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ON STOPPED FEYNMAN-KAC FUNCTIONALS

by

Kai Lai Chung*

1. Introduction

Let $X = \{x(t),\ t \geq 0\}$ be a strong Markov process with continuous paths on $R = (-\infty, +\infty)$. Such a process is often called a diffusion. For each real b, we define the hitting time τ_b as follows:

$$(1) \qquad \tau_b = \inf\{t > 0 \mid x(t) = b\} .$$

Let P_a and E_a denote as usual the basic probability and expectation associated with paths starting from a. It is assumed that for every a and b, we have

$$(2) \qquad P_a\{\tau_b < \infty\} = 1 .$$

Now let q be a bounded Borel measurable function on R, and write for brevity

$$(3) \qquad e(t) = \exp\left(\int_0^t q(x(s))ds\right).$$

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This is a multiplicative functional introduced by R. Feynman and M. Kac. In this paper we study the quantity

$$u(a,b) = E_a\{e(\tau_b)\} . \tag{4}$$

Since q is bounded below, (2) implies that $u(a,b) > 0$ for every a and b, but it may be equal to $+\infty$. A fundamental property of u is given by

$$u(a,b)\, u(b,c) = u(a,c) , \tag{5}$$

valid for $a < b < c$, or $a > b > c$. This is a consequence of the strong Markov property (SMP).

2. The Results

We begin by defining two abscissas of finiteness, one for each direction.

$$\begin{aligned} \beta &= \inf\{b \in R \,|\, \exists\ a < b : u(a,b) = \infty\} \\ &= \sup\{b \in R \,|\, \forall\ a < b : u(a,b) < \infty\}; \\ \alpha &= \sup\{a \in R \,|\, \exists\ b > a : u(b,a) = \infty\} \\ &= \inf\{a \in R \,|\, \forall\ b > a : u(b,a) < \infty\}. \end{aligned} \tag{6}$$

It is possible, e.g., that $\beta = -\infty$ or $+\infty$. The first case occurs when X is the standard Brownian motion, and $q(x) \equiv 1$; for then, $u(a,b) \geq E_a(\tau_b) = \infty$, for any $a \neq b$.

Lemma 1. We have

$$\beta = \inf\{b \in R \mid \forall\ a < b : u(a,b) = \infty\}$$
$$= \sup\{b \in R \mid \exists\ a < b : u(a,b) < \infty\} ;$$
$$\alpha = \sup\{a \in R \mid \forall\ b > a : u(b,a) = \infty\}$$
$$\inf\{a \in R \mid \exists\ b > a : u(b,a) < \infty\} .$$

Proof: It is sufficient to prove the first equation above for β, because the second is trivially equivalent to it, and the equations for α follow by similar arguments. Suppose $u(a,b) = \infty$; then for $x < a < b$ we have $u(x,b) = \infty$ by (5). For $a < x < b$ we have by SMP,

$$u(x,b) \geqslant E_x\{e(\tau_a) ; \tau_a < \tau_b\}\, u(a,b) = \infty$$

since $P_x\{\tau_a < \tau_b\} > 0$ in consequence of (2).

The next lemma is a martingale argument. Let $\mathfrak{F}_t$ be the σ-field generated by $\{x_s,\ 0 \leqslant s \leqslant t\}$ and all null sets, so that $\mathfrak{F}_{t+} = \mathfrak{F}_t$ for $t \geqslant 0$; and for any optional τ let $\mathfrak{F}_\tau$ and $\mathfrak{F}_{\tau+}$ and $\mathfrak{F}_{\tau-}$ have the usual meanings.

Lemma 2. If $a < b < \beta$, then

(7) $$\lim_{a \uparrow b} u(a,b) = 1 ;$$

(8) $$\lim_{b \downarrow a} u(a,b) = 1 .$$

Proof: Let $a < b_n \uparrow b$ and consider

$$(9) \qquad E_a\{e(\tau_b)\,|\,\mathfrak{F}(\tau_{b_n})\}\ , \qquad n \geq 1\ .$$

Since $b < \beta$, $u(a,b) < \infty$ and the sequence in (9) forms a martingale. As $n \uparrow \infty$, $\tau_{b_n} \uparrow \tau_b$ a.s. and $\mathfrak{F}(\tau_{b_n}) \uparrow \mathfrak{F}(\tau_n-)$. Since $e(\tau_b) \in \mathfrak{F}(\tau_b-)$, the limit of the martingale is a.s. equal to $e(\tau_b)$. On the other hand, the conditional probability in (9) is also equal to

$$E_a\{e(\tau_b)\exp(\int_{\tau_{b_n}}^{\tau_b} q(x(s))ds)\,|\,\mathfrak{F}(\tau_{b_n})\} = e(\tau_{b_n})u(b_n,b)\ .$$

As $n \uparrow \infty$, this must then converge to $e(\tau_b)$ a.s.; since $e(\tau_{b_n})$ converges to $e(\tau_b)$ a.s., we conclude that $u(b_n,b) \to 1$. This establishes (7).

Now let $\beta > b > a_n \downarrow a$, and consider

$$(10) \qquad E_a\{e(\tau_b)\,|\,\mathfrak{F}(\tau_{a_n})\}\ , \qquad n \geq 1\ .$$

This is again a martingale. Although $a \to \tau_a$ is a.s. left continuous, not right continuous, for each fixed a we do have $\tau_{a_n} \downarrow \tau_a$ and $\mathfrak{F}(\tau_{a_n}) \downarrow \mathfrak{F}(\tau_a)$. Hence we obtain as before $u(a_n,b) \to u(a,b)$ and consequently

$$u(a,a_n) = \frac{u(a,b)}{u(a_n,b)} \to 1\ .$$

This establishes (8).

The next result illustrates the basic probabilistic method.

Theorem 1. The following three propositions are equivalent:

(i) $\beta = +\infty$;

(ii) $\alpha = -\infty$;

(iii) For every a and b , we have

$$u(a,b)u(b,a) \leq 1 . \tag{11}$$

Proof: Suppose $x(0) = b$ and let $a < b < c$. If (i) is true then $u(b,c) < \infty$ for every $c > b$. Define a sequence of successive hitting times T_n as follows (where θ denotes the usual shift operator):

$$\begin{gathered} S = \begin{cases} \tau_a & \text{if } \tau_a < \tau_c , \\ \infty & \text{if } \tau_c < \tau_a ; \end{cases} \\ T_o = 0 , \qquad T_1 = S , \\ T_{2n} = T_{2n-1} + \tau_b \circ \theta_{T_{2n-1}} , \qquad T_{2n+1} = T_{2n} + S \circ \theta_{T_{2n}} , \end{gathered} \tag{12}$$

for $n \geq 1$. Define also

$$N = \min\{n \geq 0 \,|\, T_{2n+1} = \infty\} . \tag{13}$$

It follows from $P_b\{\tau_c < \infty\} = 1$ that $0 \leq N < \infty$ a.s. For $n \geq 0$, we have

$$
\begin{aligned}
(14)\quad E_b\{e(\tau_c) \; ; \; N = n\} &= E_b\{\exp(\sum_{k=0}^{2n} \int_{T_k}^{T_{k+1}} q(x(s))ds)\} \\
&= E_b\{e(\tau_a) \; ; \; \tau_a < \tau_c\}^n \, E_a\{e(\tau_b)\}^n \, E_b\{e(\tau_c) \; ; \; \tau_c < \tau_a\} \; .
\end{aligned}
$$

Since the sum of the first term in (14) over $n \geq 0$ is equal to $u(b,c) < \infty$, the sum of the last term in (14) must converge. Thus we have

$$
(15) \qquad E_b\{e(\tau_a) \; ; \; \tau_a < \tau_c\} \, u(a,b) < 1 \; .
$$

Letting $c \to \infty$ we obtain (11). Hence $u(b,a) < \infty$ for every $a < b$ and so (ii) is true. Exactly the same argument shows that (ii) implies (iii) and so also (i).

We are indebted to R. Durrett for ridding the next lemma of a superfluous condition.

Lemma 3. Given any $a \in R$ and $Q > 0$, there exists an $\varepsilon = \varepsilon(a,Q)$ such that

$$
(16) \qquad E_a\{e^{Q\sigma_\varepsilon}\} < \infty
$$

where

$$
\sigma_\varepsilon = \inf\{t > 0 \,|\, x(t) \notin (a - \varepsilon, \, a + \varepsilon)\} \; .
$$

Proof: Since X is strong Markov and has continuous paths, there is no "stable" point. This implies $P_a\{\sigma_\varepsilon \geq 1\} \to 0$ as $\varepsilon \to 0$ and so there exists ε such that

(17) $$P_a\{\sigma_\varepsilon \geq 1\} < e^{-(Q+1)} .$$

Now σ_ε is a terminal time, so $x \to P_x\{\sigma_\varepsilon \geq 1\}$ is an excessive function for the process X killed at σ_ε . Hence by standard theory it is finely continuous. For a diffusion under hypothesis (2) it is clear that fine topology coincides with the Euclidean. Thus $x \to P_x\{\sigma_\varepsilon \geq 1\}$ is in fact continuous. It now follows that we have, further decreasing ε if necessary:

(18) $$\sup_{|x-a|<\varepsilon} P_x\{\sigma_\varepsilon \geq 1\} < e^{-(Q+1)} .$$

A familiar inductive argument then yields for all $n \geq 1$.

(19) $$P_a\{\sigma_\varepsilon \geq n\} < e^{-n(Q+1)}$$

and (16) follows.

Lemma 4. For any $a < \beta$ we have

(20) $$u(a,\beta) = \infty ;$$

for any $b > \alpha$ we have $u(b,\alpha) = \infty$.

Proof: We will prove that if $u(a,b) < \infty$, then there exists $c > b$ such that $u(b,c) < \infty$. This implies (20) by Lemma 1, and the second assertion is proved similarly.

Let $Q = \|q\|$. Given b we choose a and b so that $a < b < d$ and

(21) $$E_b\{e^{Q(\tau_a \wedge \tau_d)}\} < \infty .$$

This is possible by Lemma 3. Now let $b < c < d$; then as $c \downarrow b$ we have

(22) $$E_b\{e(\tau_a);\ \tau_a < \tau_c\} \leq E_b\{e^{Q(\tau_a \wedge \tau_d)};\ \tau_a < \tau_c\} \to 0$$

because $P_b\{\tau_a < \tau_c\} \to 0$. Hence there exists c such that

(23) $$E_b\{e(\tau_a);\ \tau_a < \tau_c\} < \frac{1}{u(a,b)} .$$

This is just (15) above, and so reversing the argument there, we conclude that the sum of the first term in (14) over $n \geq 0$ must converge. Thus $u(b,c) < \infty$, as was to be shown.

To sum up:

Theorem 2. The function $(a,b) \to u(a,b)$ is continuous in the region $a \leq b < \beta$ and in the region $\alpha < b \leq a$. Furthermore, extended continuity holds in $a \leq b \leq \beta$ and $\alpha \leq b \leq a$, except at (β,β) when $\beta < \infty$, and at (α,α) when $\alpha > -\infty$.

Proof: To see that there is continuity in the extended sense at (a,β), where $a < \beta$, let $a < b_n \uparrow \beta$. Then we have by Fatou's lemma

$$\varliminf_{n\to\infty} u(a,b_n) \geq E_a\{\lim_{n\to\infty} e(\tau_{b_n})\} = E_a\{e(\tau_\beta)\} = u(a,\beta) = \infty .$$

If $\beta < \infty$, then $u(\beta,\beta) = 1$ by dêfinition, but $u(a,\beta) = \infty$ for all $a < \beta$; hence u is not continuous at (β,β). The case for α is similar.

3. The Connections

Now let X be the standard Brownian motion on R and q be bounded and continuous on R.

Theorem 3. Suppose that $u(x,b) < \infty$ for some, hence all, $x < b$. Then $u(\cdot,b)$ is a solution of the Schrödinger equation:

$$\frac{1}{2}\varphi'' + q\varphi = 0$$

in $(-\infty,b)$ satisfying the boundary condition

$$\lim_{x\to b} \varphi(x) = 1.$$

There are several proofs of this result. The simplest and latest proof was found a few days ago while I was teaching a course on Brownian motion. This uses nothing but the theorem by H. A. Schwarz on generalized second derivative and the continuity of $u(\cdot,b)$ proved in Theorem 2. It will be included in a projected set of lecture notes. An older proof due to Varadhan and using Ito's calculus and martingales will be published elsewhere. An even older unpublished proof used Kac's method of Laplace transforms of which an incorrect version (lack of domination!) had been communicated to me by an *ancien collègue*.

But none of these proofs will be given here partly because they constitute excellent exercises for the reader, and partly because the results have recently been established in any dimension (for a bounded open domain in lieu of $(-\infty,b)$), in collaboration with K. M. Rao. These are in the process of consolidation and extension.

I am indebted to Pierre van Moerbeke for suggesting the investigation in this note. The situation described in Theorem 1 for the case of Brownian motion apparently means the absence of "bound states" in physics!