SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

BERNARD MAISONNEUVE

On Lévy's downcrossing theorem and various extensions

Séminaire de probabilités (Strasbourg), tome 15 (1981), p. 191-205

[<http://www.numdam.org/item?id=SPS_1981__15__191_0>](http://www.numdam.org/item?id=SPS_1981__15__191_0)

© Springer-Verlag, Berlin Heidelberg New York, 1981, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON LEVY'S DOWNCROSSING THEOREM
AND VARIOUS EXTENSIONS*

B. MAISONNEUVE

Our aim is to show that the results of [7] can be extended to regenerative systems as taken in a weak sense which will be made precise. Such a generality is motivated by Lévy's downcrossing theorem, which does not fit to the framework of [7] due to a lack of homogeneity of the processes involved. The first six sections are devoted to this result.

1. FIRST NOTATIONS.

Let $X = (\Omega, \underline{F}, \underline{F}_t, X_t, \theta_t, P)$ denote the canonical one dimensional brownian motion started at the origin: Ω is the set of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$; $(X_t)_{t \geq 0}$ is the process of the coordinates; $(\theta_t)_{t \geq 0}$ is the process of the shifts; the progression $(\underline{F}_t)_{t \geq 0}$ is the P-completion of the natural progression $(\underline{F}_t^0)$ of the process (X_t) ; finally $P[X_0 = 0] = 1$.

Now let us introduce some basic notations for our problem: for each $t \geq 0$ we put

---

* This work was supported by AFOSR Grant N° 80-0252 while the author was visiting Northwestern University, Evanston.

$$(1.1) \quad C_t = \sup_{s \leq t} X_s ,$$

$$(1.2) \quad Y_t = C_t - X_t ,$$

$$(1.3) \quad M_t = I_{\{Y_t = 0\}} ,$$

$$(1.4) \quad M = \{t: M_t = 1\} = \{t: Y_t = 0\} .$$

2. LEVY'S DOWNCROSSING THEOREM.

For $\varepsilon > 0$, $t \geq 0$ let $d_t(\varepsilon)$ denote the number of down-crossings of the process Y over the interval $(0, \varepsilon]$ by time t . Lévy's downcrossing theorem asserts that

$$(2.1) \quad P\left[\lim_{\varepsilon \rightarrow 0} \varepsilon d_t(\varepsilon) = C_t, t \in \mathbb{R}_+\right] = 1 .$$

(2.2) HISTORICAL REMARK. The result (2.1) was only conjectured by P. Lévy. The first proof can be found in ITO, McKEAN [4], including some gaps that were filled by CHUNG and DURRETT [1]. Another complete proof was given simultaneously by GETTOOR [2] in a much more general context. Finally a short proof was discovered by Williams [8], [9], but his proof remains much more complicated than that of the similar result of Lévy's involving the length of the excursions, namely that there exists $\lambda \in (0, \infty)$ such that

$$(2.3) \quad P\left[\lim_{\varepsilon \rightarrow 0} \varepsilon \delta_t(\varepsilon) = \lambda C_t, t \in \mathbb{R}_+\right] = 1 ,$$

where $\delta_t(\varepsilon)$ denotes the number of contiguous intervals of length $>\varepsilon$ contained in $[0, t]$. The term "contiguous" means maximal in the complement of M . Our proof (adapted from [7]) will follow Lévy's very simple method for proving (2.3) and will apply to much more general situations.

(2.4) MATHEMATICAL REMARK. (2.1) shows that the processes (C_t) and (X_t) are (Y_t) -adapted up to null sets. (2.3) even shows that (C_t) is adapted to the smallest complete progression which makes M progressive. This can be viewed in many other ways.

3. A REGENERATIVE SYSTEM.

Let us introduce new shifts (η_t) :

$$(3.1) \quad \eta_t = \theta_t - X_t = X_{t+} - X_t.$$

With these shifts the strong Markov property of the process X can be stated as follows: for each stopping time T and each $f \in b\mathbb{F}$

$$(3.2) \quad P [f \circ \eta_T \mid \mathbb{F}_T] = P(f) \quad \text{on } \{T < \infty\}.$$

Furthermore it is immediate to check that the following M-homogeneity holds for the processes (Y_t) and (M_t) : for each $s, t \geq 0$

$$(3.3) \quad Y_{t+s} = Y_s \circ \eta_t \quad \text{on } \{t \in M\},$$

$$(3.4) \quad M_{t+s} = M_s \circ \eta_t \quad \text{on } \{t \in M\} .$$

We shall sum up these properties by saying that the collection $(\Omega, \underline{F}, \underline{F}_t, Y_t, \eta_t, M, P)$ is a regenerative system (see §8 for a more formal definition).

4. EXCURSIONS OF THE PROCESS Y .

Let Ω^0 be the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ which remain in 0 after their first hitting of 0. On Ω^0 we define the process of the coordinates (X_s^0) and the σ -field $\underline{F}^0$ generated by the X_s^0 , $s \geq 0$. For $\omega \in \Omega$, $t \geq 0$ let $i_t \omega$ be the element of Ω^0 such that for each $s \geq 0$

$$(4.1) \quad X_s^0(i_t \omega) = \begin{cases} Y_{t+s}(\omega) & \text{if } t+s < \inf\{u > t : u \in M(\omega)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be the random set of the left-end-points in $(0, \infty)$ of the M -contiguous intervals. Both the Ω -valued process (i_t) and the random set G are M -homogeneous and it follows immediately that for each $A \in \underline{F}^0$ the increasing process

$$(4.2) \quad N_t^A = \sum_{s \in G \cap (0, t]} I_A \circ i_s, \quad t \geq 0,$$

is an M -additive (non adapted) functional, that is,

$$(4.3) \quad N_{t+s}^A = N_t^A + N_s^A \circ \eta_t \quad \text{on } \{t \in M\} .$$

The random collection $\{i_t, t \in G\}$ is called the collection of the excursions of Y ; N_t^A is the number of excursions of type A which occur by time t .

5. TIME CHANGED EXCURSIONS.

The process (C_t) increases exactly on M and is M -additive with respect to the shifts η_t . Therefore its right continuous inverse (S_t) , defined by

$$(5.1) \quad S_t = \inf\{s: C_s > t\}, \quad t \geq 0,$$

satisfies the following additivity property: for all $s, t \geq 0$

$$(5.2) \quad S_{t+s} = S_t + S_s \circ \eta_{S_t} \quad \text{on } \{S_t < \infty\};$$

in fact $S_t \in M$ on $\{S_t < \infty\}$ and $C_{S_t} = t$ on $\{S_t < \infty\}$, due to the continuity of (C_t) .

(4.3) and (5.2) further imply that for each $A \in \underline{F}^0$ the process $v_t^A = N_{S_t}^A$ satisfies

$$(5.3) \quad v_{t+s}^A = v_t^A + v_s^A \circ \eta_{S_t} \quad \text{on } \{S_t < \infty\}.$$

But $S_t < \infty$ a.s. since $\lim_{r \rightarrow \infty} C_r = +\infty$ a.s.. Hence (S_t) is a subordinator, due to (5.2) and to (3.2) applied with $T = S_t$; and whenever the process (v_t^A) is a.s. finite, it has independent and homogeneous increments, due to (5.3) and (3.2); it is even a Poisson process, since it increases by unit jumps. In the

same manner, let $A_1, \dots, A_n$ be n pairwise disjoint sets in $\mathbb{F}^0$ such that the processes $(v_t^{A_i})$ are a.s. finite; then the n -dimensional process $(v_t^{A_1}, \dots, v_t^{A_n})$ has independent and homogeneous increments and its components $(v_t^{A_1}), \dots, (v_t^{A_n})$ are Poisson processes which pairwise have no common time of jump; therefore, due to a classical result of Lévy, these processes are independent. We have just extended to the present situation Ito's excursion theory [3] and this will allow us to proceed as in [7].

6. PROOF OF LEVY'S DOWNCROSSING THEOREM.

For $\varepsilon \in (0, \infty]$ let $A_\varepsilon = \{\sup_{s \text{ rational}} X_s^0 > \varepsilon\}$. For $0 < \varepsilon < \varepsilon' \leq \infty$ the process $(v_t^{A_\varepsilon \setminus A_{\varepsilon'}})$, which is a.s. finite, is a Poisson process by previous considerations. If $0 < \varepsilon_1 < \dots < \varepsilon_n \leq \infty$ the processes $(v_t^{A_{\varepsilon_i} \setminus A_{\varepsilon_{i+1}}})$, $i = 1, \dots, n-1$ are further independent. But

$$v_t^{A_{\varepsilon_i} \setminus A_{\varepsilon_{i+1}}} = v_t^{A_{\varepsilon_i}} - v_t^{A_{\varepsilon_{i+1}}}$$

and therefore the process $\varepsilon \rightarrow v_t^{A_\varepsilon}$ is a process with independent (non-homogeneous) increments for each fixed t . The strong law of large numbers applies to this process as $\varepsilon \rightarrow 0$ and yields

$$(6.1) \quad \lim_{\varepsilon \rightarrow 0} \frac{v_t^{A_\varepsilon}}{P[v_t^{A_\varepsilon}]} = 1 \quad \text{a.s. .}$$

But we shall see that the denominator in (6.1) equals t/ε ; hence (6.1) becomes

$$(6.2) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon v_t^A \varepsilon = t \quad \text{a.s.}$$

Due to the monotonicity in t of $\varepsilon v_t^A \varepsilon$ and t , the null set in (6.2) can be chosen independently of t ; therefore one has

$$P \left[\lim_{\varepsilon \rightarrow 0} \varepsilon v_{C_t}^A \varepsilon = C_t, t \in \mathbb{R}_+ \right] = 1$$

and since $v_{C_t}^A = N_t^A$, we get

$$(6.3) \quad P \left[\lim_{\varepsilon \rightarrow 0} \varepsilon N_t^A \varepsilon = C_t, t \in \mathbb{R}_+ \right] = 1.$$

Lévy's downcrossing theorem follows from the fact that

$$|d_t(\varepsilon) - N_t^A \varepsilon| \leq 1 \text{ for each } t.$$

It remains to prove that $P[v_t^A \varepsilon] = t/\varepsilon$. Put $T_\varepsilon = \inf\{s: Y_s > \varepsilon\}$. From the equality $Y_{T_\varepsilon} = \varepsilon$ a.s. and from the martingale property of X , one immediately checks that $P[C_{T_\varepsilon}] = \varepsilon$. On the other hand, C_{T_ε} is the time of the first jump of the process $(v_t^A \varepsilon)$, which is Poisson; therefore

$$P(v_t^A \varepsilon) = t/P(C_{T_\varepsilon}) = t/\varepsilon.$$

7. OTHER LIMIT RESULTS FOR THE PROCESS (C_t) .

(7.1) THEOREM. Let $\alpha \in (0, \infty]$ and let $\{A_\varepsilon, 0 < \varepsilon \leq \alpha\}$ be a decreasing right continuous family of elements of $\underline{F}^0$. Set

$$(7.2) \quad T_{A_\varepsilon} = \inf\{t \in G: i_t \in A_\varepsilon\} = \inf\{t: N_t^A > 0\}$$

and suppose that

$$(7.3) \quad P [0 < T_{A_\varepsilon} < \infty, \varepsilon \in (0, \alpha]; \lim_{\varepsilon \rightarrow 0} T_{A_\varepsilon} = 0] = 1 .$$

Then, with the notation (4.2), one has

$$(7.4) \quad P [\lim_{\varepsilon \rightarrow 0} P [C_{T_{A_\varepsilon}}] N_t^A = C_t, t \in \mathbb{R}_+] = 1 .$$

The proof is similar to the proof of Lévy's downcrossing theorem. For more details we refer to the proof of theorem 2 of [7] and to the appendix.

(7.6) REMARK. Theorem (7.1) unifies the results (2.1) and (2.3): for (2.1) choose $A_\varepsilon = \{\sup_{s \text{ rational}} X_s^0 > \varepsilon\}$, for (2.3) choose $A_\varepsilon = \{X_\varepsilon^0 > 0\}$.

8. EXTENSIONS TO REGENERATIVE SYSTEMS.

Let us consider a regenerative system $(\Omega, \underline{\mathbb{F}}, \underline{\mathbb{F}}_t, Y_t, \eta_t, M, P)$ in the sense of [5], except that the homogeneity properties are only required on M . More precisely $(\Omega, \underline{\mathbb{F}}, \underline{\mathbb{F}}_t, P)$ is a stochastic basis with usual conditions, (Y_t) is a progressive process (with state space $(E, \underline{\mathbb{E}})$), (η_t) is a measurable process with values in $(\Omega, \underline{\mathbb{F}})$, M is a right closed progressive random set. We further assume the following properties:

(8.1) M-homogeneity: for $s, t \geq 0$

$$Y_s \circ \eta_t = Y_{t+s} \quad \text{on } \{t \in M\},$$

$$M_s \circ \eta_t = M_{t+s} \quad \text{on } \{t \in M\},$$

where $M_t = I_{\{t \in M\}}$;

(8.2) Regeneration: For each stopping time T and each $f \in b\mathbb{F}$

$$P[f \circ \eta_T \mid \mathbb{F}_T] = P[f] \quad \text{on } \{T \in M\},$$

(8.3) REMARK. This weak notion of regenerative system was already introduced in [6], in order to time change a Markov process by using the inverse of a non-continuous additive functional.

Throughout this section let us assume that the random set M is perfect, unbounded, with an empty interior a.s. and that (C_t) is a local time of M , that is (C_t) is a continuous adapted M-additive functional which increases exactly on $\bar{M}$ (the closure of M).

Then all considerations of Sections 4,5,7 extend to the present framework, with the following differences: in the definition (4.1) of i_t^ω we set

$$X_s^0(i_t^\omega) = \delta \quad \text{if } t+s \geq \inf\{u > t: u \in M(\omega)\} ,$$

where δ is a distinguished point in E which is a.s. ignored by the process Y and such that $\{\delta\} \in \underline{E}$; in the definition (4.2) of N_t^A , we assume that A is a subset of the space Ω^0 of all mappings from $\mathbb{R}_+$ to E with life time and that A further belongs to the σ -field $\underline{F}^0$ generated by the coordinates of Ω^0 .

Finally under the assumptions (7.2) and (7.3) we can state the following constructive result, which is the analog of theorem 2' of [7]:

(8.4) THEOREM. There exists a local time C_t' such that

$$P \left[\lim_{\epsilon \rightarrow 0} p(\epsilon) N_t^{A_\epsilon} = C_t', t \in \mathbb{R}_+ \right] = 1 ,$$

where we set $p(\epsilon) = P \left[T_{A_\epsilon} = T_{A_\alpha} \right]$.

9. APPENDIX.

This appendix is devoted to fixing the proof of theorem 2 of [7], which is incomplete. We shall do this in the framework of theorem (7.1) of the present paper. For $A \in \underline{F}^0$, set

$Q(A) = P[v_1^A]$ and for $\varepsilon \in (0, \alpha]$ set $q(\varepsilon) = Q(A_\varepsilon)$. Let p (resp. $\bar{p}$) be the right (resp. left) continuous inverse of q :

$$p(u) = \sup\{\varepsilon \in (0, \alpha] : q(\varepsilon) > u\}, \quad u \geq 0,$$

$$\bar{p}(u) = \sup\{\varepsilon \in (0, \alpha] : q(\varepsilon) \geq u\}, \quad u \geq 0.$$

Let us fix $t \geq 0$ and define the processes $Z, \bar{Z}$ by setting

$$Z_u = v_t^p(u), \quad \bar{Z}_u = v_t^{\bar{p}}(u), \quad u \geq 0.$$

It was claimed in [7] that the restriction to the set $T = q((0, \alpha])$ of the process Z is left continuous. Here is a proof of this fact. Let D be the set of all points u in T which are not isolated from the left and which are such that $p(u) \neq \bar{p}(u)$. For each $u \in D$ one has $q(p(u)) = q(\bar{p}(u))$. Therefore the set

$$B = \bigcup_{u \in D} (A_{p(u)} \setminus A_{\bar{p}(u)})$$

is null for the measure Q and the variable v_t^B vanishes a.s. This implies that

$$P[Z_u = \bar{Z}_u, u \in D] = 1$$

and the a.s. left continuity of the process $(Z_u)_{u \in T}$ now follows from the left continuity of $\bar{Z}$ ($u_n \uparrow u \Rightarrow \bar{p}(u_n) \downarrow \bar{p}(u) \Rightarrow \bar{p}_t(u_n) \uparrow \bar{p}_t(u)$).

The proof ends like in [7]. Basically one applies the strong law of large numbers to the process $(Z_u)_{u \in T}$: this process has independent increments and for $u, v \in T$, $u \leq v$, $Z_v - Z_u$ is Poisson distributed with parameter $t(v-u)$, since $q(p(u)) = u$ for each $u \in T$. Since we have not been able to find a reference for the version of the strong law of large numbers which is needed here, we state and prove it as a

(9.1) LEMMA. Let T be a left (resp. right) closed unbounded subset of $\mathbb{R}_+$ and let $(Z_t)_{t \in T}$ be a left (resp. right) continuous integrable process with independent increment defined on $(\Omega, \underline{F}, P)$. Assume that there exists a convolution semi-group $(\mu_s)_{s \in (0, \infty)}$ of probability measures on $\mathbb{R}$ such that $Z_v - Z_u$ has the distribution μ_{v-u} for all $u, v \in T$, $u < v$. Then one has

$$(9.2) \quad \lim_{t \rightarrow \infty} \frac{Z_t}{t} = \int x \mu_1(dx) \quad P\text{-a.s.}$$

(9.3) REMARK. The result is well known if $T = \mathbb{R}_+$: See Doob [10] p. 364. The proof given below follows the martingale method indicated by Doob [10] p. 365.

PROOF. We can restrict ourselves to the case where $0 \in T$, $Z_0 = 0$. Consider, on some auxiliary space $(W, \underline{G}, Q)$ a right contin-

uous process $(Y_s)_{s \in \mathbb{R}_+}^*$ such that $Y_0 = 0$ and such that $Y_v - Y_u$ has the distribution μ_{v-u} for all $u, v \in \mathbb{R}_+$, $u < v$. One checks easily that for $k, \ell \in \mathbb{N}$ with $k \leq \ell$

$$\frac{Y_{\ell/2^n}}{\ell} = Q \left[\frac{Y_{k/2^n}}{k} \mid Y_u, u \geq \ell/2^n \right],$$

which implies that for $s, t \in \mathbb{R}_+$, with $s \leq t$

$$\frac{Y_t}{t} = Q \left[\frac{Y_s}{s} \mid Y_u, u \geq t \right].$$

Since the process $(Z_t)_{t \in T}$ has the same distribution as the process $(Y_t)_{t \in T}$ (both are markovian relative to the same semi-group), one has also for $s, t \in T$, with $s \leq t$

$$\frac{Z_t}{t} = P \left[\frac{Z_s}{s} \mid Z_u, u \geq t \right].$$

Fix $s > 0$ in T and let $t \rightarrow \infty$ in T . By the backward martingale convergence theorem, $\frac{Z_t}{t}$ converges a.s. The limit has to be constant by the 0-1 law and equal to $P \left[\frac{Z_s}{s} \right] = \int x \mu_1(dx)$ by uniform integrability.

---

* with independent increments

REFERENCES

- [1]. CHUNG, K.L., DURRETT, R.: Downcrossings and local time.
Z. Wahrscheinlichkeitstheorie verw. Gebiete 35,
147-149 (1976).
- [2]. GETTOOR, R.K.: Another limit theorem for local time.
Z. Wahrscheinlichkeitstheorie verw. Gebiete 34,
1-10 (1976).
- [3]. ITO, K.: Poisson point processes attached to Markov
processes. Proc. Sixth Berkeley Sympos. Math. Statist.
Probab. 3, 225-240 (1971).
- [4]. ITO, K., McKEAN, H.P. Jr.: Diffusion processes and their
sample paths. 2nd ed. Springer-Verlag, Berlin,
1965.
- [5]. MAISONNEUVE, B.: Systèmes Régénératifs. Astérisque 15,
Société Mathématique de France, 1974.
- [6]. MAISONNEUVE, B.: Changements de temps d'un processus
markovien additif. Séminaire de Probabilités XI
(Univ. Strasbourg), pp. 529-538. Lecture notes in
Math. 581, Springer-Verlag, Berlin, 1977.

- [7]. MAISONNEUVE, B.: Temps local et dénombrements d'excursions.
Z. Wahrscheinlichkeitstheorie verw. Gebiete 52.
109-113 (1980).
- [8]. WILLIAMS, D.: On Lévy's downcrossing theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete 40, 157-8 (1977).
- [9]. WILLIAMS, D.: Diffusions, Markov Processes and Martingales,
vol. 1: Foundations. Wiley, New York, 1979.
- [10]. DOOB, D.L.: Stochastic Processes. Wiley, New York, 1953.

B. Maisonneuve

Université de Grenoble II

I.M.S.S.

47X-38040 Grenoble Cedex, France