots T_{n-1},\Delta_{n-1},T_n)$. One then shows by induction on n that Δ_n is a Borel function of $T_1,\dots,T_n$.

<u>Remark</u>. Quasi-left-continuity hasn't been used here. Condition (2) implies that, if (Y_t) is the jump process with jump 1 at each time T_n, then (X_t) can be reconstructed from (Y_t), i.e. (Y_t) generates $(\underline{\underline{F}}_t)$.

Sufficiency. We see from the preceding remark that it reduces to a statement on jump processes with jumps equal to 1, namely to the fact that if their natural filtration is quasi-left-continuous, then it is totally continuous.

Let T be an arbitrary stopping time, and let U,V be respectively its predictable and totally inaccessible parts : $\{T=U<\infty\}=A$ belongs to $\underline{\underline{F}}_{T-}$, and therefore so does $\{T=V\}=A^c$. To show that $\underline{\underline{F}}_T=\underline{\underline{F}}_{T-}$, it is sufficient to prove that $\underline{\underline{F}}_U=\underline{\underline{F}}_{U-}$, $\underline{\underline{F}}_V=\underline{\underline{F}}_{V-}$. The first result amounts to quasi left continuity. To prove the second one, we use the fact that, since V is totally inaccessible, its graph $[V]$ is contained in $\cup_n [T_n]$ (see [4], th. 6 ; for an easier reference[1] one may deduce it from the martingale representation theorem in Sem. Prob. IX, p. 234, prop. 3). On the other hand, $\underline{\underline{F}}_V$ is generated by $\underline{\underline{F}}_{V-}$ and Y_V from (1), and since Y <u>always</u> jumps at V on $\{V<\infty\}$ we simply have $Y_V=Y_{V-}+1$ on $\{V<\infty\}$. Theorem 2.1 is proved.

THEOREM 2.2. (Lévy processes). Let X be a Lévy process as described in the introduction. Then the following statements are equivalent.

1) The filtration $\underline{F}$ is totally continuous.

2) There exist a Borel function $f(t)\neq 0$ on $\mathbb{R}_+$, a σ-finite measure Λ on $\mathbb{R}_+$, such that

$$\nu(dt,dx) = \Lambda(dt)\varepsilon_{f(t)}(dx) \tag{3}$$

3) There exists a Borel function $f(t)\neq 0$ such that $\Delta X_t=f(t)I_{\{\Delta X_t\neq 0\}}$.

PROOF. It is clear that 3)=>1) : the proof is the same as the last part in theorem 2.1. Given an arbitrary stopping time T, we split it into its predictable and totally inaccessible parts U and V, and prove separately that $\underline{\underline{F}}_U=\underline{\underline{F}}_{U-}$ by quasi-left-continuity, and $\underline{\underline{F}}_V=\underline{\underline{F}}_{V-}$. Since V is totally inaccessible, the process is known[2] to jump at V, hence on $\{V<\infty\}$ we have $\Delta X_V=f(V)$, and $X_V=X_{V-}+\Delta X_V$ is $\underline{\underline{F}}_{V-}$-measurable. Then we apply (1) to deduce that $\underline{\underline{F}}_V=\underline{\underline{F}}_{V-}$.

Let us prove that 1)=>3). Denote by W the space $\underline{\underline{D}}(\mathbb{R}_+,\mathbb{R})$ of all r.c.l.l. functions from $\mathbb{R}_+$ to $\mathbb{R}$, with its usual σ-field $\underline{\underline{W}}$ ($\underline{\underline{W}}$ is a Lusin σ-field) . Any r.c.l.l. process Z defines a measurable mapping $\omega\mapsto Z_\cdot(\omega)$ from Ω to W, which we also denote by Z. Consider a bounded closed set K in $\mathbb{R}$ disjoint from 0 , let U be the sum of jumps of X whose size belongs to K, and V be the difference X-U ; it is well known that U and V are independent increment processes, and U is a jump process. Let T be an arbitrary stopping time ; since $\underline{F}$ is totally continuous and ΔU_T is $\underline{\underline{F}}_T$-measurable, there is a Borel function a on $\overline{\mathbb{R}}_+\times W$

1. Reference [4] is in chinese (editor's note). 2. Theorem (4.52), p. 139 of [2] (also valid for the preceding reference).

such that $\Delta U_T = a(T,X^{T-})$. Since X^{T-} is itself a Borel function of U^{T-} and V^{T-}, hence of U^{T-}, V and T, we may express ΔU_T as $b(T,U^{T-},V)$, where b is a (bounded) Borel function on $\overline{\mathbb{R}}_+\times W\times W$. Assume now that $T = T_n$, the n-th jump time of U ; then ΔU_T is the corresponding Δ_n , and we have $\Delta_n = b(T_n,U^{T_n-},V)$. But V is independent of U, hence also of (U,T_n) . Taking a conditional expectation w.r. to (T_n,U^{T_n-}) integrates out V, and finally we have $\Delta_n = c(T_n,U^{T_n-})$, which shows that <u>the natural filtration of</u> U <u>is totally continuous</u> (theorem 2.1).

Let (N_t) the Poisson process (non homogeneous in general) which counts the jumps of (U_t). According to theorem 2.1 again, (U_t) and (N_t) generate the same filtration. Therefore the (square integrable) martingale $U_t-E[U_t]$ has a predictable representation w.r. to the (square integrable) martingale $N_t-E[N_t]$, and the representing predictable process, being the density of $d\langle U,N\rangle_t$ w.r. to $d\langle N,N\rangle_t$, must be deterministic since U has independent increments in this filtration. Therefore we can write for U a representation

$$U_t = \int_0^t f(s)d(N_s-E[N_s]) + E[U_t]$$

Since $E[N_t]$ and $E[U_t]$ are continuous in t, we have $\Delta U_t=c(t)\Delta N_t$, and since U and N are of pure jump type, we simply have

$$U_t = \int_0^t f(s)dN_s \tag{4}$$

Now take for K the complement of $]-1/k,+1/k[$, and denote by U^k, N^k, f_k the corresponding processes and functions as just described. The relation $\Delta U^k_t = f_k(t)\Delta N^k_t$ can be written as

$$\Delta X_t(\omega) = f_k(t) \text{ on } \{(t,\omega) : |\Delta X_t(\omega)|\geq 1/k\} \text{ (up to evanescent sets)}$$

and therefore the Borel functions $f_k(t)$ can be << pasted together >> into a single Borel function f such that

$$\Delta X_t(\omega) = f(t)I_{\{\Delta X_t\neq 0\}} \quad . \tag{5}$$

This proves property 3). Let $\Lambda^k(dt)$ be the (Radon) measure $dE[N^k_t]$. Then the Lévy measure of U^k is

$$\nu^k(dt,dt) = \Lambda^k(dt)\varepsilon_{f_k(t)}(dx) = \Lambda^k(dt)I_{\{|f(t)|\geq 1/k\}}\varepsilon_{f(t)}(dx)$$

It follows that there exists a measure $\Lambda(dt)$ on $\mathbb{R}_+$ such that

$$\Lambda^k(dt) = \Lambda(dt)I_{\{|f(t)|\geq 1/k\}} \tag{6}$$

(in particular, Λ is σ-finite), and then the Lévy measure of X is given by (3), and 2) is satisfied. We leave it to the reader to check that 2)=>3), to end the proof of the theorem.

REMARK. The filtration $\underline{F}$ really doesn't depend on the function f, but only on the σ-discrete (non homogeneous) Poisson random measure

(7) $$\Sigma_t \; I_{\{\Delta X_t \neq 0\}} \varepsilon_t$$

which, however, cannot be defined by a counting process N_t since it may have infinitely many points in finite intervals. Conversely, given such a Poisson random measure with σ-finite characteristic measure $d\Lambda(t)$, choose any function $f(t)$ on $]0,\infty[$, everywhere $\neq 0$ and such that $1 \wedge f^2(t)$ is integrable on compact sets w.r. to $d\Lambda$. Then ν defined by (3) is the Lévy measure of some purely discontinuous Lévy process X_t , from which the Poisson random measure can be reconstructed as the point measure of jump locations.

3. STRONG PREDICTABLE REPRESENTATIONS

We first show the relation between strong predictable representation and total continuity :

LEMMA 3.1. Let $\underline{F}$ be a quasi-left-continuous filtration which has the strong predictable representation property. Then it is totally continuous.

PROOF. Since $\underline{F}$ is quasi-left-continuous, the argument used in theorem 2.1 shows that it suffices to prove that $\underline{\underline{F}}_T=\underline{\underline{F}}_{T-}$ at a totally inaccessible stopping time T. It is well known that, given any $\underline{\underline{F}}_T$-measurable bounded random variable ϕ, the process $\phi I_{\{t \geq T\}}$ can be compensated to give a martingale N_t^ϕ whose jump at time T is ϕ on $\{T<\infty\}$. This martingale has a representation as a predictable stochastic integral

$$N_t^\phi = \int_0^t H_s^\phi dM_s \quad \text{(M is the basic local martingale).}$$

and therefore $\phi=H_T^\phi \Delta M_T$ on $\{T<\infty\}$. Taking $\phi=1$, we find that $\phi=H_T^\phi/H_T^1$ on $\{T<\infty\}$. Since H_t^ϕ and H_t^1 are predictable processes, ϕ is $\underline{\underline{F}}_{T-}$-measurable, and the theorem is proved.

THEOREM 3.2. Let X be a jump process generating a quasi-left continuous filtration $\underline{F}$. Then $\underline{F}$ is totally continuous if and only if it has the strong predictable representation property.

PROOF. We have just seen that (strong representation)=>(total continuity). To see the converse, we just remark (theorem 2.1) that if total continuity holds, then the filtration is generated by a jump process with jump sizes equal to 1, and such a filtration is known to have the strong representation property (see the work of Chou and Meyer in Sém. Prob. IX, p. 226-236, Lecture Notes in M. 465, 1975).

The case of Levy processes is more surprising.

THEOREM 3.3. Let X be a Lévy process without fixed discontinuities, and let (α,β,ν) be its local characteristics. Then $\underline{F}$ has the strong predictable representation property if and only if

- The Lévy measure ν has the form (3) : $\nu(dt,dx)=\Lambda(dt)\varepsilon_{f(t)}(dx)$, and
- Λ and β are mutually singular.

PROOF. Necessity. We know that strong representation property implies total continuity, and therefore (theorem 2.2), ν must be given by (3). To prove the second condition, according to the end of section 1, we may assume that X is a square integrable martingale, without changing either β or Λ (since in the transformation we change only the size of jumps, and Λ depends only on their location). Then we decompose X into its continuous and purely discontinuous parts X^c and X^d, which are processes with independents and square integrable martingales at the same time, and therefore have predictable representations

$$X^c = H\cdot M \ , \quad X^d = K.M$$

Since $H.(K.M)=K.(H.M)$ is continuous and purely discontinuous, it must be 0. The predictable process $L=I_{\{K\neq 0\}}$ satisfies the properties $L.X^d=X^d$, $L.X^c=0$, and therefore $L.\langle X^d,X^d\rangle= L^2.\langle X^d,X^d\rangle = \langle LX^d,LX^d\rangle =\langle X^d,X^d\rangle$, and $L.\langle X^c,X^c\rangle=0$. The random measures $d\langle X^c,X^c\rangle$ and $d\langle X^d,X^d\rangle$ are therefore mutually singular. On the other hand, they are deterministic, and equivalent to β and Λ respectively.

Sufficiency. We may assume as above that X is a square integrable martingale. Let N be an arbitrary square integrable martingale w.r. to $\underline{F}$ (it is well known that if strong predictable representation holds for square integrable martingales, then it holds for local martingales too). Then N has a weak predictable representation, in the L^2-sense

$$N_t = \int_0^t K_s dX_s^c + \lim_{k\to\infty} \int_{s\leq t,|x|\geq 1/k} \hat{L}(\omega,s,x)Q(\omega,ds,dx)$$

where Q is the compensated Poisson measure $\Sigma_{t:\Delta X_t\neq 0}\ \varepsilon_{t,\Delta X_t}(ds,dx)-\nu(ds,dx)$

Recalling that $X_t^d = \int_{s\leq t} xQ(\omega,ds,dx)$ and that $\Delta X_t=f(t)I_{\{\Delta X_t\neq 0\}}$, we may write the second integral as $\int_0^t L_s\, dX_s^d$, with $L_s(\omega)= \frac{1}{f(s)} \hat{L}(\omega,s,f(s))$, and we have simply $N_t=\int_0^t (K_s dX_s^c+L_s dX_s^d)$. Let now A and B be two disjoint subsets of $\mathbb{R}_+$ which carry respectively dX^c and dX^d . Then we have

$$N_t = \int_0^t H_s dX_s \quad \text{with } H=KI_A+LI_B$$

and the theorem is proved.

REMARK. In the time homogeneous case, β and Λ are proportional to Lebesgue measure, and therefore cannot be mutually singular unless one of them is 0. So we recover the well known fact that strong predictable representation can hold in this case only for the Brownian filtration and the Poisson filtration (see for instance Sém. Prob. IX, p. 235).

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(Prof. P.A. Meyer suggested also the reference :
Y. Le JAN. Temps d'arrêt stricts et martingales de sauts, ZW 44, 1978, p. 213-226,
where questions closely related to total continuity are studied).

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