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# A NON REVERSIBLE SEMI-MARTINGALE

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The time-reversal of a semi-martingale may fail to be a semi-martingale. Here is a simple example.

Let  $B_t$  be a standard Brownian motion and, inspired by Barlow's example in [1], let  $\phi$  be a measurable function which maps  $C[0,1]$  one-to-one into  $[0,1]$ . Let  $T(\omega) = \phi(\{B_t(\omega), 0 \leq t \leq 1\})$ , and define

$$X_t = \begin{cases} B_t & \text{if } 0 \leq t \leq 1 \\ B_1 & \text{if } 1 \leq t \leq T + 1 \\ B_{t-T} & \text{if } t \geq 1 + T. \end{cases}$$

Then  $X$  is just a Brownian motion with a flat spot of length  $T \leq 1$  interpolated from  $t = 1$  to  $t = T + 1$ .  $T$  is  $\sigma\{X_s, s \leq 1\}$ -measurable, so that it is easy to see that  $X$  is a martingale.

Now reverse  $X$  from  $t = 2$ : let  $\tilde{X}_t = X_{2-t}$  for  $0 \leq t \leq 2$ . Let  $(\tilde{\mathcal{F}}_t)$  be the natural filtration of  $\tilde{X}$ . Note that  $T$  is  $\tilde{\mathcal{F}}_1$ -measurable, hence so is  $\{\tilde{X}_t, t \leq 1\}$ , since it is just the time-reversal of  $\phi^{-1}(T)$ . Consequently,  $\tilde{\mathcal{F}}_t = \tilde{\mathcal{F}}_1$  for  $t > 1$ . Any martingale on these fields will be constant on  $(1,2)$  and any semi-martingale will have finite variation there. But  $\tilde{X}_t$  has infinite variation on  $(1,2)$ , so it is not a semi-martingale relative to the  $(\tilde{\mathcal{F}}_t)$ . By Stricker's theorem [2], it can't be a semi-martingale relative to any filtration whatsoever.

## References

- [1] Barlow, M.T. On Brownian Local Time. Preprint.
- [2] Stricker, C. Quasimartingales, martingales locales, semi-martingales, et filtrations naturelles, ZW 39, (1977) p 55-64.