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ON LEVY'S DOWNCROSSING THEOREM  
AND VARIOUS EXTENSIONS\*

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Our aim is to show that the results of [7] can be extended to regenerative systems as taken in a weak sense which will be made precise. Such a generality is motivated by Lévy's downcrossing theorem, which does not fit to the framework of [7] due to a lack of homogeneity of the processes involved. The first six sections are devoted to this result.

1. FIRST NOTATIONS.

Let  $X = (\Omega, \underline{\mathbb{F}}, \underline{\mathbb{F}}_t, X_t, \theta_t, P)$  denote the canonical one dimensional brownian motion started at the origin:  $\Omega$  is the set of all continuous functions from  $\mathbb{R}_+$  to  $\mathbb{R}$ ;  $(X_t)_{t \geq 0}$  is the process of the coordinates;  $(\theta_t)_{t \geq 0}$  is the process of the shifts; the progression  $(\underline{\mathbb{F}}_t)_{t \geq 0}$  is the P-completion of the natural progression  $(\underline{\mathbb{F}}_t^0)$  of the process  $(X_t)$ ; finally  $P[X_0 = 0] = 1$ .

Now let us introduce some basic notations for our problem: for each  $t \geq 0$  we put

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$$(1.1) \quad C_t = \sup_{s \leq t} X_s ,$$

$$(1.2) \quad Y_t = C_t - X_t ,$$

$$(1.3) \quad M_t = I_{\{Y_t = 0\}} ,$$

$$(1.4) \quad M = \{t: M_t = 1\} = \{t: Y_t = 0\} .$$

## 2. LEVY'S DOWNCROSSING THEOREM.

For  $\varepsilon > 0$ ,  $t \geq 0$  let  $d_t(\varepsilon)$  denote the number of downcrossings of the process  $Y$  over the interval  $(0, \varepsilon]$  by time  $t$ . Lévy's downcrossing theorem asserts that

$$(2.1) \quad P \left[ \lim_{\varepsilon \rightarrow 0} \varepsilon d_t(\varepsilon) = C_t, t \in \mathbb{R}_+ \right] = 1 .$$

(2.2) HISTORICAL REMARK. The result (2.1) was only conjectured by P. Lévy. The first proof can be found in ITO, McKEAN [4], including some gaps that were filled by CHUNG and DURRETT [1]. Another complete proof was given simultaneously by GETTOOR [2] in a much more general context. Finally a short proof was discovered by Williams [8], [9], but his proof remains much more complicated than that of the similar result of Lévy's involving the length of the excursions, namely that there exists  $\lambda \in (0, \infty)$  such that

$$(2.3) \quad P \left[ \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_t(\varepsilon) = \lambda C_t, t \in \mathbb{R}_+ \right] = 1 ,$$

where  $\delta_t(\varepsilon)$  denotes the number of contiguous intervals of length  $>\varepsilon$  contained in  $[0, t]$ . The term "contiguous" means maximal in the complement of  $M$ . Our proof (adapted from [7]) will follow Lévy's very simple method for proving (2.3) and will apply to much more general situations.

(2.4) MATHEMATICAL REMARK. (2.1) shows that the processes  $(C_t)$  and  $(X_t)$  are  $(Y_t)$ -adapted up to null sets. (2.3) even shows that  $(C_t)$  is adapted to the smallest complete progression which makes  $M$  progressive. This can be viewed in many other ways.

### 3. A REGENERATIVE SYSTEM.

Let us introduce new shifts  $(\eta_t)$ :

$$(3.1) \quad \eta_t = \theta_t - X_t = X_{t+} - X_t .$$

With these shifts the strong Markov property of the process  $X$  can be stated as follows: for each stopping time  $T$  and each  $f \in b\mathbb{F}$

$$(3.2) \quad P [ f \circ \eta_T \mid \mathbb{F}_T ] = P(f) \quad \text{on } \{T < \infty\} .$$

Furthermore it is immediate to check that the following M-homogeneity holds for the processes  $(Y_t)$  and  $(M_t)$ : for each  $s, t \geq 0$

$$(3.3) \quad Y_{t+s} = Y_s \circ \eta_t \quad \text{on } \{t \in M\} ,$$

$$(3.4) \quad M_{t+s} = M_s \circ \eta_t \quad \text{on } \{t \in M\} .$$

We shall sum up these properties by saying that the collection  $(\Omega, \underline{F}, \underline{F}_t, Y_t, \eta_t, M, P)$  is a regenerative system (see §8 for a more formal definition).

#### 4. EXCURSIONS OF THE PROCESS Y.

Let  $\Omega^0$  be the set of all functions from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  which remain in 0 after their first hitting of 0. On  $\Omega^0$  we define the process of the coordinates  $(X_s^0)$  and the  $\sigma$ -field  $\underline{F}^0$  generated by the  $X_s^0$ ,  $s \geq 0$ . For  $\omega \in \Omega$ ,  $t \geq 0$  let  $i_t \omega$  be the element of  $\Omega^0$  such that for each  $s \geq 0$

$$(4.1) \quad X_s^0(i_t \omega) = \begin{cases} Y_{t+s}(\omega) & \text{if } t+s < \inf\{u>t: u \in M(\omega)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $G$  be the random set of the left-end-points in  $(0, \infty)$  of the  $M$ -contiguous intervals. Both the  $\Omega$ -valued process  $(i_t)$  and the random set  $G$  are  $M$ -homogeneous and it follows immediately that for each  $A \in \underline{F}^0$  the increasing process

$$(4.2) \quad N_t^A = \sum_{s \in G \cap (0, t]} I_A \circ i_s, \quad t \geq 0,$$

is an  $M$ -additive (non adapted) functional, that is,

$$(4.3) \quad N_{t+s}^A = N_t^A + N_s^A \circ \eta_t \quad \text{on } \{t \in M\} .$$

The random collection  $\{i_t, t \in G\}$  is called the collection of the excursions of  $Y$ ;  $N_t^A$  is the number of excursions of type  $A$  which occur by time  $t$ .

#### 5. TIME CHANGED EXCURSIONS.

The process  $(C_t)$  increases exactly on  $M$  and is  $M$ -additive with respect to the shifts  $\eta_t$ . Therefore its right continuous inverse  $(S_t)$ , defined by

$$(5.1) \quad S_t = \inf\{s: C_s > t\}, \quad t \geq 0,$$

satisfies the following additivity property: for all  $s, t \geq 0$

$$(5.2) \quad S_{t+s} = S_t + S_s \circ \eta_{S_t} \quad \text{on } \{S_t < \infty\};$$

in fact  $S_t \in M$  on  $\{S_t < \infty\}$  and  $C_{S_t} = t$  on  $\{S_t < \infty\}$ , due to the continuity of  $(C_t)$ .

(4.3) and (5.2) further imply that for each  $A \in \underline{\mathbb{F}}^0$  the process  $v_t^A = N_{S_t}^A$  satisfies

$$(5.3) \quad v_{t+s}^A = v_t^A + v_s^A \circ \eta_{S_t} \quad \text{on } \{S_t < \infty\}.$$

But  $S_t < \infty$  a.s. since  $\lim_{r \rightarrow \infty} C_r = +\infty$  a.s.. Hence  $(S_t)$  is a subordinator, due to (5.2) and to (3.2) applied with  $T = S_t$ ; and whenever the process  $(v_t^A)$  is a.s. finite, it has independent and homogeneous increments, due to (5.3) and (3.2); it is even a Poisson process, since it increases by unit jumps. In the

same manner, let  $A_1, \dots, A_n$  be  $n$  pairwise disjoint sets in  $\underline{F}^0$  such that the processes  $(v_t^A i)$  are a.s. finite; then the  $n$ -dimensional process  $(v_t^A 1, \dots, v_t^A n)$  has independent and homogeneous increments and its components  $(v_t^A 1), \dots, (v_t^A n)$  are Poisson processes which pairwise have no common time of jump; therefore, due to a classical result of Lévy, these processes are independent. We have just extended to the present situation Ito's excursion theory [3] and this will allow us to proceed as in [7].

#### 6. PROOF OF LEVY'S DOWNCROSSING THEOREM.

For  $\varepsilon \in (0, \infty]$  let  $A_\varepsilon = \{\sup_{s \text{ rational}} X_s^0 > \varepsilon\}$ . For  $0 < \varepsilon < \varepsilon' \leq \infty$  the process  $(v_t^A \varepsilon \setminus A_{\varepsilon'})$ , which is a.s. finite, is a Poisson process by previous considerations. If  $0 < \varepsilon_1 < \dots < \varepsilon_n \leq \infty$  the processes  $(v_t^A \varepsilon_i \setminus A_{\varepsilon_{i+1}})$ ,  $i = 1, \dots, n-1$  are further independent. But

$$v_t^A \varepsilon_i \setminus A_{\varepsilon_{i+1}} = v_t^A \varepsilon_i - v_t^A \varepsilon_{i+1}$$

and therefore the process  $\varepsilon \rightarrow v_t^A \varepsilon$  is a process with independent (non-homogeneous) increments for each fixed  $t$ . The strong law of large numbers applies to this process as  $\varepsilon \rightarrow 0$  and yields

$$(6.1) \quad \lim_{\varepsilon \rightarrow 0} \frac{v_t^A \varepsilon}{P[v_t^A \varepsilon]} = 1 \quad \text{a.s. .}$$

But we shall see that the denominator in (6.1) equals  $t/\varepsilon$ ; hence (6.1) becomes

$$(6.2) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon v_t^A \varepsilon = t \quad \text{a.s.}$$

Due to the monotonicity in  $t$  of  $\varepsilon v_t^A \varepsilon$  and  $t$ , the null set in (6.2) can be chosen independently of  $t$ ; therefore one has

$$P \left[ \lim_{\varepsilon \rightarrow 0} \varepsilon v_{C_t}^A \varepsilon = C_t, t \in \mathbb{R}_+ \right] = 1$$

and since  $v_{C_t}^A = N_{C_t}^A$ , we get

$$(6.3) \quad P \left[ \lim_{\varepsilon \rightarrow 0} \varepsilon N_{C_t}^A \varepsilon = C_t, t \in \mathbb{R}_+ \right] = 1.$$

Lévy's downcrossing theorem follows from the fact that

$$|d_t(\varepsilon) - N_t^A \varepsilon| \leq 1 \text{ for each } t.$$

It remains to prove that  $P [ v_t^A \varepsilon ] = t/\varepsilon$ . Put  $T_\varepsilon = \inf\{s: Y_s > \varepsilon\}$ . From the equality  $Y_{T_\varepsilon} = \varepsilon$  a.s. and from the martingale property of  $X$ , one immediately checks that  $P [ C_{T_\varepsilon} ] = \varepsilon$ . On the other hand,  $C_{T_\varepsilon}$  is the time of the first jump of the process  $(v_t^A \varepsilon)$ , which is Poisson; therefore

$$P(v_t^A \varepsilon) = t/P(C_{T_\varepsilon}) = t/\varepsilon.$$

## 7. OTHER LIMIT RESULTS FOR THE PROCESS $(C_t)$ .

(7.1) THEOREM. Let  $\alpha \in (0, \infty]$  and let  $\{A_\varepsilon, 0 < \varepsilon \leq \alpha\}$  be a decreasing right continuous family of elements of  $\underline{F}^0$ . Set

$$(7.2) \quad T_{A_\varepsilon} = \inf\{t \in G: i_t \in A_\varepsilon\} = \inf\{t: N_t^A > \varepsilon\}$$

and suppose that

$$(7.3) \quad P [ 0 < T_{A_\varepsilon} < \infty, \varepsilon \in (0, \alpha]; \lim_{\varepsilon \rightarrow 0} T_{A_\varepsilon} = 0 ] = 1 .$$

Then, with the notation (4.2), one has

$$(7.4) \quad P [ \lim_{\varepsilon \rightarrow 0} P [ C_{T_{A_\varepsilon}} ] N_t^A = C_t, t \in \mathbb{R}_+ ] = 1 .$$

The proof is similar to the proof of Lévy's downcrossing theorem. For more details we refer to the proof of theorem 2 of [7] and to the appendix.

(7.6) REMARK. Theorem (7.1) unifies the results (2.1) and (2.3): for (2.1) choose  $A_\varepsilon = \{\sup_{s \text{ rational}} X_s^0 > \varepsilon\}$ , for (2.3) choose  $A_\varepsilon = \{X_\varepsilon^0 > 0\}$ .

## 8. EXTENSIONS TO REGENERATIVE SYSTEMS.

Let us consider a regenerative system  $(\Omega, \underline{F}, \underline{F}_t, Y_t, \eta_t, M, P)$  in the sense of [5], except that the homogeneity properties are only required on  $M$ . More precisely  $(\Omega, \underline{F}, \underline{F}_t, P)$  is a stochastic basis with usual conditions,  $(Y_t)$  is a progressive process (with state space  $(E, \underline{E})$ ),  $(\eta_t)$  is a measurable process with values in  $(\Omega, \underline{F})$ ,  $M$  is a right closed progressive random set. We further assume the following properties:

(8.1) M-homogeneity: for  $s, t \geq 0$

$$Y_s \circ \eta_t = Y_{t+s} \quad \text{on } \{t \in M\},$$

$$M_s \circ \eta_t = M_{t+s} \quad \text{on } \{t \in M\},$$

where  $M_t = I_{\{t \in M\}}$ ;

(8.2) Regeneration: For each stopping time  $T$  and each  $f \in b\mathbb{F}$

$$P [ f \circ \eta_T \mid \mathbb{F}_T ] = P[f] \quad \text{on } \{T \in M\},$$

(8.3) REMARK. This weak notion of regenerative system was already introduced in [6], in order to time change a Markov process by using the inverse of a non-continuous additive functional.

Throughout this section let us assume that the random set  $M$  is perfect, unbounded, with an empty interior a.s. and that  $(C_t)$  is a local time of  $M$ , that is  $(C_t)$  is a continuous adapted M-additive functional which increases exactly on  $\bar{M}$  (the closure of  $M$ ).

Then all considerations of Sections 4,5,7 extend to the present framework, with the following differences: in the definition (4.1) of  $i_t^\omega$  we set

$$X_s^0(i_t^\omega) = \delta \quad \text{if } t+s \geq \inf\{u > t: u \in M(\omega)\} ,$$

where  $\delta$  is a distinguished point in  $E$  which is a.s. ignored by the process  $Y$  and such that  $\{\delta\} \in \underline{\mathbb{E}}$ ; in the definition (4.2) of  $N_t^A$ , we assume that  $A$  is a subset of the space  $\Omega^0$  of all mappings from  $\mathbb{R}_+$  to  $E$  with life time and that  $A$  further belongs to the  $\sigma$ -field  $\underline{\mathbb{F}}^0$  generated by the coordinates of  $\Omega^0$ .

Finally under the assumptions (7.2) and (7.3) we can state the following constructive result, which is the analog of theorem 2' of [7]:

(8.4) THEOREM. There exists a local time  $C_t'$  such that

$$P \left[ \lim_{\varepsilon \rightarrow 0} p(\varepsilon) N_t^A \varepsilon = C_t', t \in \mathbb{R}_+ \right] = 1 ,$$

where we set  $p(\varepsilon) = P \left[ T_{A_\varepsilon} = T_{A_\alpha} \right]$ .

## 9. APPENDIX.

This appendix is devoted to fixing the proof of theorem 2 of [7], which is incomplete. We shall do this in the framework of theorem (7.1) of the present paper. For  $A \in \underline{\mathbb{F}}^0$ , set

$Q(A) = P[v_1^A]$  and for  $\varepsilon \in (0, \alpha]$  set  $q(\varepsilon) = Q(A_\varepsilon)$ . Let  $p$  (resp.  $\bar{p}$ ) be the right (resp. left) continuous inverse of  $q$ :

$$p(u) = \sup\{\varepsilon \in (0, \alpha] : q(\varepsilon) > u\}, \quad u \geq 0,$$

$$\bar{p}(u) = \sup\{\varepsilon \in (0, \alpha] : q(\varepsilon) \geq u\}, \quad u \geq 0.$$

Let us fix  $t \geq 0$  and define the processes  $Z, \bar{Z}$  by setting

$$Z_u = v_t^p(u), \quad \bar{Z}_u = v_t^{\bar{p}}(u), \quad u \geq 0.$$

It was claimed in [7] that the restriction to the set  $T = q((0, \alpha])$  of the process  $Z$  is left continuous. Here is a proof of this fact. Let  $D$  be the set of all points  $u$  in  $T$  which are not isolated from the left and which are such that  $p(u) \neq \bar{p}(u)$ . For each  $u \in D$  one has  $q(p(u)) = q(\bar{p}(u))$ . Therefore the set

$$B = \bigcup_{u \in D} (A_{p(u)} \setminus A_{\bar{p}(u)})$$

is null for the measure  $Q$  and the variable  $v_t^B$  vanishes a.s. This implies that

$$P[Z_u = \bar{Z}_u, u \in D] = 1$$



and the a.s. left continuity of the process  $(Z_u)_{u \in T}$  now follows from the left continuity of  $\bar{Z}$  ( $u_n \uparrow u \Rightarrow \bar{p}(u_n) \downarrow \bar{p}(u) \Rightarrow \sqrt{v} \bar{p}_t(u_n) \uparrow \sqrt{v} \bar{p}_t(u)$ ).

The proof ends like in [7]. Basically one applies the strong law of large numbers to the process  $(Z_u)_{u \in T}$ : this process has independent increments and for  $u, v \in T$ ,  $u \leq v$ ,  $Z_v - Z_u$  is Poisson distributed with parameter  $t(v-u)$ , since  $q(p(u)) = u$  for each  $u \in T$ . Since we have not been able to find a reference for the version of the strong law of large numbers which is needed here, we state and prove it as a

(9.1) LEMMA. Let  $T$  be a left (resp. right) closed unbounded subset of  $\mathbb{R}_+$  and let  $(Z_t)_{t \in T}$  be a left (resp. right) continuous integrable process with independent increment defined on  $(\Omega, \underline{F}, P)$ . Assume that there exists a convolution semi-group  $(\mu_s)_{s \in (0, \infty)}$  of probability measures on  $\mathbb{R}$  such that  $Z_v - Z_u$  has the distribution  $\mu_{v-u}$  for all  $u, v \in T$ ,  $u < v$ . Then one has

$$(9.2) \quad \lim_{t \rightarrow \infty} \frac{Z_t}{t} = \int x \mu_1(dx) \quad P\text{-a.s.}$$

(9.3) REMARK. The result is well known if  $T = \mathbb{R}_+$ : See Doob [10] p. 364. The proof given below follows the martingale method indicated by Doob [10] p. 365.

PROOF. We can restrict ourselves to the case where  $0 \in T$ ,  $Z_0 = 0$ . Consider, on some auxiliary space  $(W, \underline{G}, Q)$  a right contin-

uous process  $(Y_s)_{s \in \mathbb{R}_+}^*$  such that  $Y_0 = 0$  and such that  $Y_v - Y_u$  has the distribution  $\mu_{v-u}$  for all  $u, v \in \mathbb{R}_+$ ,  $u < v$ . One checks easily that for  $k, \ell \in \mathbb{N}$  with  $k \leq \ell$

$$\frac{Y_{\ell/2^n}}{\ell} = Q \left[ \frac{Y_{k/2^n}}{k} \mid Y_u, u \geq \ell/2^n \right],$$

which implies that for  $s, t \in \mathbb{R}_+$ , with  $s \leq t$

$$\frac{Y_t}{t} = Q \left[ \frac{Y_s}{s} \mid Y_u, u \geq t \right].$$

Since the process  $(Z_t)_{t \in T}$  has the same distribution as the process  $(Y_t)_{t \in T}$  (both are markovian relative to the same semi-group), one has also for  $s, t \in T$ , with  $s \leq t$

$$\frac{Z_t}{t} = P \left[ \frac{Z_s}{s} \mid Z_u, u \geq t \right].$$

Fix  $s > 0$  in  $T$  and let  $t \rightarrow \infty$  in  $T$ . By the backward martingale convergence theorem,  $\frac{Z_t}{t}$  converges a.s. The limit has to be constant by the 0-1 law and equal to  $P \left[ \frac{Z_s}{s} \right] = \int x \mu_1(dx)$  by uniform integrability.

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\* with independent increments

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