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ON LEVY'S DOWNCROSSING THEOREM
AND VARIOUS EXTENSIONS*

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Our aim is to show that the results of [7] can be extended to regenerative systems as taken in a weak sense which will be made precise. Such a generality is motivated by Lévy's down-crossing theorem, which does not fit to the framework of [7] due to a lack of homogeneity of the processes involved. The first six sections are devoted to this result.

1. FIRST NOTATIONS.

Let $X = (\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, X_t, \theta_t, P)$ denote the canonical one dimensional brownian motion started at the origin: Ω is the set of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$; $(X_t)_{t\geq 0}$ is the process of the coordinates; $(\theta_t)_{t\geq 0}$ is the process of the shifts; the progression $(\underline{\underline{F}}_t)_{t\geq 0}$ is the P-completion of the natural progression $(\underline{\underline{F}}_t^0)$ of the process (X_t); finally $P[X_0 = 0] = 1$.

Now let us introduce some basic notations for our problem: for each $t \geq 0$ we put

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(1.1) $$C_t = \sup_{s \le t} X_s ,$$

(1.2) $$Y_t = C_t - X_t ,$$

(1.3) $$M_t = I_{\{Y_t = 0\}} ,$$

(1.4) $$M = \{t: M_t = 1\} = \{t: Y_t = 0\} .$$

2. LEVY'S DOWNCROSSING THEOREM.

For $\varepsilon > 0$, $t \geq 0$ let $d_t(\varepsilon)$ denote the number of downcrossings of the process Y over the interval $(0,\varepsilon]$ by time t. Lévy's downcrossing theorem asserts that

(2.1) $$P[\lim_{\varepsilon \to 0} \varepsilon d_t(\varepsilon) = C_t,\ t \in \mathbb{R}_+] = 1 .$$

(2.2) HISTORICAL REMARK. The result (2.1) was only conjectured by P. Lévy. The first proof can be found in ITO, McKEAN [4], including some gaps that were filled by CHUNG and DURRETT [1]. Another complete proof was given simultaneously by GETOOR [2] in a much more general context. Finally a short proof was discovered by Williams [8], [9], but his proof remains much more complicated than that of the similar result of Lévy's involving the length of the excursions, namely that there exists $\lambda \in (0,\infty)$ such that

(2.3) $$P[\lim_{\varepsilon \to 0} \varepsilon\ \delta_t(\varepsilon) = \lambda C_t,\ t \in \mathbb{R}_+] = 1 ,$$

where $\delta_t(\varepsilon)$ denotes the number of contiguous intervals of length $>\varepsilon$ contained in $[0,t]$. The term "contiguous" means maximal in the complement of M. Our proof (adapted from [7]) will follow Lévy's very simple method for proving (2.3) and will apply to much more general situations.

(2.4) MATHEMATICAL REMARK. (2.1) shows that the processes (C_t) and (X_t) are (Y_t)-adapted up to null sets. (2.3) even shows that (C_t) is adapted to the smallest complete progression which makes M progressive. This can be viewed in many other ways.

3. A REGENERATIVE SYSTEM.

Let us introduce new shifts (η_t):

$$\eta_t = \theta_t - X_t = X_{t+\cdot} - X_t \quad . \tag{3.1}$$

With these shifts the strong Markov property of the process X can be stated as follows: for each stopping time T and each $f \in b\underline{\underline{F}}$

$$P\;[\; f\circ\eta_T \mid \underline{\underline{F}}_T \;] = P(f) \qquad \text{on } \{T < \infty\} \ . \tag{3.2}$$

Furthermore it is immediate to check that the following M-<u>homogeneity</u> holds for the processes (Y_t) and (M_t): for each $s,t \geq 0$

$$Y_{t+s} = Y_s\circ\eta_t \qquad \text{on } \{t \in M\} \ , \tag{3.3}$$

$$M_{t+s} = M_s \circ \eta_t \qquad \text{on } \{t \in M\} . \tag{3.4}$$

We shall sum up these properties by saying that the collection $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, Y_t, \eta_t, M, P)$ is a regenerative system (see §8 for a more formal definition).

4. EXCURSIONS OF THE PROCESS Y.

Let Ω^0 be the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ which remain in 0 after their first hitting of 0. On Ω^0 we define the process of the coordinates (X_s^0) and the σ-field $\underline{\underline{F}}^0$ generated by the X_s^0, $s \geq 0$. For $\omega \in \Omega$, $t \geq 0$ let $i_t\omega$ be the element of Ω^0 such that for each $s \geq 0$

$$X_s^0(i_t\omega) = \begin{cases} Y_{t+s}(\omega) & \text{if } t+s < \inf\{u>t:\ u \in M(\omega)\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Let G be the random set of the left-end-points in $(0,\infty)$ of the M-contiguous intervals. Both the Ω-valued process (i_t) and the random set G are M-homogeneous and it follows immediately that for each $A \in \underline{\underline{F}}^0$ the increasing process

$$N_t^A = \sum_{s \in G \cap (0,t]} I_A \circ i_s , \qquad t \geq 0 , \tag{4.2}$$

is an M-<u>additive</u> (non adapted) functional, that is,

$$N_{t+s}^A = N_t^A + N_s^A \circ \eta_t \qquad \text{on } \{t \in M\} . \tag{4.3}$$

The random collection $\{i_t, t \in G\}$ is called the collection of the excursions of Y; N_t^A is the number of excursions of type A which occur by time t.

5. TIME CHANGED EXCURSIONS.

The process (C_t) increases exactly on M and is M-additive with respect to the shifts η_t. Therefore its right continuous inverse (S_t), defined by

$$(5.1) \qquad S_t = \inf\{s : C_s > t\} , \qquad t \geq 0 ,$$

satisfies the following additivity property: for all $s,t \geq 0$

$$(5.2) \qquad S_{t+s} = S_t + S_s \circ \eta_{S_t} \qquad \text{on } \{S_t < \infty\} ;$$

in fact $S_t \in M$ on $\{S_t < \infty\}$ and $C_{S_t} = t$ on $\{S_t < \infty\}$, due to the continuity of (C_t).

(4.3) and (5.2) further imply that for each $A \in \underline{\underline{F}}^0$ the process $\nu_t^A = N_{S_t}^A$ satisfies

$$(5.3) \qquad \nu_{t+s}^A = \nu_t^A + \nu_s^A \circ \eta_{S_t} \qquad \text{on } \{S_t < \infty\} .$$

But $S_t < \infty$ a.s. since $\lim_{r\to\infty} C_r = +\infty$ a.s.. Hence (S_t) is a subordinator, due to (5.2) and to (3.2) applied with $T = S_t$; and whenever the process (ν_t^A) is a.s. finite, it has independent and homogeneous increments, due to (5.3) and (3.2); it is even a Poisson process, since it increases by unit jumps. In the same manner, let $A_1,\ldots,A_n$ be n pairwise disjoint sets in $\underline{\underline{F}}^0$ such that the processes $(\nu_t^{A_i})$ are a.s. finite; then the n-dimensional process $(\nu_t^{A_1},\ldots,\nu_t^{A_n})$ has independent and homogeneous increments and its components $(\nu_t^{A_1}),\ldots,(\nu_t^{A_n})$ are Poisson processes which pairwise have no common time of jump; therefore, due to a classical result of Lévy, these processes are independent We have just extended to the present situation Ito's excursion theory [3] and this will allow us to proceed as in [7].

6. PROOF OF LEVY'S DOWNCROSSING THEOREM.

For $\varepsilon \in (0,\infty]$ let $A_\varepsilon = \{\sup_{s \text{ rational}} X_s^0 > \varepsilon\}$. For $0 < \varepsilon < \varepsilon' \leq \infty$ the process $(\nu_t^{A_\varepsilon \setminus A_{\varepsilon'}})$, which is a.s. finite, is a Poisson process by previous considerations. If $0 < \varepsilon_1 < \ldots < \varepsilon_n \leq \infty$ the processes $(\nu_t^{A_{\varepsilon_i} \setminus A_{\varepsilon_{i+1}}})$, $i = 1,\ldots,n-1$ are further independent. But

$$\nu_t^{A_{\varepsilon_i} \setminus A_{\varepsilon_{i+1}}} = \nu_t^{A_{\varepsilon_i}} - \nu_t^{A_{\varepsilon_{i+1}}}$$

and therefore the process $\varepsilon \to \nu_t^{A_\varepsilon}$ is a process with independent (non-homogeneous) increments for each fixed t. The strong law of large numbers applies to this process as $\varepsilon \to 0$ and yields

$$\lim_{\varepsilon \to 0} \frac{\nu_t^{A_\varepsilon}}{P[\nu_t^{A_\varepsilon}]} = 1 \qquad \text{a.s.} \tag{6.1}$$

But we shall see that the denominator in (6.1) equals t/ε; hence (6.1) becomes

(6.2) $$\lim_{\varepsilon\to 0} \varepsilon\nu_t^{A_\varepsilon} = t \qquad \text{a.s.}$$

Due to the monotonicity in t of $\varepsilon\nu_t^{A_\varepsilon}$ and t, the null set in (6.2) can be chosen independently of t; therefore one has

$$P\left[\lim_{\varepsilon\to 0} \varepsilon\nu_{C_t}^{A_\varepsilon} = C_t \ , \ t \in \mathbb{R}_+ \right] = 1$$

and since $\nu_{C_t}^{A} = N_t^A$, we get

(6.3) $$P\left[\lim_{\varepsilon\to 0} \varepsilon N_t^{A_\varepsilon} = C_t, \ t \in \mathbb{R}_+ \right] = 1 \ .$$

Lévy's downcrossing theorem follows from the fact that $|d_t(\varepsilon) - N_t^{A_\varepsilon}| \leq 1$ for each t.

It remains to prove that $P\,[\, \nu_t^{A_\varepsilon} \,] = t/\varepsilon$. Put $T_\varepsilon = \inf\{s: Y_s > \varepsilon\}$ From the equality $Y_{T_\varepsilon} = \varepsilon$ a.s. and from the martingale property of X, one immediately checks that $P\,[\, C_{T_\varepsilon} \,] = \varepsilon$. On the other hand, C_{T_ε} is the time of the first jump of the process $(\nu_t^{A_\varepsilon})$, which is Poisson; therefore

$$P(\nu_t^{A_\varepsilon}) = t/P(C_{T_\varepsilon}) = t/\varepsilon \ .$$

7. OTHER LIMIT RESULTS FOR THE PROCESS (C_t).

(7.1) THEOREM. Let $\alpha \in (0,\infty]$ and let $\{A_\varepsilon, 0<\varepsilon\leq\alpha\}$ be a decreasing right continuous family of elements of $\underline{\underline{F}}^0$. Set

$$(7.2) \qquad T_{A_\varepsilon} = \inf\{t \in G : i_t \in A_\varepsilon\} = \inf\{t : N_t^{A_\varepsilon} > 0\}$$

and suppose that

$$(7.3) \qquad P\left[\, 0 < T_{A_\varepsilon} < \infty,\ \varepsilon \in (0,\alpha];\ \lim_{\varepsilon\to 0} T_{A_\varepsilon} = 0 \,\right] = 1 .$$

Then, with the notation (4.2), one has

$$(7.4) \qquad P\left[\, \lim_{\varepsilon\to 0} P\left[\, C_{T_{A_\varepsilon}} \,\right] N_t^{A_\varepsilon} = C_t,\ t \in \mathbb{R}_+ \,\right] = 1 .$$

The proof is similar to the proof of Lévy's downcrossing theorem. For more details we refer to the proof of theorem 2 of [7] and to the appendix.

(7.6) REMARK. Theorem (7.1) unifies the results (2.1) and (2.3): for (2.1) choose $A_\varepsilon = \{\sup_{s \text{ rational}} X_s^0 > \varepsilon\}$, for (2.3) choose $A_\varepsilon = \{X_\varepsilon^0 > 0\}$.

8. EXTENSIONS TO REGENERATIVE SYSTEMS.

Let us consider a regenerative system $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, Y_t, \eta_t, M, P)$ in the sense of [5], except that the homogeneity properties are only required on M. More precisely $(\Omega, \underline{\underline{F}}, \underline{\underline{F}}_t, P)$ is a stochastic basis with usual conditions, (Y_t) is a progressive process (with state space $(E, \underline{\underline{E}})$), (η_t) is a measurable process with values in $(\Omega, \underline{\underline{F}})$, M is a right closed progressive random set. We further assume the following properties:

(8.1) M-homogeneity: for $s,t \geq 0$

$$Y_s \circ \eta_t = Y_{t+s} \qquad \text{on } \{t \in M\} ,$$

$$M_s \circ \eta_t = M_{t+s} \qquad \text{on } \{t \in M\} ,$$

where $M_t = I_{\{t \in M\}}$;

(8.2) Regeneration: For each stopping time T and each $f \in b\underline{\underline{F}}$

$$P [f \circ \eta_T \mid \underline{\underline{F}}_T] = P[f] \qquad \text{on } \{T \in M\} ,$$

(8.3) REMARK. This weak notion of regenerative system was already introduced in [6], in order to time change a Markov process by using the inverse of a non-continuous additive functional.

Throughout this section let us assume that the random set M is perfect, unbounded, with an empty interior a.s. and that (C_t) is a local time of M, that is (C_t) is a continuous adapted M-additive functional which increases exactly on $\overline{M}$ (the closure of M).

Then all considerations of Sections 4,5,7 extend to the present framework, with the following differences: in the definition (4.1) of $i_t\omega$ we set

$$X_s^0(i_t\omega) = \delta \qquad \text{if } t+s \geq \inf\{u > t: u \in M(\omega)\} ,$$

where δ is a distinguished point in E which is a.s. ignored by the process Y and such that $\{\delta\} \in \underline{\underline{E}}$; in the definition (4.2) of N_t^A, we assume that A is a subset of the space Ω^0 of all mappings from $\mathbb{R}_+$ to E with life time and that A further belongs to the σ-field $\underline{\underline{F}}^0$ generated by the coordinates of Ω^0.

Finally under the assumptions (7.2) and (7.3) we can state the following constructive result, which is the analog of theorem 2´ of [7]:

(8.4) THEOREM. There exists a local time C_t' such that

$$P \; [\lim_{\varepsilon\to 0} p(\varepsilon) \; N_t^{A_\varepsilon} = C_t', \; t \in \mathbb{R}_+ \;] = 1 ,$$

where we set $p(\varepsilon) = P \; [\; T_{A_\varepsilon} = T_{A_\alpha} \;]$.

9. APPENDIX.

This appendix is devoted to fixing the proof of theorem 2 of [7], which is incomplete. We shall do this in the framework of theorem (7.1) of the present paper. For $A \in \underline{\underline{F}}^0$, set

$Q(A) = P[\nu_1^A]$ and for $\varepsilon \in (0,\alpha]$ set $q(\varepsilon) = Q(A_\varepsilon)$. Let p (resp. $\bar{p}$) be the right (resp. left) continuous inverse of q:

$$p(u) = \sup\{\varepsilon \in (0,\alpha] : q(\varepsilon) > u\}\ , \qquad u \geq 0\ ,$$

$$\bar{p}(u) = \sup\{\varepsilon \in (0,\alpha] : q(\varepsilon) \geq u\}\ , \qquad u \geq 0\ .$$

Let us fix $t \geq 0$ and define the processes Z, $\bar{Z}$ by setting

$$Z_u = \nu_t^{p(u)}\ , \qquad \bar{Z}_u = \nu_t^{\bar{p}(u)}\ , \qquad u \geq 0\ .$$

It was claimed in [7] that the restriction to the set $T = q((0,\alpha])$ of the process Z is <u>left continuous</u>. Here is a proof of this fact. Let D be the set of all points u in T which are not isolated from the left and which are such that $p(u) \neq \bar{p}(u)$. For each $u \in D$ one has $q(p(u)) = q(\bar{p}(u))$. Therefore the set

$$B = \bigcup_{u \in D} (A_{p(u)} \setminus A_{\bar{p}(u)})$$

is null for the measure Q and the variable ν_t^B vanishes a.s. This implies that

$$P[\ Z_u = \bar{Z}_u,\ u \in D\ ] = 1$$

and the a.s. left continuity of the process $(Z_u)_{u\in T}$ now follows from the left continuity of $\bar{Z}$ ($u_n \uparrow u \Rightarrow \bar{p}(u_n) \downarrow \bar{p}(u) \Rightarrow \nu_t^{\bar{p}(u_n)} \uparrow \nu_t^{\bar{p}(u)}$).

The proof ends like in [7]. Basically one applies the strong law of large numbers to the process $(Z_u)_{u\in T}$: this process has independent increments and for $u,\nu \in T$, $u \leq \nu$, $Z_\nu - Z_u$ is Poisson distributed with parameter $t(\nu-u)$, since $q(p(u)) = u$ for each $u \in T$. Since we have not been able to find a reference for the version of the strong law of large numbers which is needed here, we state and prove it as a

(9.1) LEMMA. Let T be a left (resp. right) closed unbounded subset of $\mathbb{R}_+$ and let $(Z_t)_{t\in T}$ be a left (resp. right) continuous integrable process with independent increment defined on $(\Omega,\underline{\underline{F}},P)$. Assume that there exists a convolution semi-group $(\mu_s)_{s\in(0,\infty)}$ of probability measures on $\mathbb{R}$ such that $Z_\nu - Z_u$ has the distribution $\mu_{\nu-u}$ for all $u,\nu \in T$, $u < \nu$. Then one has

$$\lim_{t\to\infty} \frac{Z_t}{t} = \int x\mu_1(dx) \qquad \text{P-a.s.} \tag{9.2}$$

(9.3) REMARK. The result is well known if $T = \mathbb{R}_+$: See Doob [10] p. 364. The proof given below follows the martingale method indicated by Doob [10] p. 365.

PROOF. We can restrict ourselves to the case where $0 \in T$, $Z_0 = 0$. Consider, on some auxiliary space $(W,\underline{\underline{G}},Q)$ a right continuous process $(Y_s)_{s \in \mathbb{R}_+}^{\star}$ such that $Y_0 = 0$ and such that $Y_\nu - Y_u$ has the distribution $\mu_{\nu-u}$ for all $u,\nu \in \mathbb{R}_+$, $u < \nu$. One checks easily that for $k,\ell \in \mathbb{N}$ with $k \leq \ell$

$$\frac{Y_{\ell/2^n}}{\ell} = Q\left[\frac{Y_{k/2^n}}{k} \mid Y_u, \quad u \geq \ell/2^n \right] ,$$

which implies that for $s,t \in \mathbb{R}_+$, with $s \leq t$

$$\frac{Y_t}{t} = Q\left[\frac{Y_s}{s} \mid Y_u, \ u \geq t \right] .$$

Since the process $(Z_t)_{t \in T}$ has the same distribution as the process $(Y_t)_{t \in T}$ (both are markovian relative to the same semi-group), one has also for $s,t \in T$, with $s \leq t$

$$\frac{Z_t}{t} = P\left[\frac{Z_s}{s} \mid Z_u, \ u \geq t \right] .$$

Fix $s > 0$ in T and let $t \to \infty$ in T. By the backward martingale convergence theorem, $\frac{Z_t}{t}$ converges a.s. The limit has to be constant by the 0-1 law and equal to $P\left[\frac{Z_s}{s} \right] = \int x\mu_1(dx)$ by uniform integrability.

$\star$ with independent increments

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