

SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

MICHAEL J. SHARPE

Local times and singularities of continuous local martingales

Séminaire de probabilités (Strasbourg), tome 14 (1980), p. 76-101

http://www.numdam.org/item?id=SPS_1980__14__76_0

© Springer-Verlag, Berlin Heidelberg New York, 1980, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*

<http://www.numdam.org/>

LOCAL TIMES AND SINGULARITIES OF CONTINUOUS
LOCAL MARTINGALES

by
M. J. Sharpe^(*)

0. INTRODUCTION.

The first section of this paper is mostly expository. Given a complete probability space (Ω, \mathcal{F}, P) and a filtration $(\mathcal{F}_t)_{t \geq 0}$ of (Ω, \mathcal{F}, P) satisfying the usual hypotheses — that is, (\mathcal{F}_t) is right continuous and \mathcal{F}_0 contains all null sets — we consider some properties of the space \mathcal{L}^C of continuous local martingales over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ related to the local time processes L_t^a ($a \in \mathbb{R}$). Though most results in Section 1 are known, they do not all seem to be well known, and they set the stage for the results of Section 2 where we study continuous local martingales having a singularity at the time origin. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ as above, let \mathcal{L}_{open}^C denote the space of all real processes $(M_t)_{t > 0}$ defined on the open interval $]0, \infty[$ such that $t \rightarrow M_t$ is a.s. continuous and

(0.1) *There exists a decreasing sequence $\{S_n\}$ of stopping times such that $P\{0 < S_n < \infty\} = 1$ for all n , and $P\{S_n \downarrow 0\} = 1$;*

(0.2) *for each n , the process $t \rightarrow M(S_n + t)$ is a local martingale over the filtration $(\mathcal{F}(S_n + t))_{t \geq 0}$.*

(*)

Research supported, in part, by NSF Grant MCS-80623

It should be emphasized that in general the sequence $\{T_n^k\}_{k \geq 1}$ which reduces $(M(S_n + t))$ — that is, such that $t \rightarrow M(S_n + t \wedge T_n^k) 1_{\{T_n^k > 0\}}$ is a uniformly integrable martingale over $(\mathcal{F}(S_n + t))$ and $T_n^k \uparrow \infty$ a.s. as $k \uparrow \infty$ — depends on n . To illustrate the possibilities, let $(B_t)_{t \geq 0}$ be a standard Brownian motion on \mathbb{R}^d ($d \geq 2$) and let f be harmonic on $\mathbb{R}^d \setminus \{0\}$. Since B_t never hits 0 at a strictly positive time, the Itô calculus shows that $f(B_t)_{t > 0}$ is in $\mathcal{L}_{\text{open}}^C$ relative to P^0 , the law of B starting at 0. The nature of the singularity of f at 0 is reflected in the behavior of $f(B_t)$ at $t \downarrow 0$. If the singularity is removable, $\lim_{t \downarrow 0} f(B_t)$ exists in \mathbb{R} . If f has a pole at 0, $\lim_{t \downarrow 0} f(B_t)$ exists in $\bar{\mathbb{R}} = [-\infty, \infty]$, while if f has an essential singularity at 0, $\liminf_{t \downarrow 0} f(B_t) = -\infty$ and $\limsup_{t \downarrow 0} f(B_t) = \infty$. Walsh [5] studied conformal local martingales on $]0, \infty[$ and showed that almost surely, either the limit as $t \downarrow 0$ exists in the Riemann sphere or the path is dense in the Riemann sphere. We consider here two aspects of the space $\mathcal{L}_{\text{open}}^C$. First of all, we shall state and prove the analogue of Walsh's Theorem for real continuous local martingales on $]0, \infty[$, with characterizations of the cases in terms of the quadratic variation and local time at zero. Following that, we consider a generalization of these results to stochastic integrals $\int_{\mathcal{C}_S} dM_S$, where the stochastic integral is meaningful over any interval bounded away from zero, but may have a singularity at time zero. In this case one may not select one single local martingale on $]0, \infty[$ whose increments give the stochastic integral over an arbitrary interval, so new methods are needed.

1. LOCAL MARTINGALES.

For the basic properties of local martingales we shall use Meyer [4] as a reference, but since we shall consider only continuous local martingales here, little is needed beyond the article of Azema and Yor [1]. Given $M \in \mathcal{L}^C$, the local time process (L_t^a) for M at a is defined to be the unique continuous

increasing process with $L_0^a = 0$ such that $|M_t - a| - L_t^a$ belongs to \mathcal{L}^C . In addition, the quadratic variation process $\langle M, M \rangle_t$ is the unique continuous increasing process with $\langle M, M \rangle_0 = 0$ such that $M_t^2 - \langle M, M \rangle_t$ belongs to \mathcal{L}^C . The following facts are very well known.

(1.1) $\langle M, M \rangle_t$ and M_t have the same intervals of constancy ([3], for example).

(1.2) If $\langle M, M \rangle_t = t$ then $M_t - M_0$ is a standard Brownian motion over (\mathcal{F}_t) (Levy's Theorem [4]).

(1.3) For all $a \in \mathbb{R}$, dL_t^a is carried by $H^a = \{t > 0: M_t = a\}$ and dL_t^a does not charge any interval contained in H^a ([1]).

(1.4) If $M \in \mathcal{L}^C$ and $T_n = \inf \{t: |M_t| \geq n\}$ then $T_n \uparrow \infty$ a.s. and for all n , $t \rightarrow M_{t \wedge T_n} 1_{\{T_n > 0\}}$ is a (bounded) martingale over (\mathcal{F}_t) .

(1.5) If $M \in \mathcal{L}^C$ and $E\langle M, M \rangle_\infty < \infty$, then $M - M_0$ is a martingale with $E[\sup_{t \geq 0} |M_t - M_0|^2] \leq 4E\langle M, M \rangle_\infty$ (Doob's inequality).

(1.6) If $M \in \mathcal{L}^C$, if $M_0 = 0$ and if M is uniformly bounded below then M_t is a supermartingale (Fatou's lemma) and $M_\infty = \lim_{t \rightarrow \infty} M_t$ exists and is finite a.s.

(1.7) One may choose the L_t^a so that $(a, t, \omega) \rightarrow L_t^a(\omega)$ is jointly measurable ([1], p.10) and then $\langle M, M \rangle_t = \int_{-\infty}^{\infty} L_t^a da$. (Assume M bounded, by stopping, so that $\int da(|M_t - a| - L_t^a)$ is a martingale, and consequently $M_t^2 - \int L_t^a da$ is a martingale.)

(1.8) $d\langle M, M \rangle_t$ does not charge H^a for any $a \in \mathbb{R}$ (by (1.7)).

(1.9) (Tanaka's formula [1]). If $M \in \mathcal{L}^C$ then

$$|M_t - a| = |M_0 - a| + \int_0^t \operatorname{sgn}(M_u - a) dM_u + L_t^a$$

where $\operatorname{sgn} x = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.

The following consequence of Tanaka's formula seems to be known, at least to experts.

(1.10) Proposition. Let $M \in \mathcal{L}^C$ with $M_0 = 0$, and set $W_t = - \int_0^t \operatorname{sgn} M_u dM_u$ so that by (1.9), $|M_t| = L_t - W_t$. Then

(i) $\langle W, W \rangle_t = \langle M, M \rangle_t$ for all $t \geq 0$;

(ii) for all $t \geq 0$, $L_t^0 = W_t^m := \max\{W_s : 0 \leq s \leq t\}$.

Proof. Statement (i) comes from the fact that

$$\langle W, W \rangle_t = \int_0^t \operatorname{sgn}^2 M_u d\langle M, M \rangle_u$$

which is equal to $\langle M, M \rangle_t$ by (1.8). To prove that $L_t^0 = W_t^m$ for all $t \geq 0$, it suffices to prove that their right continuous inverse processes are indistinguishable. Let $\sigma_r(\omega) = \inf\{t : L_t^0 > r\}$ (with $\inf \emptyset = \infty$) and $\tau_r(\omega) = \inf\{t : W_t^m > r\}$. If $\sigma_r(\omega) < \infty$, $\sigma_r(\omega)$ is a point of increase of $t \rightarrow L_t^0(\omega)$ and so by (1.3), $M_{\sigma_r(\omega)}(\omega) = 0$. Since $|M| = L - W$ and $L_{\sigma_r} = r$ if $\sigma_r < \infty$ we obtain $W_{\sigma_r(\omega)}(\omega) = r$ if $\sigma_r(\omega) < \infty$. It follows that for all $s < r$, $\tau_s(\omega) \leq \sigma_r(\omega)$ and so $\tau_{r-}(\omega) \leq \sigma_r(\omega)$ for all r . On the other hand, if $\tau_r(\omega) < \infty$, $\tau_r(\omega)$ is a point of increase of $t \rightarrow W_t^m(\omega)$ so

$W_{\tau_r(\omega)}^n(\omega) = W_{\tau_r(\omega)}(\omega)$. Using $L-W = |M| \geq 0$ this implies $L_{\tau_r(\omega)}(\omega) - r \geq 0$ if $\tau_r(\omega) < \infty$. This implies that for all $s < r$, $\sigma_s(\omega) \leq \tau_r(\omega)$ so $\sigma_{r-}(\omega) \leq \tau_r(\omega)$. By right continuity we obtain $\sigma_r(\omega) = \tau_r(\omega)$ for all $r \geq 0$ a.s. .

As a first application of (1.10), we consider the convergence of $M \in \mathcal{L}^C$ as $t \rightarrow \infty$.

(1.11) Theorem. *Let $M \in \mathcal{L}^C$. Then, almost surely*

$$\{M_\infty \text{ exists and is finite}\} = \{\limsup_{t \rightarrow \infty} M_t < \infty\} = \{\langle M, M \rangle_\infty < \infty\} = \{L_\infty^0 < \infty\}.$$

Proof. We may assume that $M_0 = 0$. The first equality is due to Doob ([2], p.382) but since the proof is a model for the other equalities we indicate its proof. Let $T_n = \inf\{t: M_t \geq n\}$ so that $U\{T_n = \infty\} = \{\limsup M_t < \infty\}$. Since $M_{t \wedge T_n}$ is bounded above, it converges a.s. (1.6), hence M_∞ exists a.s. on $U\{T_n = \infty\}$. For the next equality, first set $T_n = \inf\{t: |M_t| \geq n\}$ so that $M_t^{T_n} = M_{t \wedge T_n}$ is a uniformly bounded martingale. Since $M_{t \wedge T_n}$ is an L^2 bounded martingale, $E\langle M^{T_n}, M^{T_n} \rangle_\infty \leq n^2$, and in particular $\langle M, M \rangle_{T_n} < \infty$. Since $U\{T_n = \infty\} \supset \{M_\infty \text{ exists}\}$, this shows that $\{M_\infty \text{ exists}\} \subset \{\langle M, M \rangle_\infty < \infty\}$. In the same way, $EL_{t \wedge T_n}^0 = E|M_{t \wedge T_n}| \leq n$ shows that $\{M_\infty \text{ exists}\} \subset \{L_\infty^0 < \infty\}$. Now set $R_n = \inf\{t: \langle M, M \rangle_t \geq n\}$ so that $U\{R_n = \infty\} = \{\langle M, M \rangle_\infty < \infty\}$. Since $\langle M^{R_n}, M^{R_n} \rangle \leq n$, M^{R_n} is L^2 bounded (1.5) and so $M_\infty^{R_n}$ exists. Thus $\{\langle M, M \rangle_\infty < \infty\} \subset \{M_\infty \text{ exists}\}$. Finally, let $W = L^0 - |M|$ as in (1.10). Because $L_\infty^0 = \sup_t W_t$, $\limsup W_t < \infty$ on $\{L_\infty^0 < \infty\}$ so W_t converges on $\{L_\infty^0 < \infty\}$. But since $\{W_\infty \text{ exists}\} = \{\langle W, W \rangle_\infty < \infty\} = \{\langle M, M \rangle_\infty < \infty\} = \{M_\infty \text{ exists}\}$, we have shown that M converges on $\{L_\infty^0 < \infty\}$.

The connection between convergence of M and the local time of M at zero is not surprising because it is well known that L_t^0 can be expressed as a

normalized limit of the number of downcrossings of $[0, \varepsilon]$ up to time t . If $M \in \mathcal{L}^C$ and $M_0 = 0$, it is easy to see from (1.10) that $EL_\infty^0 = \sup\{E|M_T| : T \text{ a finite stopping time}\}$ so that in particular, if M is a martingale, M is L^1 bounded if and only if $EL_\infty^0 < \infty$. Further in this direction, if $M \in \mathcal{L}^C$ and $M_0 = 0$ then setting $W = L^0 - |M|$ so that $\langle W, W \rangle = \langle M, M \rangle$, the Burkholder-Davis-Gundy inequalities imply that for every $p > 1$, there exist absolute constants C_p such that

$$(1.12) \quad E(L_\infty^0)^p \leq C_p \sup\{E|M_T|^p : T \text{ a finite stopping time}\}.$$

Similar arguments show that if M is in BMO then

$$(1.13) \quad E[\exp(\lambda L_\infty^0)] < \infty \text{ for some } \lambda > 0.$$

The inequality (1.12) was obtained in [1] by different (more elementary) methods.

If in the situation of (1.10), M is a standard Brownian motion then $W = L^0 - |M|$ is also a standard Brownian motion since $\langle W, W \rangle_t = \langle M, M \rangle_t = t$. The fact that $L_t^0 = W_t^m (= \sup\{W_s : 0 \leq s \leq t\})$ sharpens the well known fact that L_t^0 and the one-sided maximal process of Brownian motion have the same law, by actually producing a Brownian motion for which L_t^0 is the one-sided maximal function. Similarly, (1.10) demonstrates why $|M_t|$ and $M_t^m - M_t$ are processes with the same law: one produces a Brownian motion W_t with $|M_t| = W_t^m - W_t$.

In preparation for a number of arguments in the next section, we need the following simple lemma about birthing a local martingale at a stopping time.

(1.14) Lemma. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfy the usual hypotheses and let R be a stopping time over (\mathcal{F}_t) . Then*

(i) for any stopping time T over (\mathcal{F}_t) , the random variable $(T-R)1_{\{T>R\}}$ is a stopping time over (\mathcal{F}_{R+t}) ;

(ii) if M is a local martingale over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ then $N_t = M_{t+R} 1_{\{R<\infty\}}$ is a local martingale over (\mathcal{F}_{R+t}) relative to the conditional probability measure $P\{\cdot | R<\infty\}$.

Proof. It is easy to see that (\mathcal{F}_{R+t}) satisfies the usual hypotheses. To prove (i), just observe that $\{(T-R)1_{\{T>R\}} > t\} = \{T > R+t\} \in \mathcal{F}_{R+t}$. For (ii) we may assume that $P\{R<\infty\} > 0$, for otherwise $N=0$ and there is nothing to prove, no matter how $P\{\cdot | R<\infty\}$ is defined. If M is uniformly integrable with limit M_∞ then using optional sampling, for $G \in b\mathcal{F}_{R+t}$

$$\begin{aligned} E\{N_\infty G | R<\infty\} &= E\{M_\infty 1_{\{R<\infty\}} G\} / P\{R<\infty\} \\ &= E\{M_{R+t} 1_{\{R<\infty\}} G\} / P\{R<\infty\} \\ &= E\{N_t G | R<\infty\}, \end{aligned}$$

so N is a uniformly integrable martingale relative to $(\Omega, (\mathcal{F}_{R+t}), P\{\cdot | R<\infty\})$. In the general case assume that $\{T_n\}$ reduces M , and let $T'_n = (T_n - R)1_{\{T_n > R\}}$. Then $\{T'_n\}$ is an increasing sequence of stopping times over (\mathcal{F}_{R+t}) such that $P\{\lim T'_n = \infty | R<\infty\} = 1$. For every n

$$\begin{aligned} M(R+t \wedge T'_n) 1_{\{T'_n > 0\}} &= M(R+t \wedge (T_n - R)) 1_{\{T_n > R\}} \\ &= M((t+R) \wedge T_n) 1_{\{T_n > R\}}. \end{aligned}$$

Since $t \rightarrow M_{t \wedge T_n} 1_{\{T_n > 0\}}$ is a uniformly integrable martingale and $\{T_n > R\} \in \mathcal{F}_R$, it follows that $t \rightarrow M(R+t \wedge T'_n) 1_{\{T'_n > 0\}}$ is a uniformly integrable martingale over $(\Omega, (\mathcal{F}_{R+t}), P\{\cdot | R<\infty\})$ for every $n \geq 1$.

2. LOCAL MARTINGALES OVER]0,∞[.

Observe to begin with that (1.14) implies $\mathcal{L}^C \subset \mathcal{L}_{\text{open}}^C$. It will turn out (2.16) that if $M \in \mathcal{L}_{\text{open}}^C$ and $M_0 = M_{0+}$ exists and is finite a.s. then $(M_t)_{t \geq 0}$ is in \mathcal{L}^C .

(2.1) Proposition. A continuous process $(M_t)_{t > 0}$ is in $\mathcal{L}_{\text{open}}^C$ if and only if for some sequence $\{S_n\}$ of stopping times (not necessarily finite valued) satisfying

$$(2.2) \quad P\{S_n > 0\} = 1 \text{ and } P\{S_n \text{ decreases to } 0\} = 1$$

it is the case that

(2.3) for all $n \geq 1$, the process $N_t^n = M(S_n + t) 1_{\{S_n < \infty\}}$ is a local martingale over $(\Omega, (\mathcal{F}_{S_n+t}), P\{\cdot | S_n < \infty\})$.

If $M \in \mathcal{L}_{\text{open}}^C$, then for every sequence $\{S_n\}$ satisfying (2.2) the condition (2.3) holds.

Proof. Fix one sequence $\{S_n\}$ satisfying (2.2) and (2.3) and let $\{R_n\}$ be a sequence satisfying (2.2). We shall prove then that $\{R_n\}$ satisfies (2.3).

Taking each R_n finite valued will show $M \in \mathcal{L}_{\text{open}}^C$, and the last assertion of (2.1) will obtain for general $\{R_n\}$. For $m \geq 1$ and $s > 0$, let $T(m, s) = \inf\{t > s : |M_t| \geq m\}$. Then $T(m, s)$ is a stopping time over (\mathcal{F}_t) and for all $s > 0$, $T(m, s)$ increases in m , say to $T(\infty, s)$. By hypothesis $T(\infty, S_n) = \infty$ a.s. for all n . Since $S_n \downarrow 0$ a.s. and $s \rightarrow T(m, s)$ is increasing in s for all $m \geq 1$, it follows that $P\{T(\infty, s) = \infty \text{ for all } s > 0\} = 1$. Because of (2.3), (1.4) and (1.14), the process

$$t \rightarrow M((S_n + t) \wedge T(m, S_n)) 1_{\{T(m, S_n) > S_n\}}$$

is a bounded martingale over $(\Omega, \mathfrak{F}_{S_n+t}, P\{\cdot | S_n < \infty\})$ for all $n \geq 1$ and $m \geq 1$. It follows that as $t \uparrow T(m, S_n)$, $M(t)$ converges a.s. on $\{T(m, S_n) > S_n\}$. Denoting the limit by $M(T(m, S_n)) 1_{\{T(m, S_n) > S_n\}}$, one has

$$M((S_n + t) \wedge T(m, S_n)) 1_{\{T(m, S_n) > S_n\}} = E\{M(T(m, S_n)) 1_{\{T(m, S_n) > S_n\}} | \mathfrak{F}_{S_n+t}\}.$$

By optional sampling, with t replaced by $(R_k + t - S_n) 1_{\{R_k + t > S_n\}}$ we obtain, a.s. on $\{R_k + t > S_n\}$,

$$M((R_k + t) \wedge T(m, S_n)) 1_{\{T(m, S_n) > S_n\}} = E\{M(T(m, S_n)) 1_{\{T(m, S_n) > S_n\}} | \mathfrak{F}_{R_k+t}\}.$$

This equality holds in particular on $\{R_k > S_n\}$, and one may then interpret the equality to mean that $t \rightarrow M(R_k + t) 1_{\{R_k < \infty\}}$ is a local martingale relative to $(\Omega, \mathfrak{F}_{R_k+t}, P\{\cdot | R_k < \infty\})$ having reducing times $(T(m, S_n) - R_k) 1_{\{T(m, S_n) > S_n \vee R_k\}} = T_{m,n}^k$.

(Note that for a fixed k , $P\{\sup_{m,n} T_{m,n}^k = \infty | R_k < \infty\} = 1$.)

The first important result describing the behavior at the time origin of $M \in \mathcal{L}_{\text{open}}^C$ is the following.

(2.4) Theorem. Let $M \in \mathcal{L}_{\text{open}}^C$. Then for a.a. ω , either

(i) $\lim_{t \downarrow 0} M_t(\omega)$ exists in \mathbf{R}

or

(ii) $\lim_{t \downarrow 0} M_t(\omega) = \pm \infty$

or

(iii) $\liminf_{t \downarrow 0} M_t(\omega) = -\infty$ and $\limsup_{t \downarrow 0} M_t(\omega) = \infty$.

This theorem may be deduced from Walsh's Theorem on conformal martingales since every local martingale is the real part of some conformal martingale. Because Walsh's proof is a little obscure at one point, we shall derive (2.4) from scratch. In preparation for this we need a couple of lemmas.

(2.5) Lemma. Let $M \in \mathcal{L}^C$ and suppose that $M_0 = b$ a.s.. Then if $a < b < c$, $P\{M_t \text{ hits } a \text{ before it hits } c | \mathcal{F}_0\} \leq (c-b)/(c-a)$.

Proof. Let T_a (resp., T_c) denote the first time M_t hits a (resp., c). As we mentioned in (1.4), a uniformly bounded local martingale is in fact a martingale. Since T_a and T_c are stopping times over (\mathcal{F}_t) , it follows that $M_{t \wedge (T_a \wedge T_c)}$ is a bounded martingale whose limit at infinity is

$$M(T_a \wedge T_c) \leq a 1_{\{T_a < T_c\}} + c 1_{\{T_a \geq T_c\}} .$$

Taking conditional expectations leads to

$$b \leq a P\{T_a < T_c | \mathcal{F}_0\} + c P\{T_a \geq T_c | \mathcal{F}_0\} ,$$

from which the desired inequality obtains.

(2.6) Lemma. Let $M \in \mathcal{L}^C$ and for $a < b < c$ and k a positive integer, let $R_{[a,b]}^k = \inf\{t: M_s (0 \leq s \leq t) \text{ completes } k \text{ upcrossing of } [a, b]\}$ and $T_c = \inf\{t: M_t \geq c\}$ (with $\inf \emptyset = \infty$). Then

$$P\{R_{[a,b]}^k < T_c\} \leq \left(\frac{c-b}{c-a}\right)^{k-1} .$$

Proof. In the course of the proof, we let R^k denote $R_{[a,b]}^k$. On $\{R^k < \infty\}$, $M(R^k) = b$ by definition of R^k . Let P_k denote the conditional probability measure $P_k\{\cdot\} = P\{\cdot | R^k < \infty\}$ defined in an arbitrary way if $P\{R^k < \infty\} = 0$. We showed (1.14) that the process $t \rightarrow M(R^k + t) 1_{\{R^k < \infty\}}$ under $P\{\cdot | R^k < \infty\}$ is a continuous local martingale over the filtration $\mathfrak{F}(R^k + t)$. Moreover, if $P\{R^k < \infty\} > 0$, $P_k\{M(R^k) = b\} = 1$. For all $k \geq 2$ we have

$$\begin{aligned} P\{R^k < T_c\} &\leq P\{R^{k-1} < T_c, M(R^{k-1} + t) \text{ hits } a \text{ before it hits } c\} \\ &= E\{P\{M(R^{k-1} + t) \text{ hits } a \text{ before it hits } c | \mathfrak{F}(R^{k-1})\}; R^{k-1} < T_c\} \\ &= E\{P_{k-1}\{M(R^{k-1} + t) \text{ hits } a \text{ before it hits } c | \mathfrak{F}(R^{k-1})\}; R^{k-1} < T_c\} \\ &\leq (c-b)/(c-a) P\{R^{k-1} < T_c\} \end{aligned}$$

because of (2.5) applied to the filtration $\mathfrak{F}(R^{k-1} + t)$. The conclusion of (2.6) is now clear by induction on k .

Proof of (2.4): Given $a < b$, let

$$\Gamma_{a,b} = \{\omega \in \Omega: \liminf_{t \downarrow 0} M_t(\omega) < a, \limsup_{t \downarrow 0} M_t(\omega) > b\}$$

$$\Gamma = \{\omega \in \Omega: \liminf_{t \downarrow 0} M_t(\omega) = -\infty, \limsup_{t \downarrow 0} M_t(\omega) = \infty\}$$

Obviously $\Gamma = \bigcap \{\Gamma_{a,b}: a < b \text{ rationals}\}$. In order to prove (2.4) it is enough to prove that $\Gamma \supset \Gamma_{a,b}$ a.s. for any pair of rationals $a < b$. If $c > b$, let $T_c = \inf\{t: M_t \geq c\}$. We shall prove that for all $c > b$, $T_c = 0$ a.s. on $\Gamma_{a,b}$ and this will show that $\limsup_{t \downarrow 0} M_t = \infty$ a.s. on $\Gamma_{a,b}$. Applying this result to $-M$, one will then obtain $\Gamma \supset \Gamma_{a,b}$ a.s. Fix $a < b < c$ and let $\{S_n\}$ satisfy (0.1) (and hence (0.2) also by (2.1)). For each $n \geq 1$, let

$T_n^c = \inf\{t: M(S_n + t) \geq c\}$ and let R_n^k denote the first time $M(S_n + t)$ completes k upcrossing of $[a, b]$. Now fix $k \geq 1$. As n increases, the events $\{T_c > S_n\} \cap \{R_n^k < T_c^n\}$ increase, for on $\{T_c > S_n\}$, $T_c^n = T_c - S_n$. On the other hand, their union over all n contains $\{T_c > 0\} \cap \Gamma_{a,b}$. Thus

$$\begin{aligned} P\{\{T_c > 0\} \cap \Gamma_{a,b}\} &\leq \lim_n P\{R_n^k < T_c^n\} \\ &\leq \left(\frac{c-b}{c-a}\right)^{k-1}, \end{aligned}$$

using (2.6). Since k is arbitrary, this proves that $T_c = 0$ a.s. on $\Gamma_{a,b}$, completing the proof.

We turn now to characterizing the cases (i), (ii) and (iii) of (2.4) in terms of the quadratic variation and local times for M , which we now describe.

If M is a continuous local martingale over an arbitrary filtration (\mathcal{G}_t) (satisfying the usual hypotheses) of (Ω, \mathcal{F}, P) it is easy to see, using (1.14), that for any finite stopping time R the quadratic variation process and the local time at a for $t \rightarrow M(R+t)1_{\{R < \infty\}}$ are respectively $\langle M, M \rangle_{R+t} - \langle M, M \rangle_R$ and $L_{R+t}^a - L_R^a$.

The following result obtains by an elementary covering argument.

(2.7) Proposition. *Let $M \in \mathcal{L}_{\text{open}}^c$. There exist unique random measures $Q(\omega, dt)$ and $\lambda^a(\omega, dt)$ on $]0, \infty[$ such that if $\{S_n\}$ are stopping times satisfying (0.1).*

Then for all n

(2.8) $M_{S_n+t}^2 - Q(\omega,]S_n(\omega), S_n(\omega) + t])$ is a local martingale over (\mathfrak{F}_{S_n+t}) ;

(2.9) $|M_{S_n+t} - a| - \lambda^a(\omega,]S_n(\omega), S_n(\omega) + t])$ is a local martingale over (\mathfrak{F}_{S_n+t}) .

In general, $Q(\omega, dt)$ and $\lambda^a(\omega, dt)$ blow up at the origin and so they are not always generated by continuous increasing processes normalized to vanish at the origin. However, (2.7) shows that a.s., $Q(\omega, \cdot)$ and $\lambda^a(\omega, \cdot)$ are Radon measures on $]0, \infty[$. We record now two elementary operations which preserve the class $\mathcal{L}_{\text{open}}^C$. The proofs are routine and are left to the reader.

(2.10) Proposition. If $M \in \mathcal{L}_{\text{open}}^C$ then

(2.11) for any stopping time T , $M_{t \wedge T} 1_{\{T > 0\}} \in \mathcal{L}_{\text{open}}^C$;

(2.12) if $H \in \mathfrak{F}_0$ then $1_H M \in \mathcal{L}_{\text{open}}^C$.

It is evident from (2.7) that if Q and λ^a are the quadratic variation and local time measures for M , then those for $M_{t \wedge T} 1_{\{T > 0\}}$ and $1_H M$ are respectively $1_{]0, T]}(t)Q(dt)$, $1_{]0, T]}(t)\lambda^a(dt)$ and $1_H(\omega)Q(\omega, dt)$, $1_H(\omega)\lambda^a(\omega, dt)$. For example, the last case above uses the observation that

$$|1_H M_{s+t} - a| - 1_H \lambda^a(]s, s+t]) = 1_H[|M_{s+t} - a| - \lambda^a(]s, s+t])] + 1_{H^c}|a|.$$

is a local martingale over (\mathfrak{F}_{s+t}) for all $s > 0$.

(2.13) Lemma. If $M \in \mathcal{L}_{\text{open}}^C$ and M is uniformly bounded, then $(M_t)_{t>0}$ is a martingale. Consequently, $M_0 = \lim_{t \downarrow 0} M_t$ exists a.s. and $(M_t)_{t \geq 0}$ is a martingale.

Proof. Once we prove that $(M_t)_{t>0}$ is a martingale, the assertions of the sentence will follow from the reverse martingale convergence theorem. Let $r_n \downarrow 0$. Then by (2.1), $t \rightarrow M(r_n + t)$ is a local martingale over $(\mathcal{F}(r_n + t))$, and its boundedness implies that it is in fact a uniformly integrable martingale. It follows that M_∞ exists and for all $t \geq 0$, $E\{M_\infty | \mathcal{F}(r_n + t)\} = M(r_n + t)$. That is, $(M_t)_{t>0}$ is a martingale over (\mathcal{F}_t) .

(2.14) Lemma. Let $M \in \mathcal{L}_{\text{open}}^C$ and suppose that $M_t^2 - t \in \mathcal{L}_{\text{open}}^C$. Then $\lim_{t \downarrow 0} M_t = M_0$ exists and is finite, and $(M_t - M_0)_{t \geq 0}$ is a standard Brownian motion over (\mathcal{F}_t) .

Proof. Fix a sequence of constant times $r_n \downarrow 0$ so that for all n , $M(r_n + t)$ and $M^2(r_n + t) - (r_n + t)$ are continuous local martingales over $(\mathcal{F}_{r_n + t})$. Lévy's Theorem (1.2) implies that $M(r_n + t) - M(r_n)$ is a standard Brownian motion. It follows that for $0 < u < v$, $M_v - M_u$ has a normal distribution with mean 0 and variance $v - u$, and that the increments of M_t are independent. Therefore the process $t \rightarrow M_1 - M_{1-t}$ ($0 \leq t < 1$) is a continuous martingale which is L^2 -bounded, so $\lim_{t \uparrow 1} M_1 - M_{1-t}$ exists a.s.. Consequently M_0 exists a.s., and since then $M_t - M_0 = \lim_{u \downarrow 0} M_t - M_u$ has independent Brownian increments, the result follows.

Here then is the main result of this section.

(2.15) Theorem. Let $M \in \mathcal{L}_{\text{open}}^C$, and let $\Omega_Q = \{\omega \in \Omega: Q(\omega,]0, t]) < \infty \text{ for some (and hence all) } t > 0\}$, $\Omega_a = \{\omega \in \Omega: \lambda^a(\omega,]0, t]) < \infty \text{ for some (and hence all) } t > 0\}$. For each fixed $a \in \mathbf{R}$, almost surely

- (i) $\{\omega: \lim_{t \downarrow 0} M_t(\omega) \text{ exists and is finite}\} = \Omega_Q$;
- (ii) $\{\omega: \lim_{t \downarrow 0} M_t(\omega) = \pm \infty\} = \Omega_a \setminus \Omega_Q$;
- (iii) for all $\omega \notin (\Omega_Q \cup \Omega_a)$, $\liminf_{t \downarrow 0} M_t = -\infty$ and $\limsup_{t \downarrow 0} M_t(\omega) = \infty$.

Proof. Let $\Lambda = \{\omega: \lim_{t \downarrow 0} M_t(\omega) \text{ exists and is finite}\}$. Since $\Lambda \in \mathcal{F}_{0+} = \mathcal{F}_0$, $N_t = 1_\Lambda M_t \in \mathcal{S}_{\text{open}}^C$ by (2.12). For all $\omega \in \Omega$, $\lim_{t \downarrow 0} N_t = N_0$ exists and is finite. For each $k \geq 1$ let $T_k = \inf\{t: |N_t| \geq k\}$. Then a.s. $\cup\{T_k > 0\} = \Omega$. The process $N_{t \wedge T_k} 1_{\{T_k > 0\}}$ is uniformly bounded and in $\mathcal{S}_{\text{open}}^C$ by (2.10). According to (2.13), $(N_{t \wedge T_k} 1_{\{T_k > 0\}})_{t \geq 0}$ is a bounded martingale over $(\mathcal{F}_t)_{t \geq 0}$. Thus $(N_{t \wedge T_k} 1_{\{T_k > 0\}})$ has a finite quadratic variation process A_t^k . On the other hand, by the remarks following (2.12), $A_t^k(\omega) = 1_\Lambda 1_{\{T_k(\omega) > 0\}} Q(\omega,]0, t \wedge T_k(\omega)[)$ a.s.. Since $\cup\{T_k > 0\} = \Omega$ a.s., it follows that $\Omega_Q \supset \Lambda$ a.s.. In order to show that $\Lambda \supset \Omega_Q$ a.s., we define now $Z_t = 1_{\Omega_Q} M_t$ for $t > 0$. Since $\Omega_Q \in \mathcal{F}_{0+} = \mathcal{F}_0$, $Z \in \mathcal{S}_{\text{open}}^C$ and Z has a finite quadratic variation process $A_t(\omega) = 1_{\Omega_Q}(\omega) Q(\omega,]0, t[)$. It suffices to prove that $Z \in \mathcal{S}_{\text{open}}^C$ having a finite quadratic variation process implies that Z_{0+} exists and is finite a.s.. To this end, we may adjoin to the underlying space by the usual product construction a standard Brownian motion (B_t) independent of \mathcal{F} . The quadratic variation process for Z remains the same over the augmented filtration $(\bar{\mathcal{F}}_t)$ since it is given by a limit of quadratic variational sums of Z without conditioning. Replacing Z_t by $\bar{Z}_t = Z_t + B_t$ affects neither the limiting behavior at time zero nor the finiteness of the quadratic variation. Let \bar{A}_t be the quadratic variation process for \bar{Z}_t . Then \bar{A}_t is continuous, strictly increasing, and $\bar{A}_\infty = \infty$ a.s. If $\tau_t = \inf\{s: \bar{A}_s > t\}$, then τ_t is strictly increasing, continuous, and $\tau_t < \infty$ for all $t < \infty$. In addition, $\tau_t \uparrow \infty$

as $t \uparrow \infty$. It is easy to see then that $\bar{Z}(\tau_t)_{t>0}$ is a continuous local martingale on $]0, \infty[$ relative to $(\bar{\mathcal{F}}_{\tau_t})$. Since $\bar{Z}_t^2 - A_t \in \mathcal{L}_{\text{open}}^C$, it is also the case that $\bar{Z}^2(\tau_t) - A(\tau_t) = \bar{Z}^2(\tau_t) - t$ is a continuous local martingale on $]0, \infty[$ relative to $(\bar{\mathcal{F}}_{\tau_t})$. Then (2.14) shows that $\lim_{t \uparrow \infty} \bar{Z}_{\tau_t}$ exists a.s., and hence $\lim_{t \uparrow \infty} Z_t$ exists a.s.. We have now proven (i). We now turn to (ii). On $\{M_{0+} = \pm \infty\}$, $\inf\{t: M_t = a\} > 0$ for all $a \in \mathbb{R}$. Since $\lambda^a(\omega, \cdot)$ is carried by $\{t: M_t(\omega) = a\}$ and $\lambda^a(\omega, \cdot)$ is a Radon measure on $]0, \infty[$, it follows that $\{M_{0+} = \pm \infty\} \subset \Omega_a$. On the other hand, since $\Omega_a \in \mathcal{F}_{0+} = \mathcal{F}_0$, $(1_{\Omega_a} M_t)_{t>0}$ is in $\mathcal{L}_{\text{open}}^C$ and so, by the remarks following (2.12), if we set $L_t^a(\omega) = 1_{\Omega_a}(\omega) \lambda^a(\omega,]0, t])$, then

$$N_t = |1_{\Omega_a} M_t - a| - L_t^a \in \mathcal{L}_{\text{open}}^C.$$

Obviously $\liminf_{t \uparrow \infty} N_t \geq 0$ so by (2.4), $\lim_{t \uparrow \infty} N_t$ exists a.s. in $[0, \infty]$.

Because of the alternatives (2.4) for M_t , it is clear that on Ω_a M_{0+} must exist a.s. in $[-\infty, \infty]$. That is $\Omega_a \subset \{M_{0+} \text{ exists in } \bar{\mathbb{R}}\}$. Using (2.4) again, we see that (2.15) has been proven.

(2.16) Corollary. *If $(M_t)_{t>0} \in \mathcal{L}_{\text{open}}^C$ and if $\lim_{t \uparrow \infty} M_t = M_0$ exists and is finite a.s., then $(M_t)_{t \geq 0}$ is in \mathcal{L}^C .*

Proof. The first part of the proof of (2.15) shows that if $T_n = \inf\{t \geq 0: |M_t| \geq n\}$ then $M_{t \wedge T_n} 1_{\{T_n > 0\}}$ is a bounded martingale over (\mathcal{F}_t) . Since $T_n \uparrow \infty$ a.s.

$$(M_t)_{t \geq 0} \in \mathcal{L}^C.$$

3. LOCAL MARTINGALE INCREMENTS.

The situation described in §2 does not cover the possible ways a singularity at the time origin can manifest itself. Consider the following examples.

(3.1) Let $M \in \mathcal{L}^C$ (a genuine continuous local martingale) and let C be a predictable process such that for all $t > 0$, $\int_t^{t+h} C_s^2 d\langle M, M \rangle_s < \infty$ for all $h > 0$ but $\int_0^1 C_s^2 d\langle M, M \rangle_s = \infty$ with positive probability. One may then define the stochastic integral $\int_t^{t+h} C_s dM_s$ as a local martingale on $[t, \infty[$ for all $t > 0$, but it is not possible in general to find one single normalization at $t=0$ which makes $\int_t^{t+h} C_s dM_s$ the increment over $]t, t+h]$ of one local martingale on $]0, \infty[$, simultaneously for all $t > 0$.

(3.2) Let $M \in \mathcal{L}_{\text{open}}^C$ and let C be a bounded predictable process. Then $\int_t^{t+h} C_s dM_s$ is well defined for all $t > 0$ and $h \geq 0$ but, as in (3.1), there is no way to define $N \in \mathcal{L}_{\text{open}}^C$ such that $\int_t^{t+h} C_s dM_s = N_{t+h} - N_t$.

(3.3) Let $M \in \mathcal{L}_{\text{open}}^C$ and let Q be its quadratic variation measure. Though we can define the process $M_t^2 - M_s^2 - Q(\omega,]s, t])$ for $t \geq s$, there is no $N \in \mathcal{L}_{\text{open}}^C$ having the same increments on $[s, \infty]$ for all $s > 0$.

These examples motivate the following definition.

(3.4) Definition. A local martingale increment process $(M_{s,t})$ is a family of real random variables indexed by pairs $0 < s \leq t$ such that

(3.5) for all $s > 0$, $t \rightarrow M_{s,t}$ ($t \geq s$) is a local martingale relative to $(\Omega, (\mathcal{F}_t)_{t \geq s}, P)$;

(3.6) for all triples $0 < r \leq s \leq t$, $M_{r,t} = M_{r,s} + M_{s,t}$.

Note that (3.6) forces $M_{t,t} = 0$ for all $t > 0$. The examples (3.1) – (3.3) obviously fit into the above scheme. Let $\mathcal{L}_{\text{inc}}^C$ denote the space of all local martingale increment processes relative to $(\Omega, \mathfrak{F}_t, P)$ such that for all $s > 0$, $t \mapsto M_{s,t}$ is a.s. continuous on $[s, \infty[$. If $M \in \mathcal{L}_{\text{inc}}^C$ then for all $s > 0$, $t \mapsto M_{s,t}$ ($t \geq s$) has an associated quadratic variation process $\langle M_{s,\cdot}, M_{s,\cdot} \rangle_t$ ($t \geq s$). If $0 < r < s$, then since $t \mapsto M_{r,t}$ and $t \mapsto M_{s,t}$ have the same increments over intervals in $[t, \infty[$, one has

$$\langle M_{r,\cdot}, M_{r,\cdot} \rangle_t - \langle M_{r,\cdot}, M_{r,\cdot} \rangle_s = \langle M_{s,\cdot}, M_{s,\cdot} \rangle_t.$$

It follows that there is a well defined random measure $Q(\omega, dt)$ defined on \mathbb{R}^{++} such that if $0 < s < t$, then

$$Q(\cdot,]s, t]) = \langle M_{r,\cdot}, M_{r,\cdot} \rangle_t - \langle M_{r,\cdot}, M_{r,\cdot} \rangle_s$$

for all $r \in]0, s]$. The random measure Q will be called the quadratic variation measure for M . For obvious reasons it is not in principle possible to define local times for $M \in \mathcal{L}_{\text{inc}}^C$.

(3.7) Lemma. Let $M \in \mathcal{L}_{\text{inc}}^C$ and let R be a stopping time with $P\{0 < R < \infty\} = 1$. Then $Y_t = M_{R, R+t}$ ($t \geq 0$) is a local martingale over the filtration (\mathfrak{F}_{R+t}) .

Proof. Let $s_n \uparrow 0$. If we show that for every n , $1_{\{R \geq s_n\}} Y_t$ is a local martingale over (\mathfrak{F}_{R+t}) . Then since $P\{R \geq s_n\} \uparrow 1$ as $n \rightarrow \infty$, the claimed result

will follow from the following argument. Let $\Lambda_n = \{R \geq s_n\} \in \mathfrak{F}_R$ and for $k \geq 1$ let $T_k = \inf\{t: |Y_t| \geq k\}$. Since $1_{\Lambda_n} Y_t$ is a continuous local martingale, $1_{\Lambda_n} Y(t \wedge T_k) 1_{\{T_k > 0\}}$ is a uniformly bounded martingale for all $k \geq 1$ and $n \geq 1$. Now let $n \rightarrow \infty$ to see that $Y(t \wedge T_k) 1_{\{T_k > 0\}}$ is a bounded martingale for all k . That is, $(Y_t)_{t \geq 0}$ is a local martingale. Fix now $s > 0$ and let $\Lambda = \{R > s\}$. Then $t \rightarrow M_{s, s+t}$ is a continuous local martingale over the filtration (\mathfrak{F}_{s+t}) , and hence because R_{Vs-s} is a stopping time over (\mathfrak{F}_{s+t}) we see from (1.14) that $M_{s, s+(R_{Vs-s})+t} = M_{s, R_{Vs}+t}$ is a local martingale over $(\mathfrak{F}_{R_{Vs}+t})$. But since $M_{R_{Vs}, R_{Vs}+t} = M_{s, R_{Vs}+t} - M_{s, R_{Vs}}$ and $M_{s, R_{Vs}} \in \mathfrak{F}_{R_{Vs}}$, it follows that $M_{R_{Vs}, R_{Vs}+t}$ is a local martingale over $(\mathfrak{F}_{R_{Vs}+t})$. However, $1_{\Lambda} Y_t = 1_{\Lambda} M_{R_{Vs}, R_{Vs}+t}$ is therefore a local martingale over $(\mathfrak{F}_{R_{Vs}+t})$, and since the trace of $\mathfrak{F}_{R_{Vs}+t}$ on Λ is equal to the trace of \mathfrak{F}_{R+t} on Λ , we are done.

(3.8) Proposition. Suppose that $M \in \mathcal{L}_{\text{inc}}^C$ and that for some $r > 0$, $M_{0,r} = \lim_{s \downarrow 0} M_{s,r}$ exists and is finite almost surely. For arbitrary $t \geq 0$, define $M_t = (M_{0,r} - M_{t,r}) 1_{[0,r]}(t) + (M_{0,r} + M_{r,t}) 1_{]r,\infty]}(t)$. Then $M \in \mathcal{L}^C$ and for all $0 < s < t$, $M_{s,t} = M_t - M_s$.

Proof. The fact that $M_{s,t} = M_t - M_s$ for all $0 < s < t$ is evident, as is the fact that $M_t = \lim_{s \downarrow 0} M_{s,t}$ for all $t > 0$. Because the increments $M_{s,t}$ form a local martingale in $t \geq s$ for $s > 0$ it follows that $(M_t)_{t > 0} \in \mathcal{L}_{\text{open}}^C$. Since $M_t \rightarrow 0$ a.s. as $t \downarrow 0$, the result follows from (2.16).

We turn now to a criterion which guarantees that $\lim_{s \downarrow 0} M_{s,r}$ exists for some (and hence all) $r > 0$.

(3.9) Proposition. Suppose that $M \in \mathcal{L}_{inc}^C$ has quadratic variation measure Q such that $Q(\omega,]0, t]) < \infty$ for all $t > 0$ a.s. . Then there exists $N \in \mathcal{L}^C$ such that for all $0 < s < t$, $M_{s,t} = N_t - N_s$.

Proof. Let $A_t(\omega) = Q(\omega,]0, t])$ for $t \geq 0$, so that $A_0 = 0$ and A is a continuous increasing process adapted to (\mathcal{F}_t) . We show that $\lim_{s \downarrow 0} M_{s,1}$ exists in \mathbb{R} almost surely. To this effect we may assume that A is strictly increasing and $A_\infty = \infty$ a.s., for if this is not so, adjoin an independent Brownian motion B_t so that $M_{s,t}$ is replaced by $\bar{M}_{s,t} = M_{s,t} + (B_t - B_s)$. See the proof of (2.15). With the above assumption on A in force, let $\tau_t = \inf\{s: A_s > t\}$. Just as in the proof of (2.15), for every $s > 0$ the process M_{τ_s, τ_t} is a local martingale increment process over the filtration (\mathcal{F}_{τ_t}) . We are also using (3.7) at this point. For every $s > 0$, the process M_{τ_s, τ_s+t} is a local martingale over (\mathcal{F}_{τ_s+t}) and so is $M_{\tau_s, \tau_s+t}^2 - (A(\tau_s+t) - A(\tau_s))$. By time change, it follows that $M_{\tau_s, \tau_t}^2 - (t-s)$ is a local martingale for $t \geq s$ over the filtration $(\mathcal{F}_{\tau_t})_{t \geq s}$. It follows from Lévy's theorem (1.2) that $t \rightarrow M_{\tau_s, \tau_t}$ ($t \geq s$) is a standard Brownian motion relative to (\mathcal{F}_{τ_t}) . In particular, for $s \leq t$, M_{τ_s, τ_t} has a normal distribution with mean zero and variance $t-s$. Consequently $s \rightarrow M_{\tau_{1-s}, \tau_1}$ ($0 \leq s < 1$) is an L^2 -bounded martingale so $\lim_{s \downarrow 1} M_{\tau_{1-s}, \tau_1}$ exists and is finite almost surely. In other words, $\lim_{u \downarrow 0} M_{u, \tau_1}$ a.s. exists and is finite, and one concludes that $M_{0,1} = \lim_{s \downarrow 0} M_{s,1}$ exists and is finite.

(3.10) Theorem. Let $M \in \mathcal{L}_{inc}^C$ with quadratic variation measure Q . Then $\Lambda = \{\omega \in \Omega: \lim_{s \downarrow 0} M_{s,1}(\omega) \text{ exists and is finite}\}$ and $\Gamma = \{\omega \in \Omega: Q(\omega,]0, 1]) < \infty\}$ are almost surely equal.

Proof. Since $\Gamma = \{Q(\omega,]0, \varepsilon] < \infty\}$ for every $\varepsilon > 0$, $\Gamma \in \mathfrak{F}_{0+} = \mathfrak{F}_0$. In addition, $\Lambda = \{\omega \in \Omega: \lim_{s \downarrow 0} M_{s,r}(\omega)$ exists and is finite $\}$ for every $r > 0$ so $\Lambda \in \mathfrak{F}_{0+} = \mathfrak{F}_0$. Obviously $1_{\Lambda} M_{s,t} \in \mathcal{L}_{inc}^C$ and since $\lim_{s \downarrow 0} 1_{\Lambda} M_{s,1}$ exists and is finite almost surely, (3.8) shows that $1_{\Lambda} M_{s,t}$ is obtained from the increments of a genuine continuous local martingale N . Since the quadratic variation of $1_{\Lambda} M$ is given on one hand by $1_{\Lambda} Q$ and on the other hand by $\langle N, N \rangle$, this proves that $\Lambda \subset \Gamma$ a.s.. Going the other way, $1_{\Gamma} M_{s,t} \in \mathcal{L}_{open}^C$ has quadratic variation measure $1_{\Gamma} Q$, which is a.s. finite near zero. Then (3.9) shows that $\lim_{s \downarrow 0} 1_{\Gamma} M_{s,t}$ exists a.s. and is finite, so $\Gamma \subset \Lambda$ almost surely.

We show next that all continuous local martingale increment processes may be obtained as increments of stochastic integrals in the manner of example (3.1).

(3.11) Theorem. Let $M \in \mathcal{L}_{inc}^C$. Then there exists $N \in \mathcal{L}^C$ and a predictable process C such that for all $0 < s < t$

$$(3.12) \quad \int_s^t C_u^2 d\langle N, N \rangle_u < \infty \quad \text{a.s. ;}$$

$$(3.13) \quad M_{s,t} = \int_s^t C_u dN_u$$

Proof. Let Q be the quadratic variation measure for M . Fix a sequence $t_n \downarrow 0$ and let D_t denote the predictable process

$$D_t(\omega) = \sum_{n \geq 1} 2^{-n} 1_{[t_n, t_{n-1}[}(\omega) e^{-Q(\omega,]t_n, t]})$$

where t_0 is set equal to $+\infty$. Obviously $D_t(\omega) > 0$ for all $t > 0$.

We have then, since $A_t^n = Q([\cdot]t_n, t]$ is continuous and $A_{t_n}^n = 0$

$$\begin{aligned} \int_0^\infty D_t^2 Q(dt) &= \sum_{n \geq 1} 2^{-2n} \int_{t_n}^{t_{n-1}} e^{-Q([\cdot]t_n, t]} Q(dt) \\ &= \sum_{n \geq 1} 2^{-2n} \int_{t_n}^{t_{n-1}} e^{A_t^n} dA_t^n \\ &= \sum_{n \geq 1} 2^{-2n} (1 - e^{-A^n(t_{n-1})}) \\ &\leq 1. \end{aligned}$$

For any $s > 0$, the stochastic integral

$$N_{s,t} = \int_s^t D_u dM_{s,u}$$

is therefore defined, and has quadratic variation process

$$\int_s^t D_u^2 d\langle M_{s,\cdot}, M_{s,\cdot} \rangle_u = \int_s^t D_u^2 Q(du)$$

bounded by one. If $0 < r < s < t$

$$N_{r,t} = \int_r^t D_u dM_{r,u} = \int_r^s D_u dM_{r,u} + \int_s^t D_u dM_{r,u}$$

$$\begin{aligned}
 &= N_{r,s} + N_{s,t} + \int_s^t D_u d(M_{r,u} - M_{s,u}) \\
 &= N_{r,s} + N_{s,t} .
 \end{aligned}$$

That is, $N \in \mathcal{L}_{inc}^C$. The quadratic variation measure for N is the measure $D_u^2 Q(du)$

which is bounded by one. Consequently, (3.10) shows that $N_{0,t} = \lim_{s \downarrow 0} N_{s,t}$

exists a.s. and defines a local martingale. It is clear then that since

$$M_{s,t} = \int_s^t D_u^{-1} dN_u, \text{ one obtains (3.12) and (3.13), setting } C_u = D_u^{-1} .$$

Our final results on \mathcal{L}_{inc}^C concerns the behavior of $M_{s,t}$ as $s \downarrow 0$ when it is known that convergence does not occur. The result here is rather less precise than either (2.4) or (2.15). Given $M \in \mathcal{L}_{inc}^C$, we define the maximal increment process $W_{s,t} (0 < s < t)$ for M by

$$(3.14) \quad W_{s,t} = \sup\{|M_{u,v}| : s \leq u < v \leq t\} .$$

It is clear that for $0 < s < t$, $W_{s,t} \in \mathfrak{F}_t$, and for fixed $s > 0$, $t \rightarrow W_{s,t}$ is continuous and increasing, while for fixed $t > 0$, $s \rightarrow W_{s,t}$ is continuous on $]0, t[$ and it increases as s decreases. The quantity

$$(3.15) \quad W_0 = \lim_{t \downarrow 0} \sup_{s < t} W_{s,t}$$

is in $\mathfrak{F}_{0+} = \mathfrak{F}_0$. It is clear that on $\{W_0 = 0\}$, $\lim_{s \downarrow 0} M_{s,t}$ exists and is finite.

On $\{W_0 > 0\}$, $\lim_{s \downarrow 0} M_{s,t}$ does not exist in \mathbb{R} , though it may exist in $\overline{\mathbb{R}}$.

(3.16) Theorem. Let $M \in \mathcal{L}_{\text{inc}}^C$ and let W_0 be the limiting oscillation of M , defined in (3.15). Then $P\{0 < W_0 < \infty\} = 0$.

Proof. As in the proof of (3.9) we may assume, adding an independent Brownian motion of M if necessary, that the quadratic variation measure Q for M has the property that a.s., for all $s > 0$, $t \rightarrow Q([s, t])$ is strictly increasing on $[s, \infty[$ and tends to infinity as $t \rightarrow \infty$. We shall prove that for all $a > 0$, $W_0 \geq 2a$ a.s. on $\{W_0 > a\}$, and from this the assertion follows trivially. Observe first that on $\{W_0 > a\}$, for every $t > 0$ there exist $0 < u < v < t$ with $|M_{u,v}| > a$. For $c > 0$ and $s > 0$ let

$$\begin{aligned} R^1(c, s) &= \inf\{t > s : |M_{u,t}| = c \text{ for some } u \in [s, t[\} \\ &= \inf\{t > s : \max_{s \leq u \leq t} M_{s,u} \geq \min_{s \leq u \leq t} M_{s,u} + c\}. \end{aligned}$$

Recursively, for $k \geq 1$ set

$$\begin{aligned} R^{k+1}(c, s) &= R^1(c, R^k(c, s)) \\ &= \inf\{t > R^k(c, s) : |M_{u,t}| = c \text{ for some } u \in [R^k(c, s), t[\}. \end{aligned}$$

On $\{W_0 > a\}$ since M must have oscillations of size $> a$ in arbitrarily small time intervals, it must be that for every $k \geq 1$

$$R^k(a, s) \rightarrow 0 \text{ as } s \downarrow 0.$$

On $\{W_0 < 2a\}$ there exists $t > 0$ such that $|M_{u,v}| < 2a$ for all $0 < u < v \leq t$. Consequently, on $\{W_0 < 2a\} \cap \{W_0 > a\}$, for each fixed $k \geq 1$, $R^k(a, s) < R^1(2a, s)$

for all sufficiently small $s > 0$. We prove that there exists a sequence $\varepsilon_k(a)$, independent of $s > 0$, such that

$$(3.17) \quad P\{R_a^k(a, s) < R_a^1(2a, s)\} \leq \varepsilon_k(a) \text{ and } \varepsilon_k(a) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Once we prove (3.17), since k is arbitrary, it will follow that $P\{a < W_0 < 2a\} = 0$.

In order to prove (3.17), we set $A_t = Q(]s, s+t])$ and let $\tau_t = \inf\{u: A_u > t\}$.

Then the process $B_t = M_{s, s+\tau_t}$ is a standard Brownian motion. Since $M_{s, t}$ ($t \geq s$) and B_t ($t \geq 0$) run through the same points in the same order, it is enough to prove that $P\{R_a^k < R_{2a}^1\}$ is dominated by a suitable sequence $\varepsilon_k(a)$, where R_c^k denotes $R^k(c, 0)$ for the process (B_t) ($t \geq 0$). By definition of the R_a^k , $|B(R_{2a}^1)| \leq 2a$, and the discrete parameter process $Y_k = B(R_a^k)$ ($k \geq 1$) is a random walk. Then

$$P\{R_a^k < R_{2a}^1\} \leq P\{\sup_{j \leq k} |Y_j| \leq 2a\}.$$

Letting $\varepsilon_k(a) = P\{\sup_{j \leq k} |Y_j| \leq 2a\}$, the fact that the law of Y_1 is not degenerate shows that $\varepsilon_k(a) \rightarrow 0$ as $k \rightarrow \infty$, completing the proof.

(3.18) Corollary. Let $M \in \mathcal{L}^C$ and let C be a predictable process such that

$\int_s^t C_u^2 d\langle M, M \rangle_u < \infty$ a.s. for all $0 < s < t < \infty$. Then for any $t > 0$,

$\Lambda = \{\omega: \int_s^t C_u(\omega) dM_u(\omega) \text{ converges to a finite limit as } s \downarrow 0\}$ is almost surely

equal to $\Gamma = \{\omega: \int_0^\varepsilon C_u^2(\omega) d\langle M, M \rangle_u < \infty \text{ for some } \varepsilon > 0\}$, and a.s. on Γ^C ,

$s \rightarrow \int_s^t C_u(\omega) dM_u$ has unbounded oscillations as $s \downarrow 0$.

Proof. The first assertion is just (3.10) applied to $N_{s,t} = \int_s^t C_u dM_u$ and the second assertion follows from (3.16) since $\Lambda^C = \{W_0(N) > 0\}$.

References

1. J. Azéma et M. Yor. En guise d'introduction. Astérisque 52-53 (Temps Locaux) 3-16 (1978).
2. J. L. Doob. Stochastic Processes. Wiley, New York (1953).
3. R. K. Gettoor and M. J. Sharpe. Conformal martingales. Invent. Math. 16, 271-308 (1972).
4. P. A. Meyer. Un cours sur les intégrales stochastiques. Sémin. de Probab. X, Springer lecture notes 511 (1976).
5. J. B. Walsh. A property of conformal martingales. Sémin. de Probab. XI, Springer lecture notes 581 (1977).

Department of Mathematics
University of California, San Diego
La Jolla, California 92093