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Transience and Recurrence of Markov Processes

by

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1. INTRODUCTION.

The purpose of this paper is to present an elementary exposition of some various conditions that have been used to define transience or recurrence of a Markov process. Transience is discussed in Proposition 2.2 and Corollary 2.3 while Proposition 2.4 deals with recurrence. In several important papers ([1] and [2]) Azema, Kaplan-Duflo, and Revuz have treated the question of recurrence and transience. Most of the results discussed here may be found in those papers or are easy consequences of them. Moreover, Proposition 2.4 follows in outline the series of somewhat opaque exercises II-(4.17) through II-(4.22) in [3]. In spite of this I think that an elementary and unified discussion of these ideas may be worthwhile.

2. STATEMENT OF RESULTS.

In this paper we assume that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a Markov process with state space $(E, \mathcal{E})$ which satisfies the right hypotheses as stated in Section 9 of [4]. In particular E is a universally measurable subset of a

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compact metric space $\hat{E}$, $\mathcal{E}$ is the σ-algebra of Borel subsets of the metric space E, and $\mathcal{E}^*$ the σ-algebra of universally measurable subsets of E. The process X is a right continuous strong Markov process such that if f is α-excessive for some $\alpha \geq 0$, then $t \to f(X_t)$ is almost surely right continuous. Let $(P_t)_{t\geq 0}$ and $(U^\alpha)_{\alpha\geq 0}$ be the semigroup and resolvent of X. As usual we write $U = U^0$ for the potential kernel of X. Each P_t and U^α sends universally measurable functions into universally measurable functions, but we do not assume that they map Borel functions to Borel functions. Moreover, we do not assume that excessive functions are nearly Borel, although they are nearly Borel in the Ray topology. However, we shall make no explicit use of the Ray topology or the Ray-space of X. We do *not* assume that $P_t 1 = 1$ and so we suppose the existence of a point $\Delta \notin E$ that acts as a cemetery. As usual we write $E_\Delta = E \cup \{\Delta\}$ and $\zeta = \inf\{t: X_t = \Delta\}$ for the lifetime of X. Then $P_t 1(x) = P^x(t < \zeta)$.

We adopt the following notational conventions. A function f is a non-negative universally measurable function on E and a set B is a universally measurable subset of E unless stated otherwise. Also any function f is automatically extended to E_Δ by setting $f(\Delta) = 0$. The statements $f > 0$ or $f < \infty$ mean $f(x) > 0$ or $f(x) < \infty$ for all x in E. For each $\alpha \geq 0$, $\mathcal{S}^\alpha$ denotes the cone of α excessive functions. These are functions on E which vanish at Δ by our convention. Let $\mathcal{E}^e$ be the σ-algebra generated by $\bigcup_{\alpha\geq 0} \mathcal{S}^\alpha$. It is immediate from the resolvent equation that $\mathcal{E}^e = \sigma(\mathcal{S}^\alpha)$ for each fixed $\alpha > 0$. Under our assumptions $\mathcal{E} \subset \mathcal{E}^e \subset \mathcal{E}^*$. If $B \in \mathcal{E}^e$, define

$$T_B = \inf\{t > 0: X_t \in B\}$$

(2.1)

$$L_B = \sup\{t: X_t \in B\}$$

where the infimum, resp. supremum, of the empty set is taken to be ∞, resp. 0. Then T_B is the hitting time of B and is an $(\mathcal{F}_t)$ stopping, and L_B is the last exit time from B and is $\mathcal{F}$ measurable.

A set $B \in \mathcal{E}^e$ is called *transient* if $L_B < \infty$ almost surely. *We shall denote by* (LSC) *the condition that for some* $\alpha > 0$ *the* α*-excessive functions are lower-semi-continuous* (ℓsc). If $\beta < \alpha$, $\mathcal{S}^\beta \subset \mathcal{S}^\alpha$ and so under (LSC) all excessive functions ($\alpha = 0$) are ℓsc. In Proposition 2.4 we shall use the (apparently weaker) condition that *all excessive functions* ($\alpha = 0$) *are* ℓsc *which we denote by* (LSC_0).

We are now prepared to state our results.

(2.2) PROPOSITION. *The following conditions are equivalent.*

(i) *There exists a bounded* h *with* Uh *bounded and* $Uh > 0$.

(ii) *There exists a bounded finely continuous* $\mathcal{E}^e$ *measurable* $h > 0$ *with* Uh *bounded. (Under* (LSC) h *may be chosen* ℓsc.*)*

(iii) *There exists a sequence* (h_n) *with each* h_n *and* Uh_n *bounded and* $Uh_n \uparrow \infty$.

(iv) *There exists a sequence* (B_n) *with* $B_n \uparrow E$ *and* $U(\cdot, B_n)$ *bounded for each* n.

(i') *There exists* $h < \infty$ *with* $0 < Uh < \infty$.

(ii') *There exists a finely continuous* $\mathcal{E}^e$ *measurable* h *with* $0 < h < \infty$ *and* $Uh < \infty$. *(Under* (LSC) h *may be chosen* ℓsc.*)*

(iii') *There exists a sequence* (h_n) *with* $h_n < \infty$ *and* $Uh_n < \infty$ *for each* n *and* $Uh_n \uparrow \infty$.

(iv') *There exists a sequence* (B_n) *with* $B_n \uparrow E$ and $U(\cdot, B_n) < \infty$ *for each* n.

(v) *There exists a sequence* (B_n) *of transient sets with* $B_n \uparrow E$; *that is, each* $B_n \in \mathcal{E}^e$ *and* $L_{B_n} < \infty$ *almost surely.*

If X satisfies any, and hence all, of the conditions in (2.2) we shall say that X is *transient*. It will be evident from the proof of (2.2) that the sets B_n in (iv), (iv'), and (v) may be chosen finely open and $\mathcal{E}^e$ measurable, and even open under (LSC). If one assumes a bit more about X, then the statements in (2.2) are equivalent to statements involving compact sets. For example, the following result is an immediate consequence of (2.2) and the fact that the sets B_n in (iv), (iv'), and (v) may be chosen open under (LSC).

(2.3) <u>COROLLARY</u>. *Assume* (LSC) *and that* E *is a countable union of compact sets. Then each of the following conditions is equivalent to* X *being transient.*

(vi) $U(\cdot, K)$ *is bounded for all compact* K.

(vi') $U(\cdot, K) < \infty$ *for all compact* K.

(vii) *Each compact* K *is transient.*

We turn now to recurrence. If f is a function and $c \in \overline{\mathbb{R}}$, then the statement $f = c$ means $f(x) = c$ for all x in E. If $B \in \mathcal{E}^e$, define

$$\phi_B(x) = P^x(T_B < \infty).$$

It is well known that ϕ_B is excessive and ϕ_B is called the equilibrium potential of B. Clearly $\phi_B(x) = P^x(L_B > 0)$.

(2.4) <u>PROPOSITION</u>. *If* E *has at least two points the following conditions are equivalent.*

(i) *For each* B, $U(\cdot, B) = 0$ *or* $U(\cdot, B) = \infty$.

(ii) *If* B *is nonvoid and finely open,* $U(\cdot, B) = \infty$.

(iii) *If* $B \in \mathcal{E}^e$ *is not polar, then* $\phi_B = 1$.

(iv) *If* $B \in \mathcal{E}^e$ *is nonvoid and finely open (or only open under* (LSC_0)), *then* $\phi_B = 1$.

(v) *Each excessive function is constant.*

(vi) *If* $B \in \mathcal{E}^e$ *is not polar, then* $L_B = \infty$ *almost surely.*

(vii) *If* $B \in \mathcal{E}^e$ *is nonvoid and finely open (or only open under* (LSC_0)), *then* $L_B = \infty$ *almost surely.*

If X satisfies any, and hence all, of the conditions in (2.4) we shall say that X is *recurrent*.

3. PROOFS. We begin by establishing Proposition 2.2. Clearly the statements (i), (ii), (iii), and (iv) imply respectively the corresponding primed statements (i'), (ii'), (iii'), and (iv'). In addition it is obvious that (ii) ⇒ (i) and (ii') ⇒ (i') since $Uh > 0$ if $h > 0$.

(a) (i) ⇒ (ii). Let h be bounded with Uh bounded and $Uh > 0$. Let $g = U^{\alpha}h$ with $\alpha > 0$. Then $Uh > 0$ implies that $g = U^{\alpha}h > 0$. Clearly $g \le Uh$ and so g is bounded. From the resolvent equation

$$Ug = UU^{\alpha}h = \alpha^{-1}[Uh - U^{\alpha}h] \le \alpha^{-1}Uh$$

and so Ug is bounded. Since g is α-excessive it is finely continuous and $\mathcal{E}^e$ measurable. Under (LSC) one may choose $\alpha > 0$ such $g = U^{\alpha}h$ is ℓsc. Thus (i) ⇒ (ii).

(b) (i) ⇔ (iii). If (i) holds let $h_n = nh$. Then $Uh_n = nUh \uparrow \infty$ since $Uh > 0$. If (iii) holds, let $b_n = \sup\{1, \|h_n\|, \|Uh_n\|\}$ where $\|f\| = \sup_x |f(x)|$ for any function f, and define $h = \Sigma\, (2^n b_n)^{-1}\, h_n$. Then h and Uh are bounded, and since for each $x \in E$ there exists an n with $Uh_n(x) > 0$, it follows that $Uh > 0$.

(c) (iv) ⇒ (iii). Let $h_n = n1_{B_n}$. Since for each x, $U(x, B_n) \uparrow U(x, E) > 0$ it follows that $Uh_n \uparrow \infty$.

(d) (ii) ⇒ (iv). Let $B_n = \{h > 1/n\}$. Then $B_n \uparrow E$ since $h > 0$. Clearly B_n is finely open and $\mathcal{E}^e$ measurable, and even open under (LSC). Now $1_{B_n} \le nh$ and so $U(\cdot, B_n) \le n\, Uh$ is bounded for each n.

We have now established the equivalence of (i), (ii), (iii), and (iv). The equivalence of the primed statements is established by exactly the same arguments except for the implication (iii') $\Rightarrow$ (i'). We shall actually show that (iii') $\Rightarrow$ (i) establishing the implications from the primed to unprimed statements. To this end if f is a function, $a > 0$, and $A = \{Uf \le a\}$, then

$$U(1_A f)(x) = E^x \int_0^\infty 1_A(X_t) f(X_t) dt$$

$$\le E^x \int_{T_A}^\infty f(X_t) dt = P_A Uf(x) \le a \ .$$

Let (h_n) be the sequence in (iii'). Set

$$h_n' = \sup_{1 \le k \le n} h_k \le h_1 + \ldots + h_n \ .$$

Then (h_n') is increasing, $Uh_n' < \infty$, and $Uh_n' \uparrow \infty$. Now replacing h_n' by $h_n' \wedge n$ we see that we may suppose that the sequence (h_n) in (iii') is increasing and that each h_n is bounded. For $n \ge 1$ and $k \ge 1$ let $A_{n,k} = \{Uh_n \le k\}$ and $g_{n,k} = 1_{A_{n,k}} h_n$. Then by the above estimate $Ug_{n,k} \le k$. For each fixed n, $A_{n,k} \uparrow E$ as $k \uparrow \infty$, and so

$$\lim_n \lim_k Ug_{n,k} = \lim_n Uh_n = \infty \ .$$

Hence for each x there exist n and k with $Ug_{n,k}(x) > 0$. Therefore

$$h = \sum_{n,k} (2^{n+k} b_{n,k})^{-1} g_{n,k} \ ,$$

where $b_{n,k} = \sup(1, \|g_{n,k}\|, \|Ug_{n,k}\|)$, is a bounded function with Uh bounded and $Uh > 0$. Hence (iii') ⇒ (i).

This completes the equivalence of all the statements in (2.2) except (v).

(e) (i) ⇒ (v). Let $B_n = \{Uh > 1/n\}$. Then $B_n \in \mathcal{E}^e$, B_n is finely open (even open under (LSC)), and $B_n \uparrow E$. Thus it suffices to show that each B_n is transient. To this end let $B = \{Uh > a\}$ where $a > 0$. Then if $t > 0$,

$$P_t Uh(x) \geq P_{t+T_B \circ \theta_t} Uh(x) \geq a\, P^x(t + T_B \circ \theta_t < \infty) .$$

But $P_t Uh \to 0$ as $t \to \infty$ since Uh is bounded, and consequently $P^x(n + T_B \circ \theta_n < \infty$ for all $n) = 0$ for each x. Therefore $L_B < \infty$ almost surely. Hence each B_n is transient.

Before coming to the final implication we state and prove a well-known fact that will also be needed in the proof of (2.4).

(3.1) LEMMA. *Let* g *be a bounded excessive function with* $P_t g \to 0$ *as* $t \to \infty$. *Define* $g_n = n(g - P_{1/n}\, g)$. *Then each* g_n *is a bounded, nonnegative, finely continuous* $\mathcal{E}^e$ *measurable function such that*

$$Ug_n = n \int_0^{1/n} P_t g \, dt \uparrow g$$

as $n \to \infty$.

PROOF. Since g is excessive, $P_{1/n}\, g$ is also excessive and so each g_n is bounded, nonnegative, finely continuous, and $\mathcal{E}^e$ measurable. Now

$$\int_0^t P_s g_n ds = n \int_0^t P_s g \, ds - n \int_{1/n}^{t+1/n} P_s g \, ds =$$

$$= n\int_0^{1/n} P_s g\, ds - n\int_t^{t+1/n} P_s g\, ds$$

provided $t > 1/n$. But $s \to P_s g$ is decreasing and so the last term is dominated by $P_t g$. Thus letting $t \to \infty$ and using the hypothesis we obtain $Ug_n = n \int_0^{1/n} P_s g\, ds$. Making the change of variable $t = ns$ it is clear that $Ug_n \uparrow g$.

(f) (v) ⇒ (i). Let $B_n \in \mathcal{E}^e$ be transient with $B_n \uparrow E$. Fix $B = B_n$ and let $\phi(x) = P^x(L_B > 0) = P^x(T_B < \infty)$. Then ϕ is excessive and

$$P_t\phi(x) = P^x(L_B \circ \theta_t > 0) = P^x(L_B > t) \to 0$$

as $t \to \infty$ since $L_B < \infty$ almost surely. Now let $\phi_K(x) = P^x(T_{B_k} < \infty)$. Since $B_k \uparrow E$, it follows that $T_{B_k} \downarrow 0$ and so $\phi_k \uparrow 1$. Let $g_{n,k} = n(\phi_k - P_{1/n}\, \phi_k)$. Then $g_{n,k} \le n$ and by (3.1), $Ug_{n,k} \uparrow \phi_k \le 1$ as $n \to \infty$. Therefore for each x there exist n and k with $Ug_{n,k}(x) > 0$, and so we can construct a bounded h with Uh bounded and $Uh > 0$ as before. This completes the proof of Proposition 2.2.

We turn now to the proof of (2.4). The following implications are obvious: (i) ⇒ (ii), (iii) ⇒ (iv), and (v) ⇒ (iv). We shall break up the proof into a series of steps.

(3.2) LEMMA. *Assume* (ii). *If* $B \in \mathcal{E}^e$ *is such that* $\phi_B(x) = 1$ *for some* x *in* E, *then* $\phi_B = 1$.

PROOF. Since ϕ_B is excessive, $t \to P_t\phi_B$ is decreasing. Let $\psi = \lim_{t\to\infty} P_t\phi_B \le \phi_B$. Then $P_s\psi = \lim_{t\to\infty} P_{t+s}\,\phi_B = \psi$ for all $s \ge 0$. Let $g = \phi_B - \psi$. Then

$$P_tg = P_t\phi_B - P_t\psi = P_t\phi_B - \psi \uparrow \phi_B - \psi$$

as $t \to 0$ and so g is excessive. Clearly $P_tg \to 0$ as $t \to \infty$ and $g \le 1$. Thus by (3.1) there exists a sequence (g_n) of bounded, finely continuous, $\mathcal{E}^e$ measurable functions with $Ug_n \uparrow g$. If $B_n = \{g_n > 0\}$ is not empty, then (ii) implies that $Ug_n = \infty$. Since $g \le 1$ we conclude that each $g_n = 0$ and so $g = 0$. That is $\phi_B = \psi$, and hence $P_t\phi_B = \phi_B$ for all t.

So far we have not used the hypotheses on B. Let $D = \{\phi_B < 1\}$. Then D is finely open and $\mathcal{E}^e$ measurable. By assumption there exists an x with $\phi_B(x) = 1$, and so

$$1 = \phi_B(x) = P_t\phi_B(x) = E^x[\phi_B(X_t)].$$

Consequently for each $t \ge 0$, $P^x(X_t \in D) = 0$, which in turn implies that $U(x, D) = 0$. But by (ii) if D is not empty, $U(\cdot, D) = \infty$, and this establishes (3.2).

(ii) $\Rightarrow$ (iv). Let $B \in \mathcal{E}^e$ be nonvoid and finely open. If $x \in B$, $\phi_B(x) = 1$, and so $\phi_B = 1$ by (3.2).

(iv) $\Rightarrow$ (v). Let f be a non-constant excessive function. Then there exist $0 < a < b$ and $x \in E$ with $f(x) < a$ and $B = \{f > b\}$ nonvoid. Also B is finely open (open under (LSC_0)). Hence $\phi_B = 1$ by (iv). But

$$\begin{aligned} a > f(x) \ge P_Bf(x) &= E^x[f(X_{T_B});\ T_B < \infty] \\ &\ge b\, P^x(T_B < \infty) = b\phi_B(x) = b, \end{aligned}$$

establishing (v).

In view of the obvious implications mentioned above (3.2) we now have established the following:

(3.3) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v) and (iii) $\Rightarrow$ (iv).

The next lemma is the only place where we explicitly use the assumption that E has at least two points.

(3.4) LEMMA. *Assume* (iv) *and that* E *has at least two points. Then* $P^x(\zeta < \infty) = 0$ *for all* $x \in E$.

PROOF. Let $\psi(x) = E^x(1 - e^{-\zeta})$. Observe that

$$P_t\psi(x) = E^x\{\psi(X_t);\ t < \zeta\} = E^x\{1 - e^{-(\zeta - t)};\ t < \zeta\} \uparrow \psi(x)$$

as $t \to 0$. Hence ψ is constant since (iv) $\Leftrightarrow$ (v). Therefore $E^x(e^{-\zeta}) = c$ for all x in E. Since E is a metric space with at least two points there exist x in E and a nonvoid open set $G \subset E$ with x not in $\overline{G}$, the closure of G. By (iv), $\phi_G = 1$. Of course, since $G \subset E$, $\{T_G < \infty\} = \{T_G < \zeta\}$. Hence

$$\begin{aligned} c = E^x(e^{-\zeta}) &= E^x(e^{-\zeta};\ T_G < \zeta) \\ &= E^x(e^{-T_G} E^{X(T_G)}[e^{-\zeta}];\ T_G < \zeta) \\ &= cE^x(e^{-T_G}). \end{aligned}$$

But $x \notin \overline{G}$ and so $E^x(e^{-T_G}) < 1$. Therefore $c = 0$, establishing (3.4).

(3.5) (iv) $\Rightarrow$ (i). Let $B \in \mathcal{E}^*$ and suppose that $b = \sup U(\cdot, B) > 0$. Let $0 < a < b$ and $A = \{U(\cdot, B) > a\}$. Then by (iv), $\phi_A = 1$, and from (3.4), $\zeta = \infty$ almost surely. Now fix $x \in E$ and $t > 0$, and observe that

$$E^x \int_{t+T_A \circ \theta_t}^{\infty} 1_B(X_s)ds = E^x\{E^{X(t)} \int_{T_A}^{\infty} 1_B(X_s)ds\}$$

$$= E^x\{E^{X(t)}[U(X_{T_A}, B); T_A < \infty]\}$$

$$\geq aP^x(t < \zeta) = a.$$

Moreover

$$U(x, B) \geq E^x \int_0^t 1_B(X_s)ds + E^x \int_{t+T_A \circ \theta_t}^{\infty} 1_B(X_s)ds$$

$$\geq E^x \int_0^t 1_B(X_s)ds + a,$$

and letting $t \to \infty$ we obtain $U(x, B) \geq U(x, B) + a$. This yields $U(x, B) = \infty$, establishing (i) since x is arbitrary.

(3.6) (ii) $\Rightarrow$ (iii). Since (ii) $\Rightarrow$ (iv), $\zeta = \infty$ almost surely by (3.4). Suppose $B \in \mathcal{E}^e$ is not polar. Then by (iv), $\phi_B = c$ where c is a strictly positive constant. Hence for each $x \in E$ and $t > 0$

$$c = P^x(T_B < \infty) = P^x(T_B \leq t) + P^x(t < T_B, T_B < \infty)$$

$$= P^x(T_B \leq t) + E^x[P^{X(t)}(T_B < \infty); t < T_B]$$

$$= P^x(T_B \leq t) + cP^x(t < T_B),$$

and letting $t \to \infty$ we obtain $c(1-c) = 0$. Since $c \neq 0$ we must have $c = 1$ proving (iii).

In view of (3.3), (3.5), and (3.6) we have now established the equivalence of the first five statements in (2.4).

(v) $\Rightarrow$ (vi). Let $B \in \mathcal{E}^e$ and assume that B is not polar. By (iii), $P^x(L_B > 0) = \phi_B(x) = 1$ for all x. Let $\psi(x) = P^x(L_B < \infty) = P^x(0 < L_B < \infty)$. Then $P_t\psi(x) = P^x(t < L_B < \infty)$. Thus ψ is excessive and $P_t\psi \to 0$ as $t \to \infty$. By (v) $\psi = c$, and since $\zeta = \infty$ almost surely according to (3.4), $c = P_t c$. But $P_t c = P_t \psi \to 0$ as $t \to \infty$. Therefore $c = 0$ proving (vi).

It remains to show that (vii) $\Rightarrow$ (i) in order to complete the proof of (2.4). Let $B \in \mathcal{E}^*$ and suppose that for some $x \in E$, $U(x, B) < \infty$. Then for this x, $P_t U1_B(x) \to 0$ as $t \to \infty$. If $a > 0$ let $G = \{U(\cdot, B) > a\}$. If G is not empty (vii) implies that $L_G = \infty$ almost surely and consequently for each t, $P^x(t + T_G \circ \theta_t < \infty) = 1$. But

$$P_t U1_B(x) \geq E^x[U1_B(X_{t+T_G\circ\theta_t}); \; t + T_G \circ \theta_t < \infty]$$

$$= E^x\{E^{X(t)}[U1_B(X_{T_G}); \; T_G < \infty]\}$$

$$\geq aP^x(t + T_G \circ \theta_t < \infty) = a.$$

Hence G must be empty and since $a > 0$ was arbitrary $U(\cdot, B) = 0$. This completes the proof of (2.4).

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