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A DIRECT PROOF OF THE RAY-KNIGHT THEOREM

P. Mc GILL (*)

Of the several known proofs of the Ray-Knight theorem, the martingale stopping argument of ([3], problem 5, p. 74) and [2] is arguably the most elementary. Here we exploit what is essentially the same idea to give a proof which avoids explicit computations.

Let $(B_t)_{t \geq 0}$ be a Brownian motion started at zero. Its local time L_t^a is given by the Doob-Meyer decomposition

$$|B_t - a| = |a| + L_t^a + \beta_t^a$$

where β_t^a is a Brownian motion. The occupation density formula [1] gives

$$\int_0^t g(B_s)ds = \int_R g(a)\, L_t^a\, da.$$

Here g is a bounded Borel function and we always assume that L_t^a is the jointly continuous version. In the following, T will always denote the stopping time : $\inf\{t \,/\, B_t = 1\}$.

Now define a process $(Z_a)_{a \geq 0}$ as the unique (positive) solution of the S.D.E

$$Z_a = 2\int_0^a \sqrt{|Z_b|}\, d\tilde{\beta}_b + 2\int_0^a 1_{\{0 \leq b \leq 1\}} db$$

Here $\tilde{\beta}_a$ is a Brownian motion and by [8] this equation has a unique $\tilde{\beta}_a$ adapted solution. $(Z_a)_{a \geq 0}$ is a diffusion. Let

$$m_0 = \inf\{a > 0 : Z_a = 0\}.$$

As remarked in [9], $(Z_a)_{0 \leq a \leq 1}$ is equivalent in law to $(X_t)_{0 \leq t \leq 1}$ where X_t is a $BES^2(2)$ process. Therefore, by path continuity, m_0 is greater than one almost surely.

Lemma 1 : m_0 is finite almost surely.

Proof : For $\alpha > 0$, let u be the decreasing (strictly positive on R^+) solution of

$$2a \frac{d^2u}{da^2} = \alpha u$$

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We can take $u(a) = \sqrt{2\alpha a}\ K_1(\sqrt{2\alpha a})$ where K_1 is the modified Bessel function so that (see [6] 5.7.12) $\lim_{a\to 0} u(a) = 1$. Now for $0 < \varepsilon < c$ if $m_\varepsilon^1 = \inf\{b \geq 1 \ ;\ Z_b = \varepsilon\}$, we can use Itô's formula to check that

$$1_{\{a \geq 1\}}\ 1_{\{c < Z_1\}}\ \left[u(Z_a)\ \exp\{-\alpha(a-1)\} - u(Z_1)\right]$$

is a bounded martingale. By stopping at m_ε^1, we get

$$E\left[\exp\{-\alpha m_\varepsilon^1\}\ ;\ c < Z_1\right] = e^{-\alpha}\ E\left[\frac{u(Z_1)}{u(\varepsilon)}\ ;\ c < Z_1\right]$$

Letting $\varepsilon \to 0$, $\alpha \to 0$ and $c \to 0$ we find that $P[m_0 < +\infty] = 1$.

The next result is well-known.

Lemma 2 : Let $g \geq 0$ be continuous with compact support on $\mathbb{R}$. The equation

$$f'' = 2fg \ ;\ f'(-\infty) = 0 \ ;\ f(0) = \delta > 0$$

has a unique (strictly positive, convex, increasing) solution.

Theorem (Ray [7], Knight [5]) : The process $(L_T^{1-a}, a \geq 0)$ has the same law as $(Z_a, a \geq 0)$.

Proof : We show that for every continuous function $g \geq 0$ with compact support in $(-\infty,1)$,

$$E\left[\exp\{-\int_{-\infty}^{1} g(a)\ L_T^a\ da\}\right] = E\left[\exp\{-\int_0^\infty g(1-a)\ Z_a\ da\}\right]$$

To do this, we use martingale stopping to calculate a more explicit form for each side of this equation.

L.H.S.: By *Lemma 2*, find f with

$$f'' = 2gf \ ;\ f'(-\infty) = 0 \ ;\ f(0) = 1$$

By convexity, the martingale

$$f(B_t)\ \exp\{-\int_0^t g(B_s)ds\}$$

is uniformly integrable up to time T so by the occupation density formula we get

$$E\left[\exp\{-\int_{-\infty}^{1} g(a)\ L_T^a\ da\}\right] = \frac{1}{f(1)}$$

R.H.S. : By *Lemma 2* choose v with

$$v'' = 2\ g(1-a)v \ ;\ v(1) = 1 \ ;\ v'(+\infty) = 0$$

Then $v(a) = f(1-a)$ for $a \geq 0$ and by Itô's formula

$$\frac{1}{v(a \wedge 1)} \exp\{Z_a \frac{v'}{2v}(a) - \int_0^a g(1-b)\, Z_b\, db\}$$

is a local martingale. It is uniformly integrable since $v' \leq 0$ hence by stopping at m_0 we have (since $Z_0 = 0$, $m_0 > 1$)

$$E\left[\exp\{-\int_0^\infty g(1-b)\, Z_b\, db\}\right] = \frac{1}{v(0)}$$

This completes the proof.

Final Remark : The above method applies equally well to any suitable diffusion X_t with generator

$$\mathcal{G} = \frac{1}{2}\, \sigma^2(x) \frac{d^2}{dx^2} + \tau(x) \frac{d}{dx}$$

In this case we choose f such that

$$\frac{1}{2}\, \sigma^2 f'' + \tau f' = fg\, \sigma^2 \; ; \quad f(0) = 1 \; ; \quad f'(-\infty) = 0$$

and we replace Z_a by the solution of

$$W_a = 2 \int_0^a \sqrt{|W_b|}\, d\beta_b + 2 \int_0^a \left[1_{\{0 \leq b \leq 1\}} - W_b \frac{\tau(b)}{\sigma^2(b)}\right] db$$

The argument now proceeds as before. See [4], *Proposition 5*.

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