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On Countable Dense Random Sets

by

D. J. Aldous and M. T. Barlow

We shall discuss point processes whose realisations consist typically of a countable dense set of points. In particular, we discuss when such a process may be regarded as Poisson.

The most primitive way to describe a point process on $[0, \infty)$ is as a subset B of $\Omega \times [0, \infty)$, where the section B_ω represents the times of the "points" in realisation ω . In the locally finite case, there are the more familiar descriptions using the counting process

$$N_t(\omega) = \#(B_\omega \cap [0, t]) \quad (\text{as in [BJ]})$$

or using the random measure

$$\xi(\omega, D) = \#(B_\omega \cap D) \quad (\text{as in [K]})$$

Our point processes will generally not be locally finite, so we cannot use these familiar descriptions: we revert to describing a process as a subset B .

We first describe an (obvious) construction of a countable dense Poisson process. Let Θ be a countable infinite set. Let (F_t) be a filtration (all filtrations are assumed to satisfy the usual conditions). Suppose $\{S_i^\theta : i \geq 1, \theta \in \Theta\}$ are optional times such that each counting process $N_t^\theta = \sum_{i=1}^\infty 1_{(S_i^\theta \leq t)}$ is a Poisson process of rate 1 with respect to (F_t) , and suppose the processes N^θ are independent. Let ξ be the random measure on $\Theta \times [0, \infty)$ whose realisation $\xi(\omega)$ has the set of atoms $\{(\theta, S_1^\theta(\omega)) : i \geq 1, \theta \in \Theta\}$. Then ξ describes a uniform Poisson

process on $\Theta \times [0, \infty)$, with respect to (F_t) . But we can also think of ξ as a marked point process on the line. That is, each realisation is an a.s. countable dense set $\{S_i^\theta(\omega) : i \geq 1, \theta \in \Theta\}$ of points in $[0, \infty)$, and each point is marked by some θ . The corresponding unmarked process can be described by

$$(1) \quad B = \{(\omega, t) : S_i^\theta(\omega) = t \text{ for some } i, \theta\} = \{(\omega, t) : \xi(\omega, \Theta \times \{t\}) = 1\}.$$

Think of B as a σ -finite Poisson process. We are concerned with the converse procedure: given a set B , when can we assign marks θ to the points of B to construct a uniform Poisson process ξ satisfying (1)? To allow external randomisation in assigning marks, we make the following definitions:

(2) Definition. (G_t) is an extension of (F_t) if for each t

$$(i) \quad G_t \supset F_t$$

(ii) G_t and F_∞ are conditionally independent given F_t

(3) Definition B is a σ -finite Poisson process with respect to (F_t) if

(i) B is (F_t) -optional

(ii) There exists a uniform Poisson process ξ on $\Theta \times [0, \infty)$ with respect to some extension (G_t) of (F_t) such that (1) holds.

Theorem 4 below gives a more intrinsic description of σ -finite Poisson processes. First we recall some notation. An optional time T has conditional intensity $a(\omega, s)$ if T has compensator $A_t = \int_0^t a(s) ds$. We may assume $a(\omega, s)$ is previsible by [D.V. 19]. Replacing (F_t) by an extension does not alter the conditional intensity of an (F_t) -optional time T .

Recall also the notation

$$\begin{aligned} T_D &= T \quad \text{on } D \\ &= \infty \quad \text{elsewhere.} \end{aligned}$$

Let λ be Lebesgue measure on $[0, \infty)$.

(4) THEOREM. Let (F_t) be a filtration. Let B be an optional set whose sections B_ω are a.s. countable. The following are equivalent

- (a) B is a σ -finite Poisson process
- (b) There exists a family (T^n) such that

(5) T^n is optional; the graphs $[T^n]$ are disjoint;
 $B = U[T^n]$ a.s.;

(6) T^n has a conditional intensity, say $a_n(\omega, s)$;

(7) $\sum_n a_n(\omega, s) = \infty$ a.e. $(P \times \lambda)$

- (b') Every family (T^n) satisfying (5) also satisfies (6) and (7)

(c) For every previsible set C

$$\{\omega : C_\omega \cap B_\omega = \emptyset\} = \{\omega : \lambda(C_\omega) = 0\} \quad \text{a.s.}$$

Remark Families satisfying (5) certainly exist, by the section theorem and transfinite induction [D. VI. 33] .

The next result comes out of the proof of Theorem 4.

(8) PROPOSITION. Let μ be a probability measure on $[0, \infty)$ which is equivalent to Lebesgue measure .

(a) Let (Y_i) be i.i.d. with law μ , and let (F_t) be the smallest filtration making each Y_i optional - that is, the filtration generated by the processes $1_{[Y_i, \infty)}$. Then $B = U[Y_i]$ is a σ -finite Poisson process with respect to (F_t) .

(b) Conversely, let B be a σ -finite Poisson process with respect to some (F_t) . Then there exist times (Y_i) such that $B = U[Y_i]$ a.s., (Y_i) are i.i.d. with law μ , and (Y_i) are optional with respect to some extension of (F_t) .

Before the proofs, here is an amusing example.

Example There exists a process X_t and filtrations (F_t) , (G_t) such that X is optional with respect to each of (F_t) and (G_t) , but X is not optional with respect to $F_t \cap G_t$.

To construct the example, let $(Y_i), B, (F_t)$ be as in

part (a) of Proposition 8, and let $X = 1_B$. Let Π be the set of finite permutations $\pi = (\pi(1), \pi(2), \dots)$ of $(1, 2, \dots)$. Since Π is countable we can construct a random element π^* of Π such that $P(\pi^* = \pi) > 0$ for each $\pi \in \Pi$. Take π^* independent of $\underline{Y} = (Y_1, Y_2, \dots)$. Define $\underline{V} = (V_1, V_2, \dots) = (Y_{\pi^*(1)}, Y_{\pi^*(2)}, \dots)$. Let (F_t) be the smallest filtration making each V_i optional. Since $X_t = \sum 1_{\{Y_i = t\}} = \sum 1_{\{V_i = t\}}$, plainly X is both (F_t) - and (G_t) -optional. But $F_\infty \cap G_\infty$ is trivial! For let $D \in F_\infty \cap G_\infty$. Then there exist measurable functions f, g such that

$$1_D = f(\underline{Y}) = g(\underline{V}) \quad \text{a.s.}$$

So $f(\underline{Y}) = h(\underline{Y}, \pi^*)$ a.s., where $h(y_1 y_2, \dots; \pi) = g(y_{\pi(1)}, y_{\pi(2)}, \dots)$. But π^* is independent of $\underline{Y}$ with support Π , so

$$f(\underline{Y}) = h(\underline{Y}, \pi) \quad \text{a.s., each } \pi \in \Pi.$$

So, putting $G = \{g = 1\}$,

$$D = \{(\underline{Y}_{\pi(1)}, \underline{Y}_{\pi(2)}, \dots) \in G\} \quad \text{a.s., each } \pi \in \Pi.$$

Thus D is exchangeable, and so is trivial by the Hewitt-Savage zero-one law.

We now start the proof of Theorem 4. The lemma below shows that (b) and (b') are equivalent.

(9) LEMMA. Let (T^n) be optional times whose graphs $[T^n]$ are disjoint. Let $(\hat{T}^m)$ be a similar family, and suppose $U[T^n] = U[\hat{T}^m]$. Suppose T^n has conditional intensity a_n .

Then $\hat{T}^m$ has a conditional intensity, $\hat{a}_m$ say, and
 $\Sigma \hat{a}_m = \Sigma a_n$ a.e. $(P \times \lambda)$.

Proof Put $U_{m,n} = T^n_{(T^n = \hat{T}^m)}$. Then $U_{m,n}$ has a conditional intensity, $a_{m,n}$ say. It is easy to verify

$$a_n = \Sigma_m a_{m,n} \text{ a.e.}$$

$\hat{a}_m \equiv \Sigma_n a_{m,n}$ is the conditional intensity of $\hat{T}^m$, where the sum is a.e. finite because

$$E \int \sum_{n=1}^N a_{m,n}(s) ds = \sum_{n=1}^N P(U_{m,n} < \infty) \leq P(T^m < \infty) \leq 1.$$

Hence $\Sigma a_n = \Sigma \Sigma a_{m,n} = \Sigma \hat{a}_m < \infty$ a.e.

Lemmas 10 and 13 show that conditions (b') and (c) are equivalent.

(10) LEMMA. For B as in theorem 4, the following are equivalent

- (i) $\{\omega : C_\omega \cap B_\omega = \emptyset\} \supset \{\omega : \lambda(C_\omega) = 0\}$ a.s., each previsible C.
- (ii) Each family (Y^n) satisfying (5) also satisfies (6).

Proof: (ii) implies (i) Let C be previsible. Put $T = \inf \{t : \lambda(C_\omega \cap [0, t]) > 0\}$. Then T is optional, so $C' = C \cap [0, T]$ is previsible. Now $\lambda(C'_\omega) = 0$ a.s. We must prove

$$(11) \quad C'_\omega \cap B_\omega = \emptyset \text{ a.s.}$$

Let (T^n) satisfy (5) and (6). Then

$$\begin{aligned} P(T^n \in C'_\omega) &= E \int l_{C'} d1_{[T^n, \infty)} \\ &= E \int l_{C'}(s) a_n(s) ds \\ &= 0. \end{aligned}$$

Since $B = U[T^n]$, (11) follows.

(i) implies (ii). Let T be optional, $[T] \subset B$. Let A_t be the compensator of T . From the proof of the Lebesgue decomposition theorem, we can write $A_t = \hat{A}_t + \int_0^t a(s) ds$, where there exists a progressive set D such that

$$(12) \quad \lambda(D_\omega) = 0 \text{ a.s.}; \text{ the measure } d\hat{A}(\omega) \text{ is carried on } D_\omega \text{ a.s.}$$

Let $C = \{P(1_D) > 0\}$; then C is previsible and since

$$\hat{A}_t \geq \int_0^t l_C(s) d\hat{A}_s \geq \int_0^t P(1_D)(s) d\hat{A}_s = \int_0^t l_D(s) d\hat{A}_s = \hat{A}_t,$$

and

$$\int_0^t P(1_D)(s) ds = \int_0^t l_D(s) ds = 0,$$

C satisfies (12). However

$$\begin{aligned} E\hat{A}_\infty &= E \int l_C(s) d\hat{A}_s \\ &= E \int l_C(s) dA_s \\ &= P(T \in C_\omega) = 0 \text{ by (11)}. \end{aligned}$$

So $\hat{A} \equiv 0$.

(13) LEMMA. For B as in Theorem 4, the following are equivalent.

- (i) $\{\omega : C_\omega \cap B_\omega \neq \emptyset\} \supset \{\omega : \lambda(C_\omega) > 0\}$ a.s., each previsible C .
- (ii) Each family (T^n) satisfying (5) and (6) also satisfies (7).

Proof. (ii) implies (i) Let C be previsible. Define optional times :

$$T = \inf \{t : \lambda(C_\omega \cap [0, t]) > 0\}$$

$$S = \inf \{t : t \in B_\omega \cap C_\omega\}.$$

It is sufficient to prove

$$(14) \quad S \leq T \text{ a.s.}$$

Consider the previsible set $C' = C \cap (T, S]$

Let (T^n) satisfy (5), (6). By definition of S , the sets

$\{\omega : T^n \in C'_\omega\}$ are disjoint. So $\sum_n P(T^n \in C'_\omega) \leq 1$. But

$$\begin{aligned} \sum P(T^n \in C'_\omega) &= \sum E \int_{C'} 1_{C'} d1_{[T^n, \infty)} \\ &= \sum E \int_{C'} 1_{C'}(s) a_n(s) ds \\ &= \sum E \int_{C'} 1_{C'}(s) \sum a_n(s) ds. \end{aligned}$$

But $\sum a_n = \infty$ a.e., and so $\lambda(C'_\omega) = 0$ a.s. But by definition of T we have $\lambda(C'_\omega) > 0$ on $\{T < S\}$. This proves 14.

(i) implies (ii) Let (T_n) satisfy (5), (6). Fix $N < \infty$. Consider the previsible set $H = \{(\omega, s) : \sum a_n \leq N-1\}$. We must prove $P \times \lambda(H) = 0$. Suppose not : then for some $\varepsilon > 0$ we have

$$P(\Omega_0) \geq \varepsilon, \text{ where } \Omega_0 = \{\omega : \lambda(H_\omega) > \varepsilon\}$$

Define optional times

$$S_i = \inf \{t : \lambda(H_\omega \cap [0, t]) > i\varepsilon/N\} \quad i = 0, \dots, N.$$

Consider the previsible sets

$$H^i = H \cap (S_{i-1}, S_i] \quad i = 1, \dots, N$$

$$\bar{H} = H \cap (S_0, S_N] .$$

By construction, $\lambda(H_\omega^i) = \varepsilon/N$ on Ω_0 . So by (i),

$B_\omega \cap H_\omega^i$ is a.s. non-empty on Ω_0 . So

$$E \sum_n 1_{(T_n \in \bar{H}_\omega)} = E \sum_n \sum_i 1_{(T_n \in H_\omega^i)} \geq N P(\Omega_0) \geq N\varepsilon$$

$$\text{But } E \sum_n 1_{(T_n \in \bar{H}_\omega)} = E \sum_n \int 1_{\bar{H}} dI_{[T_n, \infty)}$$

$$= E \int 1_{\bar{H}}(s) \cdot \sum a_n(s) ds$$

$$\leq (N-1) \varepsilon$$

because $\sum a_n \leq N-1$ on H , and $\lambda(\bar{H}_\omega) \leq \varepsilon$ by construction.

This contradiction establishes the result.

It remains to prove that (b) and (a) are equivalent. Recall from [BJ] that optional times $0 < S_1 < S_2 < \dots$ form a Poisson process of rate 1 with respect to (F_t) iff S_n has conditional intensity $1_{(S_{n-1} < s \leq S_n)}$. If moreover this condition holds for each family $(S_i^\theta)_{i \geq 1}$, $\theta \in \Theta$, and if the graphs $\{[S_i^\theta] : i \geq 1, \theta \in \Theta\}$ are disjoint, then the families $\{(S_i^\theta)_{i \geq 1} : \theta \in \Theta\}$ are independent.

The proof that (a) implies (b) is easy. The family (S_i^θ) in (1) plainly satisfies the conditions of (b) with respect to the extension (G_t) . Because (b) implies (b'), we deduce that any (G_t) -optional family satisfying (5) will also satisfy (6) and (7) with respect to (G_t) . Now, as remarked before, there exists a family satisfying (5) with respect to (F_t) ; and since conditional intensities are unchanged by extension, this family satisfies (6) and (7) with respect to (F_t) .

The proof that (b) implies (a) is harder. There are only two ideas. First, we show how to construct S_1 with $[S_1] \subset B$ such that S_1 has exponential law (Lemma 19). Then we can proceed inductively to construct a uniform Poisson process (S_i^θ) . Finally, we must show that $\bigcup_{i, \theta} [S_i^\theta]$ exhausts B .

Here is a straightforward technical lemma.

- (14) LEMMA. Let (Q_i) be optional times with conditional intensities a_i . Suppose $Q_i \rightarrow \infty$ a.s. and $[Q_i]$ are disjoint. Let $T = \min(Q_i)$. Then
- $T_{(T=Q_i)}$ has conditional intensity $a_i 1_{(s \leq T)}$
- T has conditional intensity $\sum a_i 1_{(s \leq T)}$.

Here is an informal description of the external randomisation. Suppose

- (15) T is optional, with conditional intensity a ,
 $p(\omega, s)$ is previsible, $0 \leq p \leq 1$.

Then we can define Q such that:

$$\begin{aligned} \text{if } T = t \text{ then } Q = t \text{ with probability } p(\omega, t) \\ = \infty \text{ otherwise} \end{aligned}$$

It is intuitively obvious that Q has conditional intensity $p.a.$ Here is the formal construction and proof.

- (16) LEMMA. Let T, a, p be as in (15), on a filtration $(\hat{F}_t)$. Let U be uniform on $[0, 1]$, independent of $\hat{F}_\infty$. Define

$$\begin{aligned} Q = T \quad \text{if } U \leq p(T) \equiv p(\omega, T(\omega)) \\ = \infty \quad \text{otherwise.} \end{aligned}$$

Let G_t be the usual augmentation of $G_t^0 = \sigma(\hat{F}_t, Q_{(Q \leq t)})$. Then (G_t) is an extension of $(\hat{F}_t)$, and Q is (G_t) -optional with conditional intensity $p.a.$

Proof $Q_{(Q \leq t)} \in \sigma(\hat{F}_t, U)$, and hence $G_t^0 \subset \sigma(\hat{F}_t, U)$, so (G_t) is indeed an extension of $(\hat{F}_t)$. Plainly Q is (G_t) -optional. To prove the final assertion, let $S < \infty$ be a (G_t) -optional time. It is sufficient to prove

$$(17) \quad P(Q \leq S) = E \int_0^S a(s)p(s)ds.$$

We assert

$$(18) \quad R = S_{(S < T)} \quad \text{is} \quad (F_t)\text{-optional}.$$

For $\{R < u\} = U \{S < t < T\}$, and $\{S < t < T\}$ is in F_t since

$$\begin{array}{c} t < u \\ t \text{ rational} \end{array}$$

$G_t \cap \{T > t\} = F_t \cap \{T > t\}$. To prove (17), note that

$$\{Q \leq S\} = \{T \leq S, Q < \infty\} = \{T \leq R, Q < \infty\} = \{T \leq R, T < \infty, U \leq p(T)\}.$$
 So

$$\begin{aligned} P(Q \leq S) &= P(T \leq R, T < \infty, U \leq p(T)) \\ &= E(1_{(T \leq R, T < \infty)} P(U \leq p(T) | F_\infty)) \\ &= E 1_{(T \leq R, T < \infty)} p(T) \quad \text{by the independence of } U \\ &= E \int 1_{(s \leq R)} p(s) \, d1_{[T, \infty)} \\ &= E \int 1_{(s \leq R)} p(s) \, a(s) \, ds. \end{aligned}$$

(17) now follows, as $[S, R] \subset [T, \infty)$, and $a = 0$ on this set.

(19) LEMMA. Let $(\hat{F}_t)$ be an extension of (F_t) . Suppose (T^n) satisfies condition (b) with respect to $(\hat{F}_t)$. Let $S_0 < \infty$ be $(\hat{F}_t)$ -optional. Then there exists an extension (G_t) of $(\hat{F}_t)$ and a (G_t) -optional time S with conditional intensity $1_{(S_0 < s \leq S)}$ such that $[S] \subset U[T^n]$.

Proof Define $\phi(x) = 1 \quad x \geq 1$
 $= x \quad 0 \leq x \leq 1$
 $= 0 \quad x \leq 0$

Define inductively

$$p_1(\omega, s) = \phi\left(\frac{1}{a_1(\omega, s)}\right) 1_{(s > S_0)}$$

$$p_j = \phi\left(\frac{1 - \sum_{i=1}^{j-1} a_i p_i}{a_j}\right) 1_{(s > S_0)}$$

Then p_j is predictable, $0 \leq p_j \leq 1$, and

$$(20) \quad \sum_{j=1}^N a_j p_j = (1 \wedge \sum_{j=1}^N a_j) \cdot 1_{(s > S_0)}$$

By Lemma 16 we can construct extensions (G_t^j) of (F_t) and (G_t^j) -optional times Q_j such that

$$[Q_j] \subset [T^j],$$

Q_j has conditional intensity $p_j a_j$.

Then

$$\begin{aligned} \sum_j P(Q_j < t) &= \sum_j E \int_0^t p_j(s) a_j(s) ds \\ &= E \int_0^t \sum_j a_j(s) p_j(s) ds \\ &\leq t \quad \text{by (20).} \end{aligned}$$

By the Borel-Cantelli lemma, $Q_j \rightarrow \infty$ a.s.

Set $S = \min (Q_j)$, and let (G_t) be the filtration generated by $(G_t^j, j \geq 1)$. By Lemma 14, S has conditional intensity $\sum a_j p_j 1_{(s \leq S)}$, and by (20) this equals $1_{(S_0 < s \leq S)}$.

For later use, note that, by Lemma 14, $S_{(S=T^n)}$ has conditional intensity $p_n a_n 1_{(s \leq S)}$. In other words, using (20),

(21) $T^n_{(T^n=S)}$ has conditional intensity

$$\left[(1 \wedge \sum_{i=1}^N a_i) - (1 \wedge \sum_{i=1}^{N-1} a_i) \right] 1_{(S_0 < s \leq S)}.$$

We can now prove (b) implies (a). Let $(T^{1,n})$ satisfy condition (b). By Lemma 19 we can construct extensions $G_t^1, G_t^2, \dots$ of F_t and (G_t^1) -optional times S_i^1 such that $[S_i^1] \subset B$ and such that S_i^1 has conditional intensity $1_{(S_{i-1}^1 < s \leq S_i^1)}$. Let F^1 be the filtration generated by $(G^i : i \geq 1)$. Then $(S_i^1)_{i \geq 1}$ is a Poisson process of rate 1 with respect to F^1 .

Now let $T^{2,n} = T^{1,n}_{(T^{1,n} \neq S_i^1 \text{ for any } i)}$.

We assert that $(T^{2,n})$ satisfies (b) with respect to (F_t^1) , for a certain set B' . We need only check (7). Write $a_{k,n}$ for the conditional intensity of $T^{k,n}$. Write

$$R_{n,i} = T^{1,n}_{(T^{1,n} = S_i^1)}$$

$$R_n = T^{1,n}_{(T^{1,n} = S_i^1 \text{ for some } i)}.$$

Then

(22) R_n has conditional intensity $a_{1,n} - a_{2,n} \geq 0$.

But $U[R_n] = \bigcup_{i=1}^n [R_{n,i}] = \bigcup_{i=1}^n [S_i^1]$, so by Lemma 9

$$\sum_i (a_{1,n} - a_{2,n}) = \sum_i 1_{(S_{i-1}^1 < s \leq S_i^1)} = 1 \text{ a.e.}$$

Thus condition (7) extends from $(T^{1,n})$ to $(T^{2,n})$.

Now we may apply Lemma 19 again to construct an extension F^2 and F^2 -optional times (S_i^2) with $[S_i^2] \subset \bigcup_n [T^{2,n}]$ and such that $(S_i^2)_{i \geq 1}$ is again a Poisson process of rate 1.

Continuing, we obtain a uniform Poisson process

$(S_i^k : i, k \geq 1)$ on $\{1, 2, \dots\} \times [0, \infty)$. By construction $\bigcup_{i,k} [S_i^k] \subset B$, but we must show there is a.s. equality. Thus we must show that, for each n ,

$$(23) \quad P(T^{k,n} < \infty) = E \int a_{k,n}(s) ds \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Define

$$R_n^k = T^{k,n}, \quad (T^{k,n} = S_i^k \text{ for some } i)$$

As at (22), R_n^k has conditional intensity $a_{k,n} - a_{k+1,n}$. But from (21), R_n^k has conditional intensity $(1 \wedge \sum_{j=1}^{n-1} a_{k,j}) - (1 \wedge \sum_{j=1}^n a_{k,j})$. So

$$(24) \quad E \int (a_{k,n} - a_{k+1,n}) ds = E \int \left((1 \wedge \sum_{j=1}^N a_{k,j}) - (1 \wedge \sum_{j=1}^{N-1} a_{k,j}) \right) ds$$

Now $a_{k,m} \rightarrow a_{\infty,n}$, say, as $k \rightarrow \infty$. Suppose, inductively, that (23) holds for $n < N$. As $k \rightarrow \infty$ the left side of (24) tends to 0, and the right side tends to $E \int (1 \wedge a_{\infty,N}) ds$ by the inductive hypothesis. Thus $a_{\infty,N} = 0$ a.e., so (23) holds for N .

Proof of Proposition 8. Put $f(t) = \frac{F'(t)}{1-F(t)}$, where F is the distribution function of μ .

From [BJ], if Y has conditional intensity $f(s)1_{(s \leq Y)}$ then Y has law μ : conversely, if Y has law μ then Y has conditional intensity $f(s)1_{(s \leq Y)}$ with respect to the smallest filtration making Y optional. Thus the random variables (Y_i) in part (a) of Proposition 8 satisfy condition (b) of Theorem 4, so $U[Y_i]$ is indeed a σ -finite Poisson process.

Part (b) is similar to, but simpler than, the proof that (b) implies (a) in Theorem 4. Let B be a σ -finite Poisson process, and let $(T^{1,n})$ satisfy condition (b) of Theorem 4. Lemma 19 showed how to construct an optional time S with conditional intensity $1_{(s \leq S)}$. Essentially the same argument shows we can construct Y_1 with conditional intensity $f(s)1_{(s \leq Y_1)}$, and hence with law μ . Put $T^{2,n} = T^{1,n}_{(T^{1,n} \neq Y_1)}$, and continue. We obtain i.i.d. variables (Y_k) , with $U[Y_k] \subset B$: arguing as at (23), we show that there is a.s. equality.

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